PROCEEDINGS OF SCIENCE

Determining $|V_{ub}|$ and $|V_{cb}|$ via hadronic form factors: An overview of the progress in the last two years

Aoife Bharucha*
Physik Department T31, Technische Universität München, James-Franck-Straße 1, D-85748 Garching, Germany
E-mail: aoife.bharucha@tum.de

We summarise the current status of those lattice QCD and QCD sum rules (on the light cone) form factor calculations for semileptonic B meson decays relevant to the extraction of the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, $|V_{ub}|$ and $|V_{cb}|$.

The 15th International Conference on B-Physics at Frontier Machines at the University of Edinburgh, 14 -18 July, 2014
University of Edinburgh, UK

*Speaker.

PoS(Beauty2014)010

1. Introduction

Although at first sight, the picture of the Cabibbo-Kobayashi-Maskawa (CKM) matrix may appear to be complete [1, 2], on closer examination a number of tensions are revealed. At the summer conferences this year, the current status of the average inclusive and exclusive measurements of $|V_{ub}|$ and $|V_{cb}|$ were presented by the UTFit collaboration revealing of 1.9σ for $|V_{ub}|$, $V_{ub}^{\text{excl.}} = (3.42 \pm 0.22) \cdot 10^{-3}$ and $V_{ub}^{\text{incl.}} = (4.40 \pm 0.31) \cdot 10^{-3}$, and 2.5σ for $|V_{cb}|$, $V_{cb}^{\text{excl.}} = (39.55 \pm 0.88) \cdot 10^{-3}$ and $V_{cb}^{\text{incl.}} = (41.7 \pm 0.7) \cdot 10^{-3}$ [3] (similar results were also found by CKMFitter [1]). Interestingly, while for $|V_{ub}|$, on removing the central measured value from the fit it was found that the exclusive result was preferred, for $|V_{cb}|$ the inclusive value is preferred. Note that this is not completely surprising, due to the difficulty of the inclusive measurement for $|V_{ub}|$ due to the large $|V_{cb}|$ background. In this talk I will review the latest progress in the exclusive channels, i.e. on form factor calculations, mentioning directions currently being pursued that might lead to a better understanding of this issue in the future.

2. Methods to determination of $|V_{xb}|$ via form factors

One obtains $|V_{xb}|$, where $x = u, c$ from exclusive measurements making use of the form factors that describe the relevant hadronic transition. Schematically, the differential branching fraction can be expressed (ignoring kinematics) via,

$$\frac{d\mathscr{B}}{dq^2} \propto |V_{xb}|^2 |f(q^2)|^2, \tag{2.1}$$

which depends on q^2, the squared momentum difference between the initial and final state hadrons and $f(q^2)$, the form factor for the process which characterises the hadronic matrix element containing the relevant quark current. Results from the B factories either involve measurements of:

- the integrated spectrum over range in q^2:

$$\Delta\zeta(0, q^2_{\text{max}}) = \frac{1}{|V_{xb}|^2} \int_0^{q^2_{\text{max}}} dq^2 \frac{d\Gamma}{dq^2}$$

- the kinematical endpoint where form factors simplify:

$$|V_{ub}|^2 f_+^{B\to\pi}(0)^2 \qquad \text{or} \qquad |V_{cb}|^2 \mathscr{F}^{B\to D^*}(q_{\text{max}})^2$$

- the entire spectrum using an extrapolation technique.

In order to make optimal use of the experimental results, it is helpful to fit either the experimental data, the theoretical predictions or both to a parameterisation which describes the shape of the differential branching fraction or the form factor as a function of $t = q^2$. These parameterisations generally fall under two categories, those based on a simple pole and those based on a series expansion.

- Simple pole-type (BK or BZ [4]): e.g. $f(t) = \dfrac{r_1}{1 - t/m_R^2} + \dfrac{r_2}{1 - t/m_{\text{fit}}^2}$

- (Simplified) Series expansion (BGL or BCL [5]): $f(t) = \frac{1}{N}\sum_k \alpha_k z^k(t)$

For the simple pole-type expansions, r_i and $m_{\rm fit}$ are fit parameters. The first term describes any low-lying resonances of mass m_R (i.e. m_{B^*} for $B \to \pi$) via a simple pole $1/P(t)$ where $P(t) = 1 - t/m_R^2$. For the series expansions, t is mapped onto a complex variable $z(t)$ via the transformation

$$z(t) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}} \tag{2.2}$$

where $t_\pm = \sqrt{m_B^2 \pm m_\pi^2}$ and t_0 is a free parameter within the range t_- to t_+. t_0 is often chosen to optimise the convergence of the series by the choice $t_0 = t_+\left(1 - \sqrt{1 - \frac{t_-}{t_+}}\right)$. In principle, the series expansions have the advantage that it is possible to impose a unitarity bound on the coefficients α_k (see [5]). The BGL and BCL expansions are distinguished by their modelling of the low-lying resonances. In the former $N = \phi(t)z(t, m_{\rm R}^2)$, and in the latter $N = P(t)$. The function $\phi(t)$ is chosen to simplify the form of the unitarity bound to $\sum_k \alpha_k^2 < 1$, and therefore the bound in the BCL case is more complicated. In practice however, this bound is very weak, and almost all parameterisation methods model the data equally well, particularly when both sum rules and Lattice results are available.

3. Progress in $|V_{ub}|$

The most competitive determination of $|V_{ub}|^{\rm \bf excl}$ is possible using $B \to \pi l \nu$, the branching fraction for which was precisely measured at the B factories [6, 7]. This depends on $f_+(q^2)$ (the only relevant form factor in the limit $m_l \to 0$) calculable from Lattice QCD (in the range $q^2 \gtrsim 15\,{\rm GeV}^2$) or QCD sum rules on the light-cone (LCSR) (in the range $q^2 \lesssim 6-7\,{\rm GeV}^2$). $|V_{ub}|^{\rm \bf excl}$ can also be obtained via other B decays, i.e. $B_{(s)} \to P l \nu$ where $P = \omega$ or K is a pseudoscalar meson, $B_{(s)} \to V l \nu$ where $V = \rho$ or K^* is a vector meson, or even via Λ_b decays, i.e. $\Lambda_b \to p l \nu$. There has been recent interest in the prospects to measure $|V_{ub}|$ at LHCb via B_s or Λ_b decays due to the larger production rates compared to that at the B factories, as well as the fact that the predominant final states are heavier than for B decays and therefore may be easier resolved in the hadronic LHC environment. A summary of the theory status for the form factors required for V_{ub} is given in table 1.

In LCSR one considers a correlator Π_μ of the time-ordered product of two quark currents, sandwiched between the final state hadron, which is on shell, and the vacuum [18], i.e. for a B decaying to a π of momenta p_B and p,

$$\Pi_\mu = i m_b \int d^D x e^{-i p_B \cdot x} \langle \pi(p) | T\{\bar{u}(0)\gamma_\mu b(0) \bar{b}(x) i\gamma_5 d(x)\} | 0 \rangle. \tag{3.1}$$

This can be expressed on one hand by a light-cone expansion via perturbative hard scattering kernels convoluted with non-perturbative light-cone distribution amplitudes (LCDAs), ordered in increasing twist, or by inserting a sum over excited states, i.e. the b hadron and a continuum of heavier states.

Decay	Calculation	Collaboration	Ref.
$B\to\pi$	LCSR (PS)	Ball and Zwicky	[4]
	LCSR ($\overline{\rm MS}$)	Khodjamirian et al	[8]
	LCSR ($@q^2=0$)	Bharucha	[9]
	LQCD	FNAL/MILC	[10]
	LQCD	RBC/UKQCD	[11]
	LQCD	HPQCD	[12]
$B\to\rho$	LCSR (PS)	Ball and Zwicky	[13]
	LCSR (corr. errors)	Bharucha, Straub and Zwicky	[14]
$B_s\to K^{(*)}$	LCSR (PS)	Ball and Zwicky	[4, 13]
	LCSR (corr. errors)	Bharucha, Straub and Zwicky	[14]
	LQCD	FNAL/MILC	[15]
	LQCD	HPQCD	[12]
$\Lambda_b\to p$	LCSR	Khodjamirian et al	[16]
	LQCD	Meinel	[17]

Table 1: A summary of the theory status of the calculations of the form factors required to calculate $|V_{ub}|$ exclusively.

For the case of $B\to\pi$, in the physical region the correlator $\Pi_+(p_B^2,q^2)$ is expressed as

$$\Pi_+(p_B^2,q^2)=f_Bm_B^2\frac{f_+(q^2)}{m_B^2-p_B^2}+\int_{s>m_B^2}ds\frac{\rho_{\rm had}}{s-p_B^2}, \tag{3.2}$$

where p_B and m_B are the mass and momentum of the B meson, f_B is the B decay constant and ($\rho_{\rm had}$ is the spectral density of the higher-mass hadronic states described earlier. In the Euclidean region where $p_B^2-m_B^2$ is large and negative, one can light-cone expand about $x^2=0$[1]

$$\Pi_+(p_B^2,q^2)=\sum_n\int du\,\mathscr{T}_+{}^{(n)}(u,p_B^2,q^2,\mu^2)\phi^{(n)}(u,\mu^2)=\int ds\frac{\rho_{\rm LC}}{s-p_B^2}, \tag{3.3}$$

where $\mathscr{T}_+^{(n)}(u,\mu^2)$ are perturbatively calculable hard kernels, $\phi^{(n)}(u,\mu^2)$ are non-perturbative LCDAs at twist n. Above the continuum threshold s_0, a continuum of states contribute and approximation of quark-hadron duality is thought to be reasonable, such that $\rho_{\rm had}=\rho_{\rm LC}\Theta(s-s_0)$. Subtracting from both sides, and Borel transforming (M^2=Borel parameter): This leads to the sum rule for $f_+(q^2)$,

$$f_+(q^2)=\frac{1}{f_Bm_B^2}\int_{m_b^2}^{s_0}ds\,\rho_{\rm LC}\,e^{-(s-m_B^2)/M^2}. \tag{3.4}$$

Note that the Borel transform ensures that assuming quark-hadron duality and truncating the series have a minimal effect on the resulting sum rule.

The next-to-leading order (NLO) corrections to $f_+(q^2)$ at leading twist (twist-2) were first calculated in LCSR in ref. [19] and LO corrections up to twist-4 were calculated in ref. [20]. Since

[1]This factorisation theorem is not proven to all orders, verified at given order by cancellation of IR and soft divergences

the LO twist-3 contribution was found to be large, it was confirmed that the NLO corrections are under control, using both the pole and $\overline{\text{MS}}$ mass for m_b [4, 8]. The subset of two-loop corrections proportional to β_0 to the form factor $f_+(0)$ at twist-2 were calculated in 2012 in ref. [9] in LCSR. This work aimed, partly, at testing the argument that, in obtaining $f_+(q^2)$ via LCSR, radiative corrections to $f_+ f_B$ and f_B should cancel when both calculated in sum rules, as the two-loop sum rules corrections to f_B are known to be large ref. [21]. Further details can be found in ref. [9]. The results show that despite the $\sim 9\%$ positive NNLO corrections to the QCD sum rules result for f_B, the LCSR prediction for $f_+(0)$ is stable and only increases by $\sim 2\%$ to $f_+(0) = 0.261^{+0.020}_{-0.023}$. This suggests the stability of LCSR with respect to higher order corrections, and could provide confirmation that f_B from sum rules, not Lattice should be used here. A recent analysis by BaBar [7] finds $|V_{ub}| = (3.34 \pm 0.10 \pm 0.05 + ^{+0.29}_{-0.26}) 10^{-3}$ using this result, and $|V_{ub}| = (3.46 \pm 0.06 \pm 0.08^{+0.37}_{-0.32}) 10^{-3}$ using $\Delta\zeta(0, 12\,\text{GeV}^2)$ from ref. [8], which are clearly in good agreement.

There have been several recent updates to lattice QCD calculations relevant to $|V_{ub}|$, as well as a number of calculations undertaken for the first time, see table 1. The most promising for the moment are the preliminary $N_f = 2+1$ results for $B \to \pi$ [10, 11, 12], which are complementary being based on different heavy quark actions (including Fermilab, NRQCD, RHQ, and HQET). The smallest uncertainty on $|V_{ub}|$ comes from FNAL/MILC (note that the calculation is still normalization blinded), where the error on $|V_{ub}|$ is now 4.1% as compared to 8.8% in 2008 [10]. Preliminary results from the RBC-UKQCD collaboration from a 3+1 parameter BCL fit to BaBar and Belle data were presented at Lattice 2014, finding $|V_{ub}|$=$(3.54 \pm 0.36) \cdot 10^{-3}$ [11]. $B \to K_s$ form factors are also being pursued by FNAL/MILC [15] and HPQCD [12] collaborations, which might allow a measurment of $|V_{ub}|$ at LHCb.

Finally, as mentioned earlier, there has of late been a number of efforts to improve the understanding of the form factors describing semileptonic Λ_b decays, despite a number of complications which arise when baryons are considered instead of mesons. One such complication is the choice of the heavy-light baryon interpolating current η described by Γ_b and $\tilde{\Gamma}_b$, $\eta = \varepsilon^{ijk}(u_i C \Gamma_b d_j)\tilde{\Gamma}_b c_k$, debated since the 1980s. Additionally, the contribution of the negative parity Λ_b^* baryon, with $J^P = 1/2^-$, which has a similar mass to Λ_b is difficult to isolate, and in the literature was often included in the continuum [22]. Recently however it was found to be possible to separate the Λ_b^* from the Λ_b contribution in the sum rule, and on comparing results for both $\Gamma_b = \gamma_5(\gamma_5\gamma_\lambda)$ and $\tilde{\Gamma}_b = 1(\gamma_\lambda)$, it was found that the resulting form factors show a reduced dependence on the choice of Γ_b and $\tilde{\Gamma}_b$ [16]. Further there has been some encouraging progress on the Lattice, where using RHQ b quarks (instead of Eichten-Hill static b quarks) and domain wall u, d, s quarks resulted in the uncertainty shrinking by more than 50% [17].

4. Exclusive determination of $|V_{cb}|$

The current status of form factor calculations enabling an extraction of V_{cb} from exclusive channels is given in table 2. The decay $B \to D^*$ is generally preferred due to the higher experimental rate and lack of nonperturbative $\mathscr{O}(1/m_b)$ corrections to the form factor, but $B_{(s)} \to D_{(s)}$ decays are also of interest. The relevant form factors for $B \to D^{(*)}$ are defined via

$$\frac{d\Gamma}{d\omega}(B \to D^* \ell \bar{\nu}_\ell) = \frac{G_F^2}{48\pi^3} |V_{cb}|^2 M_{D^*}^3 (\omega^2 - 1)^{1/2} P(\omega) (\mathscr{F}(\omega))^2$$

Decay	Calculation	Collaboration	Ref.	$\lvert V_{cb}\rvert \cdot 10^{-2}$
$B \to D^*$	SR	Gambino et al	[23, 24]	$41.6 \pm 0.6_{\rm exp} \pm 1.9_{\rm th}$
	LQCD	FNAL/MILC	[25]	$39.04 \pm 0.49_{\rm exp} \pm 0.53_{\rm QCD} \pm 0.19_{\rm QED}$
	LQCD	HPQCD	[12]	–
$B \to D$	SR	Uraltsev et al	[26]	$40.6 \pm 1.5_{\rm exp} \pm 0.8_{\rm th}$
	LQCD	FNAL/MILC	[25]	$38.5 \pm 1.9_{\rm exp+lat} \pm 0.2_{\rm QED}$
	LQCD	HPQCD	[12]	–
$B_s \to D_s^{(*)}$	LQCD	ETM	[27]	–
	LQCD	HPQCD	[28, 12]	–

Table 2: A summary of the current theory status of the calculations of the form factors required to calculate $|V_{cb}|$ exclusively.

$$\frac{d\Gamma}{d\omega}(B \to D\,\ell\bar{\nu}_\ell) = \frac{G_F^2}{48\pi^3}|V_{cb}|^2(M_B+M_D)^2 M_D^3(\omega^2-1)^{3/2}(\mathscr{G}(\omega))^2$$

where $\omega = v \cdot v' = E_{D^{(*)}}/M_{D^{(*)}}$ (in the B rest frame), and $P(\omega)$ is a known phase space factor. Note that in heavy quark limit: $\mathscr{F}(1) = \mathscr{G}(1) = 1$. The above also hold for the $B_s \to D_s^{(*)}$ channels by making the obvious replacements.

In 2012, the sum rule calculation of the zero-recoil form factor for $B \to D^*$ was revisited [24]. It was found that from the precise measurement of the hyperfine splitting, one can deduce a lower limit on the D^*'s zero momentum non-local correlation, which was found to be larger than expected. This enhances the predicted contribution from inelastic operators in the sum rule for the $B \to D^*$ form factor. This resulted in a lower value of $\mathscr{F}(\infty)$, and further predicts a larger than expected branching fraction of the B to the radially excited D mesons. This could explain the puzzling discrepancy between the inclusive and the sum over the exclusive $B \to D(*)X$ measured rates.[2] The question of the discrepancy with the current lattice QCD results for $\mathscr{F}(\infty)$ was also discussed.

At present, the only $N_f = 2+1$ Lattice result for $\mathscr{F}(\infty)$ comes from the FNAL/MILC collaboration, with Fermilab bottom and charm valence quarks. Recently in ref. [25] the uncertainty on the form factor at maximum recoil dropped from 2.6% to 1.4$ ($\mathscr{F}(1) = 0.906 \pm 0.004 \pm 0.01$), resulting in an error on $|V_{cb}|$ of 1.9% (see table 2), where the reduction in uncertainty is due to the smaller lattice spacings ($a \simeq 0.045 - 0.15$ fm), smaller light quark masses and increased statistics. Note that while the $N_f = 2+1$ Lattice error has decreased, the central value of $\mathscr{F}(1)$ for $B \to D^*$ is unchanged, increasing the tension both with the inclusive result (by 3σ), and with the result from sum rules $\mathscr{F}(1) = 0.86 \pm 0.02$ [24]. Therefore cross-checks both on the lattice, sum rules and experimental sides would be crucial, and the HPQCD collaboration is pursuing this using NRQCD heavy and highly improved action (HISQ) light valence quarks on the MILC 2+1 dynamical asqtad configurations.

Alternative cross-checks can be obtained via the decays to pseudoscalars $B \to D$ and $B_s \to D_s$. In addition to the results from FNAL/MILC and ongoing work on $B \to D$ by the HPQCD collaboration (using HISQ charm quarks), see table 2, the European twisted mass (ETM) collaboration

[2] Searches for these modes are under way at BaBar, see e.g. Bill Gary's talk at Beauty 2014.

has recently published results for the $B_{(s)} \to D_{(s)}$ for $N_f = 2$ dynamical light quarks. They employ the ratio method at four different lattice spacings within the range $a \simeq 0.054$ to 0.098 fm, using the maximally twisted Wilson quark action and obtain the result for $B_s \to D_s\, \mathscr{G}(1) = 1.052 \pm 0.046$, i.e. to an uncertainty of 4% [27]. At present the corresponding error on $\mathscr{G}(1)$ for $B \to D$ is $\sim 9\%$, however providing the statistics are increased, they should achieve a competitive prediction in the future.

5. Summary

In summary, there has been exciting progress in the exclusive determination of $|V_{ub}|$ and $|V_{cb}|$ over the last two years. For exclusive $|V_{ub}|$, the most precise determination at present is from $B \to \pi$. It was found that in LCSR at $\mathscr{O}(\alpha_s^2 \beta_0)$, despite $\sim 9\%$ increase in f_B from QCDSR, the prediction for $f_+(0)$ increases by $\sim 2\%$ to $f_+(0) = 0.262^{+0.020}_{-0.023}$, $|V_{ub}| = (3.34 \pm 0.10 \pm 0.05 +^{0.29}_{-0.26}) \cdot 10^{-3}$. This is in agreement with the results from the FNAL/MILC, on which the uncertainty has dropped to $\sim 4.1\%$. A BCL type analysis making use of the latest experiment and theory yields $(3.41 \pm 0.22) \cdot 10^{-3}$ as compared to the PDG 2012 inclusive result $(4.41^{+0.21}_{-0.23}) \cdot 10^{-3}$. On removing $|V_{ub}|$ from the fit, the CKMfitter and UTfit results are in agreement with exclusive result. There have also been interesting advances in the calculation of the form factors for alternative channels i.e. $B_s \to K^{(*)}$ and $\Lambda_b \to p$ from LCSR and lattice QCD, increasing the chances to obtain a competitive measurement of $|V_{ub}|$ from the LHC.

For exclusive $|V_{cb}|$, there has been a major reduction in the uncertainty from the FNAL/MILC collaboration in calculating the $B \to D^{(*)}$ form factors, increasing the significance of the discrepancy with inclusive to 3σ. Lattice calculations from the HPQCD collaboration are currently being pursued to confirm this result. It might be possible to explain the increasing tension with the inclusive result by larger-than-anticipated branching ratios for B decays to the 'radial' and/or D-wave states. A recent measurement from BABAR of $\mathscr{B}(B \to D^{(*)} X l \nu)$ supports this idea, however further measurments of branching ratios to radial $D^{(*)}$ mesons are required.

Acknowledgements

Thanks to Patricia Ball for her guidance and ideas as well as to Stefan Meinel, Sascha Turczyk, Aida El-Khadra and Andreas Kronfeld for very helpful discussions.

References

[1] J. Charles *et al.* [CKMfitter Group Collaboration], Eur. Phys. J. C **41** (2005) 1 [hep-ph/0406184].

[2] M. Bona *et al.* [UTfit Collaboration], JHEP **0610** (2006) 081 [hep-ph/0606167].

[3] D. Derkach, Talk given at ICHEP 2014, Valencia, Spain, July 4th, 2014.

[4] P. Ball and R. Zwicky, JHEP **0110** (2001) 019 [hep-ph/0110115] and Phys. Rev. D **71** (2005) 014015 [arXiv:hep-ph/0406232], G. Duplancic *et al.*, J. Phys. Conf. Ser. **110** (2008) 052026.

[5] C. G. Boyd, B. Grinstein and R. F. Lebed, Phys. Rev. Lett. **74**, 4603 (1995) [arXiv:hep-ph/9412324], C. Bourrely, I. Caprini and L. Lellouch, Phys. Rev. D **79**, 013008 (2009) [arXiv:0807.2722 [hep-ph]], A. Bharucha, T. Feldmann and M. Wick, JHEP **1009** (2010) 090 [arXiv:1004.3249 [hep-ph]].

[6] A. Sibidanov *et al.* [Belle Collaboration], Phys. Rev. D **88** (2013) 3, 032005 [arXiv:1306.2781 [hep-ex]].

[7] J. P. Lees *et al.* [BaBar Collaboration], Phys. Rev. D **86** (2012) 092004 [arXiv:1208.1253 [hep-ex]].

[8] A. Khodjamirian *et al.*, Phys. Rev. D **83** (2011) 094031 [arXiv:1103.2655 [hep-ph]].

[9] A. Bharucha, JHEP **1205** (2012) 092 [arXiv:1203.1359 [hep-ph]].

[10] D. Du *et al.*, PoS LATTICE **2013** (2013) 383 [arXiv:1311.6552 [hep-lat]], D. Du, Talk at LATTICE **2014**.

[11] T. Kawanai, R. S. Van de Water and O. Witzel, arXiv:1311.1143 [hep-lat], T. Kawanai, Talk at LATTICE **2014**.

[12] C. M. Bouchard *et al.*, arXiv:1310.3207 [hep-lat].

[13] P. Ball and R. Zwicky, Phys. Rev. D **71**, 014029 (2005) [hep-ph/0412079].

[14] A. Bharucha, D. Straub and R. Zwicky, In preparation, Edinburgh/14/17, TUM-HEP-957/14.

[15] Y. Liu *et al.*, PoS LATTICE **2013** (2013) 386 [arXiv:1312.3197 [hep-lat]].

[16] A. Khodjamirian *et al.*, JHEP **1109** (2011) 106 [arXiv:1108.2971 [hep-ph]]

[17] S. Meinel, PoS LATTICE **2013** (2014) 024 [arXiv:1401.2685 [hep-lat]], Talk at FPCP 2014.

[18] I. Balitsky, V. Braun and A. Kolesnichenko, Nucl. Phys. B **312** (1989) 509.

[19] A. Khodjamirian *et al.*, Phys. Lett. B **410** (1997) 275 [arXiv:hep-ph/9706303];
E. Bagan, P. Ball and V. M. Braun, Phys. Lett. B **417** (1998) 154 [arXiv:hep-ph/9709243].

[20] A. Khodjamirian *et al.*, Phys. Rev. D **62** (2000) 114002 [arXiv:hep-ph/0001297].

[21] M. Jamin and B. O. Lange, Phys. Rev. D **65** (2002) 056005 [arXiv:hep-ph/0108135].

[22] Y. -M. Wang, Y. -L. Shen and C. -D. Lu, Phys. Rev. D **80** (2009) 074012 [arXiv:0907.4008 [hep-ph]], K. Azizi *et al.*, Phys. Rev. D **80** (2009) 096007. [arXiv:0908.1758 [hep-ph]]

[23] P. Gambino, T. Mannel and N. Uraltsev, Phys. Rev. D **81** (2010) 113002 [arXiv:1004.2859 [hep-ph]].

[24] P. Gambino, T. Mannel and N. Uraltsev, JHEP **1210** (2012) 169 [arXiv:1206.2296 [hep-ph]].

[25] S. W. Qiu *et al.* [Fermilab Lattice and MILC Collaboration], PoS LATTICE **2013** (2014) 385 [arXiv:1312.0155 [hep-lat]].

[26] N. Uraltsev, Phys. Lett. B **585** (2004) 253 [hep-ph/0312001].

[27] M. Atoui, V. Morénas, D. Bečirevic and F. Sanfilippo, Eur. Phys. J. C **74** (2014) 2861 [arXiv:1310.5238 [hep-lat]].

[28] C. M. Bouchard *et al.*, PoS LATTICE **2012** (2012) 118 [arXiv:1210.6992 [hep-lat]].