TEK twisted gradient flow running coupling

Margarita García Pérez

Instituto de Física Teórica UAM/CSIC, Universidad Autónoma de Madrid, E-28048-Madrid, Spain
E-mail: margarita.garcia@uam.es

Antonio González-Arroyo

Instituto de Física Teórica UAM/CSIC and Departamento de Física Teórica, C-15, Universidad Autónoma de Madrid, E-28049-Madrid, Spain
E-mail: antonio.gonzalez-arroyo@uam.es

Liam Keegan[*]

PH-TH, CERN, CH-1211 Geneva 23, Switzerland
E-mail: liam.keegan@cern.ch

Masanori Okawa

Graduate School of Science, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan
E-mail: okawa@sci.hiroshima-u.ac.jp

We measure the running of the twisted gradient flow coupling in the Twisted Eguchi-Kawai (TEK) model, the SU(N) gauge theory on a single site lattice with twisted boundary conditions in the large N limit.

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[*]Speaker.

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1. Introduction

The Twisted Eguchi-Kawai (TEK) model [1, 2] provides a single site formulation of SU(N) lattice gauge theory, which in the infinite N limit at fixed bare 't Hooft coupling reproduces the infinite volume theory. Moreover, the corrections to the TEK model at finite N take the form (at least in perturbation theory) of finite volume corrections for an effective lattice size of $\sqrt{N}$. For example, propagators are identical to those of a normal $(\sqrt{N})^4$ lattice [1, 3]. If this relation holds in the continuum limit, it would be possible to use the rank of the gauge group as a size parameter and determine the running of the coupling with respect to it. Essentially, we could simply use a standard finite volume step scaling procedure and apply it to the single-site model replacing the linear size of the finite volume, $l = La$, by $\tilde{l} = \sqrt{N}a$. Usually the renormalization scale is defined in terms of the linear size of the finite volume, $l = La$, and the change of scale $l \to sl$ is accomplished by changing the number of points in the lattice, $L \to sL$. For the TEK model, the lattice always consists of a single site, and we implement the change of scale $\tilde{l} \to s\tilde{l}$ by scaling the rank of the gauge group, SU(N) $\to$ SU(s^2N). To extract the continuum renormalized coupling constant, one should take the $a \longrightarrow 0$ ($N \longrightarrow \infty$) limit. Thus, according to volume independence, we expect that the renormalized coupling coincides with that of the ordinary pure gauge theory at $N = \infty$.

To be specific, we based our method on a finite volume running coupling scheme defined using the gradient flow [6] in a four dimensional torus with twisted boundary conditions in one plane [4, 5]. Here we define an analogous scheme in the TEK single site model, which has twisted boundary conditions in all directions, and use it to perform a step scaling study and determine the running of the coupling in SU(N) gauge theory in the large N limit.

2. TEK Volume Independence

The TEK model consists of 4 SU(N) matrices U_μ, with the action

$$S = bN \sum_{\mu\nu} \left(N - z_{\mu\nu}\text{Tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \right] \right), \quad z_{\mu\nu} = z_{\nu\mu}^* = e^{2\pi ik/\sqrt{N}} \text{ for } \mu < \nu, \qquad (2.1)$$

where b is the lattice analog of the inverse 't Hooft coupling, $1/(Ng^2)$, and the flux k is an integer coprime with $\sqrt{N}$. In order for volume independence to hold in the large N limit, center symmetry must not be spontaneously broken, i.e. the trace of all open Wilson loops on the lattice should go to zero in this limit. This is the case if the flux k is chosen to satisfy $k/\sqrt{N} > 1/9$. The left–hand plot of Fig. 1 shows the quantity $\sqrt{b}\left|\text{Tr}\,U_\mu\right|$ as a function of $k/\sqrt{N}$ for many values of N and b, along with the perturbative prediction, $\sqrt{b}\left|\text{Tr}\,U_\mu\right| \propto 1/\sin(\pi k/\sqrt{N})$, which is in good agreement with the data for $b \gtrsim 2.0$. This quantity divided by N is shown as a function of $1/N$ for the weakest and strongest values of the coupling used for the step scaling analysis in the right–hand plot of Fig. 1. Since this goes to zero in the large N limit, reduction should hold.

To all orders in perturbation theory, the TEK model is equivalent to ordinary lattice gauge theory, up to corrections that depend on the parameter $\tilde{\theta}$, where $\tilde{\theta} = 2\pi\bar{k}/\sqrt{N}$, and $\bar{k}$ is defined as the integer that satisfies $k\bar{k} = 1 \pmod{\sqrt{N}}$. To eliminate this effect one should scale k and $\sqrt{N}$ so as to keep $\tilde{\theta}$ constant in the large N limit. Strictly speaking this is not possible as $\bar{k}$ and $\sqrt{N}$ have to be coprime. This is a source of systematic error in our results. However, the variations in $\tilde{\theta}$ can
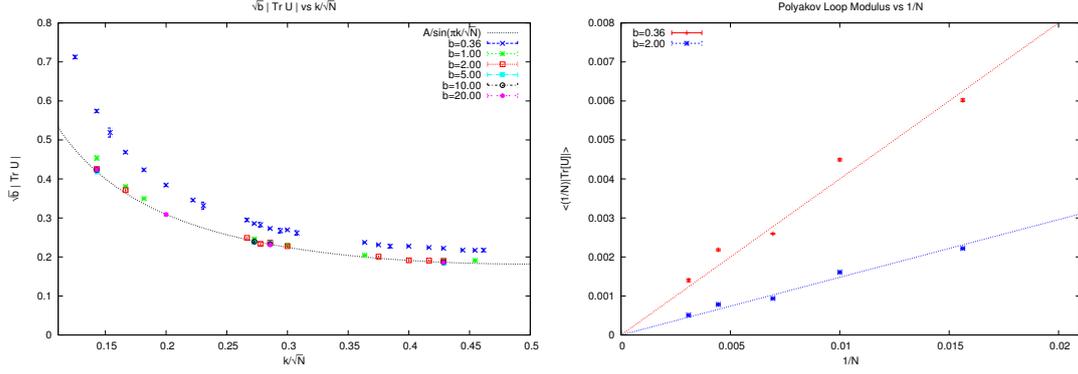
Figure 1: Left: $\sqrt{b}\left|\mathrm{Tr}\,U_\mu\right|$ vs $k/\sqrt{N}$ compared to the pertubative expectation, for many values of b and N. Right: $\frac{1}{N}\left|\mathrm{Tr}\,U_\mu\right|$ vs $1/N$ at $b=0.36$ and $b=2.00$. For all values of the coupling this quantity seems to go to zero in the large N limit. The scatter of the points is caused by $k/\sqrt{N}$ varying somewhat with N.

be made smaller for larger values of N, since there are more possible choices for k. The values of k and N used in this work are listed in Tab. 1.

3. Gradient Flow Coupling

At positive Yang–Mills gradient flow time, the action density of SU(N) gauge theory is a renormalized quantity which, in infinite volume, has a perturbative expansion [6],

$$\langle E(t)\rangle = \frac{1}{4}\left\langle G^a_{\mu\nu}(t)G^a_{\mu\nu}(t)\right\rangle = \frac{3(N^2-1)}{128\pi^2 t^2}g^2_{MS} + \mathcal{O}(g^4_{MS}). \tag{3.1}$$

If we work at finite box size l and fix the flow time to a constant fraction of this size, $\sqrt{8t}=cl$, we obtain a quantity that depends on a single length scale and that to leading order is proportional to the coupling constant. This allows us to define a renormalized coupling as follows [4, 5]:

$$\lambda_{TGF}(l) \equiv \mathcal{N}_T^{-1}(c)\frac{1}{N}t^2\langle E\rangle\big|_{t=c^2 l^2/8} = \lambda_{\overline{MS}} + \mathcal{O}(\lambda^2_{\overline{MS}}) \tag{3.2}$$

where $\lambda = Ng^2$ is the 't Hooft coupling. The constant $\mathcal{N}_T(c)$ is a kinematic factor that ensures that to leading order $\lambda_{TGF} = \lambda_{\overline{MS}}$. In the previous definition the constant c is kept fixed as the scale is changed. A change in c can be considered a change of renormalization scheme.

In this work, we use this idea to construct the equivalent renormalized coupling definition for the TEK model where $\tilde{l} = a\sqrt{N}$ replaces l. We also need to define the observable that will be used to estimate the action density. The simplest choice is the plaquette E_P:

$$E_P = \sum_{\mu\nu}\left(N - z_{\mu\nu}\mathrm{Tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right]\right) \tag{3.3}$$

It is convenient to adjust $\mathcal{N}_T(c)$ to preserve the equality of the bare and renormalized couplings at leading order on the lattice. This gives

$$\mathcal{N}_T^P(c) = \frac{3c^4}{128}\sum_n{}' e^{-c^2 N\sum_\rho \sin^2(\pi n_\rho/\sqrt{N})}. \tag{3.4}$$

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where $n_\rho = 0, 1, \ldots, \sqrt{N} - 1$ and the prime in the sum means that we do not include the term with all $n_\rho = 0$.

Alternatively, one can take a different observable E_S, which we will call *symmetric*, as follows

$$E_S = -\tfrac{1}{128} \sum_{\mu,\nu} \text{Tr} \left\{ \left[z_{\nu\mu} U_\nu U_\mu U_\nu^\dagger U_\mu^\dagger + z_{\nu\mu} U_\mu U_\nu^\dagger U_\mu^\dagger U_\nu + z_{\nu\mu} U_\nu^\dagger U_\mu^\dagger U_\nu U_\mu + z_{\nu\mu} U_\mu^\dagger U_\nu U_\mu U_\nu^\dagger - h.c. \right]^2 \right\}.$$
(3.5)

We can compute the corresponding factor $\mathscr{N}_T(c)$ which is given by

$$\mathscr{N}_T^S(c) = \tfrac{c^4}{512} \sum_{\mu \neq \nu} \sum_n e^{-c^2 N \sum_\rho \sin^2(\pi n_\rho / \sqrt{N})} \frac{\sin^2(2\pi n_\mu / \sqrt{N}) \cos^2(\pi n_\nu / \sqrt{N})}{\sum_\rho \sin^2(\pi n_\rho / \sqrt{N})}.$$
(3.6)

We emphasize that choosing $\mathscr{N}_T$ for each observable as we have done considerably reduces lattice artefacts as compared to using the continuum factor

$$\mathscr{N}_T(c) = \tfrac{3c^4}{128} \sum_{n \in Z^4 - \{0\}} e^{-\pi^2 c^2 n^2} = \tfrac{3c^4}{128} \left[(\theta_3(0, i\pi c^2))^4 - 1 \right]$$
(3.7)

where the sum is over all non-zero integer 4-vectors, and $\theta_3(0, z)$ is the Jacobi function.

We define a continuum step scaling function in the usual way:

$$\sigma(u, s) = \lambda_{TGF}(s\tilde{l}) \big|_{\lambda_{TGF}(\tilde{l}) = u}$$
(3.8)

In order to obtain this quantity from the TEK model, one starts with the lattice equivalent

$$\Sigma(u, s, \sqrt{N}) = \lambda_{TGF}(s\sqrt{N}, b) \big|_{\lambda_{TGF}(\sqrt{N}, b) = u}$$
(3.9)

and then take the continuum limit ($a \longrightarrow 0 \Leftrightarrow N \longrightarrow \infty$) keeping u fixed.

4. Numerical Determination of λ_{TGF}

We choose a step scaling factor $s = 3/2$, and simulate a series of pairs of N; $\sqrt{N} = 8, 10, 12$ and $s\sqrt{N} = 12, 15, 18$. The corresponding run parameters are listed in Table 1. For each N we simulate a series of bare couplings going from weak ($b = 2.00$) to strong ($b \simeq 0.36$) coupling. Each configuration generated in the simulation is separated by a number of sweeps, where each sweep consists of one heat–bath and 5 over–relaxation updates. The number of sweeps is chosen such that autocorrelations are negligible. This number increases both with N and as the coupling is made stronger, and goes up to 1600 sweeps between each measurement for $N = 324$ at $b = 0.37$. The Wilson flow is integrated using the 3rd order Runge–Kutta scheme proposed in Ref. [6], choosing the integration stepsize between 0.01 and 0.03, and ensuring that the resulting integration errors are much smaller than the statistical uncertainties.

The parameter c is in principle arbitrary, and different values correspond to different renormalization schemes. In general, a smaller value of c will result in smaller statistical uncertainties, but at the cost of larger lattice artefacts, and vice versa. Here we take $c = 0.30$ as a good compromise between these two effects. The measured couplings using the symmetric definition are listed in Tab. 2, and have statistical errors $\mathscr{O}(0.3 - 0.5\%)$.

	$\sqrt{N}=8$	$\sqrt{N}=10$	$\sqrt{N}=12$	$\sqrt{N}=15$	$\sqrt{N}=18$
$(k,\bar{k})$	(3,3)	(3,3)	(5,5)	(4,4)	(5,7)
$\tilde{\theta}/2\pi = \bar{k}/\sqrt{N}$	0.375	0.300	0.417	0.267	0.389

Table 1: Run parameters for each *N*, where *k* is the flux, and $\tilde{\theta}/2\pi = \bar{k}/\sqrt{N}$ is the quantity we would ideally keep constant for all *N*.

b	$\sqrt{N}=8$	$\sqrt{N}=10$	$\sqrt{N}=12$	$\sqrt{N}=15$	$\sqrt{N}=18$
0.360	16.643(77)	21.05(10)	25.60(12)	-	-
0.365	14.383(61)	17.492(82)	20.755(88)	-	-
0.370	12.979(53)	15.445(67)	17.857(81)	23.52(11)	-
0.375	11.843(45)	13.672(58)	15.698(63)	19.788(94)	24.17(11)
0.380	10.986(40)	12.469(51)	14.051(57)	17.350(81)	20.496(97)
0.390	9.624(33)	10.685(40)	11.801(44)	13.882(59)	15.626(66)
0.400	8.601(28)	9.402(33)	10.246(37)	11.579(44)	13.001(54)
0.420	7.091(22)	7.652(25)	8.190(28)	8.966(29)	9.727(35)
0.450	5.718(17)	6.075(19)	6.351(19)	6.796(20)	7.185(24)
0.500	4.370(12)	4.546(13)	4.726(14)	4.971(14)	5.156(15)
0.600	2.9878(76)	3.0635(79)	3.1536(87)	3.2498(86)	3.3256(88)
0.800	1.8556(46)	1.8815(48)	1.9041(47)	1.9419(47)	1.9720(49)
1.000	1.3434(33)	1.3618(32)	1.3747(33)	1.3990(35)	1.4123(36)
1.200	1.0603(26)	1.0712(26)	1.0747(26)	1.0842(26)	1.0990(27)
1.500	0.8030(19)	0.8101(20)	0.8127(20)	0.8227(20)	0.8255(20)
2.000	0.5716(13)	0.5752(13)	0.5771(14)	0.5805(13)	0.5826(14)

Table 2: Measured coupling λ_{TGF} for each *b* and *N* (symmetric definition).
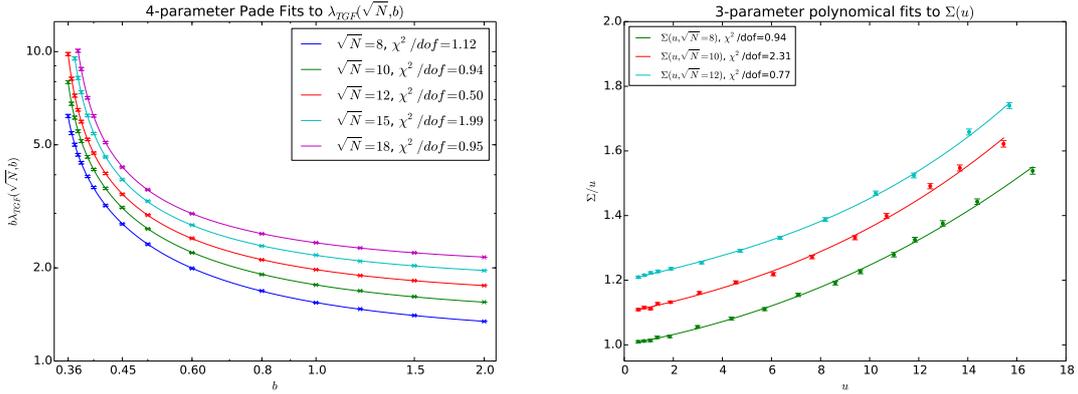

Figure 2: To take the continuum limit at fixed *u* we need to interpolate the data, which is done in two ways. Left: 4–parameter Padé interpolation of $\lambda_{TGF}(\sqrt{N},b)$ in *b* (data at different *N* displaced vertically by 0.2). Right: 3–parameter polynomial interpolation of $\Sigma(u,s,\sqrt{N})/u$ in *u* (data displaced vertically by 0.1).

5. Continuum Extrapolation of the Step Scaling Function

To take the continuum limit of the step scaling function at fixed u we need to interpolate the data. In order to check for systematics we do this in two different ways. The first is to interpolate the coupling as a function of b, for each N, using a 4–parameter Padé function of the form

$$\lambda_{TGF}(N,b) = \frac{1}{b}\frac{a_0 + a_1 b + b^2}{a_2 + a_3 b + b^2},\tag{5.1}$$

as proposed in Ref. [4]. This allows us to determine the coupling for each N at any value of b, and hence the step scaling function at any value of u. The second interpolation strategy is to first construct the lattice step scaling function directly from the data for the available values of the coupling, then to interpolate this quantity as a function of u using a 3–parameter polynomial of the form

$$\Sigma(u,\sqrt{N})/u = 1 + a_0 u + a_1 u^2 + a_2 u^3.\tag{5.2}$$

Both these fit functions are constructed to have the correct leading order behaviour in the weak coupling limit, i.e. $\lambda_{TGF}(N,b) \to 1/b$ and $\Sigma(u,\sqrt{N}) \to u$. Examples of both fits are shown in Fig. 2, where the data at different N have been displaced vertically for clarity, and have a similar χ^2/d.o.f. ~ 1. Some examples of the resulting continuum extrapolation in $1/N$ are shown in the left hand plot of Fig. 3. At each value of u there are two separate continuum extrapolations. The data for the symmetric definition of the coupling are shown as crosses, while those using the plaquette definition are shown as points. The difference between the two definitions at finite N is a measure of the size of lattice artefacts, and the two definitions should extrapolate to consistent values in the continuum limit. The errorbars are determined using bootstrap replicas of the data, and using both interpolation strategies, so they include both the statistical errors and the systematic errors due to the interpolation. They do not however include the systematic error due to $\tilde{\theta}$ not being kept exactly constant as we take the continuum limit. Indeed the fact that the $\sqrt{N} = 10$ points are systematically higher than those at other values of N in the extrapolations is presumably due to this effect.

The final continuum determination of $\sigma(u)/u$ is shown in the right hand plot of Fig. 3 as a function of u, along with the 1–loop and 2–loop perturbative predictions.

6. Conclusion

We define a scale-dependent renormalized coupling constant for the SU(N) single-site TEK model by replacing the space-time size parameter of certain definitions by an effective size determined uniquely by the rank N of the group. We use standard methods to determine the running of the coupling and the step-scaling function over a wide range of scales. The lattice step-scaling is extrapolated to the continuum limit by taking the $N \longrightarrow \infty$ limit at fixed values of the coupling. An optimal extrapolation should be done keeping $\tilde{\theta} = 2\pi\bar{k}/\sqrt{N}$ approximately constant. This is a source of systematic errors which does not seem to have a strong impact on the result.

The resulting extrapolated step-scaling function shows a similar behaviour to standard $SU(N)$ definitions and matches the perturbative prediction at weak coupling. This result provides evidence that the relation between finite rank and finite volume is preserved in the continuum limit.
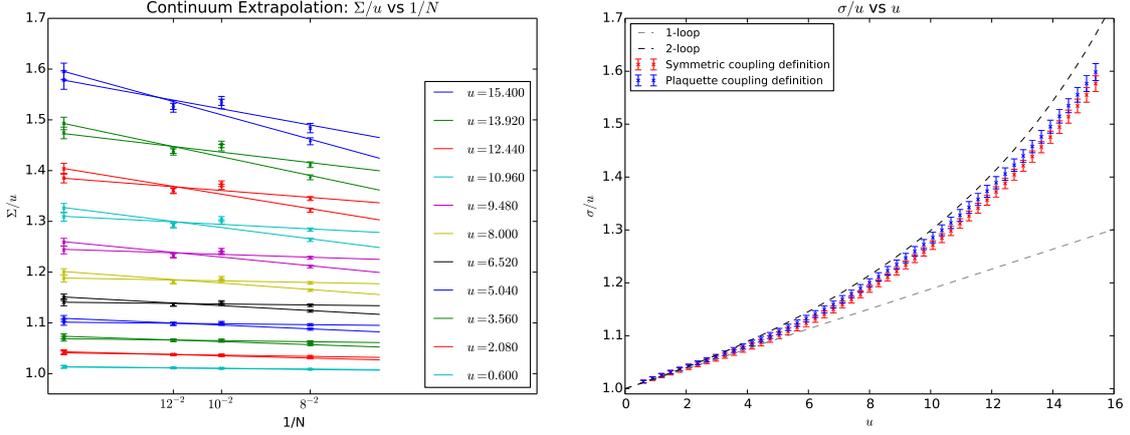
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Figure 3: Left: Some examples of the continuum extrapolation of $\Sigma(u, \sqrt{N})/u$ in $1/N$, using both the plaquette (points) and symmetric (crosses) definitions, which extrapolate to continuum values that are consistent within errors. Right: The final continuum determination of $\sigma(u, s = 3/2)/u$ vs u, using the plaquette (blue) and symmetric (red) definitions, along with the 1–loop and 2–loop perturbative predictions. The agreement between our data at weak coupling and the perturbative prediction is very good.

Acknowledgments

References

[1] A. González-Arroyo and M. Okawa, Phys. Rev. D **27**, 2397 (1983).

[2] A. González-Arroyo and M. Okawa, JHEP **1007**, 043 (2010) [arXiv:1005.1981 [hep-th]].

[3] M. García Pérez, A. González-Arroyo and M. Okawa, PoS LATTICE 2013 (2013) 342, [arXiv:1311.3465 [hep-lat]]; International Journal of Modern Physics A Vol. 29, No. 25 (2014) 1445001 [arXiv:1406.5655 [hep-th]].

[4] A. Ramos, PoS Lattice **2013** (2013) 053 [arXiv:1308.4558 [hep-lat]].

[5] A. Ramos, arXiv:1409.1445 [hep-lat].

[6] M. Luscher, JHEP **1008**, 071 (2010) [arXiv:1006.4518 [hep-lat]].