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Some advances in gauged linear sigma models

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Gauged linear sigma models (GLSM's) are simple generalizations of the supersymmetric $\mathbb{CP}^n$ model which have played a surprisingly important role in string compactifications over the last twenty years. The last six years have seen a resurgence of interest in GLSM's and some new technologies that have significantly advanced our understanding of these tools. In this talk, we will first review the basic properties of GLSM's, and then briefly discuss a few of the recent advances in their understanding.

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1. Introductions

Some of the most important technical tools used by physicists in the study of compactifications of string theory are 'gauged linear sigma models' (GLSM's) [1], generalizations of the classic supersymmetric $\mathbb{CP}^n$ model which provide a UV window into the quantum mechanics of string propagation on compactification spaces. Over the two decades since their introduction, GLSM's have provided a wealth of insight into questions ranging from global structures in moduli spaces of string compactifications and conformal field theories to worldsheet instanton corrections to matter couplings in low-energy effective field theories. Furthermore, as two-dimensional gauge theories generalizing a classic toy model in quantum field theory, GLSM's also serve as interesting quantum field theories in their own right. By learning about gauged linear sigma models, we not only learn about string compactifications, but also about quantum field theory itself.

Over the last few years there have been a number of striking advances in the understanding of GLSM's, ranging from non-Kähler compactifications (see *e.g.* [2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein) to new developments in Calabi-Yau compactifications, including realizations of non-complete-intersection target spaces [11, 12, 13, 14, 15], non-birational phases [11, 12, 15, 16, 17, 18, 19, 20] (leading to modern conjectures that all phases of GLSM's are related by a new notion known as "homological projective duality" [17, 18, 19, 20, 21, 22, 23]), nonperturbative realizations of geometry [11, 12, 15, 16, 24, 25, 26], realizations of noncommutative resolutions [12, 16, 25, 26], and localization techniques [27, 28]. Those localization techniques have been applied to deduce new methods for computing Gromov-Witten invariants [24, 29], computations of elliptic genera directly in GLSM's [30, 31, 32, 33, 34, 35], as well as many other results, see *e.g.* [30, 31, 32, 36, 37, 38, 39, 40, 41, 42, 43]. More recently, localization was applied to compute partition functions for hemispheres [44, 45, 46, 47]. One of the results of that work was an expression for central charges of D-branes involving a new multiplicative characteristic class Γ [48, 49, 50, 51], which has been applied to systematically generate arbitrarily-high order loop corrections to higher-dimensional Calabi-Yau geometries [52]. More recent work such as [53, 54] has studied dualities and dynamical supersymmetry breaking in (0,2) GLSM's. (Our own work, including [12, 16, 25, 26, 43], has spanned a number of the areas above.) Nor has this progress been confined to the physics community, as for example, work in progress on mathematics of gauged linear sigma models by T. Jarvis, Y. Ruan, and their collaborators was reported at a conference on the subject in Michigan this past March. Suffice it to say, especially within the last year, the pace of progress has been extremely rapid.

In these lectures, we will give an overview of one small aspect of this story. We will begin by reviewing pertinent aspects of nonlinear sigma models, which directly describe CFT's for Calabi-Yau manifolds but can be rather complicated to manipulate, and then gauged linear sigma models (GLSM's), which are used to construct CFT's for Calabi-Yau manifolds. Then, we will describe the construction of QFT's for Calabi-Yau's involving exotic QFT effects and CFT's for some exotic Calabi-Yau-like mathematical objects.

2. Review of nonlinear sigma models

Let us begin by reviewing how one constructs CFT's for string compactification. Briefly, a

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string propagating on some Calabi-Yau manifold is described by a nonlinear sigma model, with Lagrangian density

$$\mathscr{L} = g_{i\bar{\jmath}}(\phi)\partial_\alpha \phi^i \partial^\alpha \phi^{\bar{\jmath}} + i g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} D_{\bar{z}} \psi_+^i + i g_{i\bar{\jmath}} \psi_-^{\bar{\jmath}} D_z \psi_-^i + R_{i\bar{\jmath}k\bar{\ell}} \psi_+^i \psi_+^{\bar{\jmath}} \psi_-^k \psi_-^{\bar{\ell}}.$$

The field ϕ is a map from the worldsheet into the Calabi-Yau manifold, expressed in local coordinates on either side. The $\psi_\pm$ are Grassmann-valued chiral fermions in two dimensions, coupling to the tangent bundle of the Calabi-Yau.

Techniques are know for computing massless spectra, Gromov-Witten invariants, and a few other things, but further results quickly become painful to extract. To understand why, recall that to perturbatively quantize a nonlinear sigma model, one first Taylor expands ϕ about a constant map ϕ_0 in a local patch:

$$\phi = \phi_0 + \delta\phi.$$

Then, one expands the metric as

$$g_{i\bar{\jmath}}(\phi) = g_{i\bar{\jmath}}(\phi_0) + \delta\phi^\mu \partial_\mu g_{i\bar{\jmath}}(\phi_0) + \frac{1}{2}\delta\phi^\mu \delta\phi^\nu \partial_\mu \partial_\nu g_{i\bar{\jmath}}(\phi_0) + \cdots.$$

Plugging into the Lagrangian, we recover a purely quadratic term

$$g_{i\bar{\jmath}}(\phi_0)\partial_\alpha \delta\phi^i \partial^\alpha \delta\phi^{\bar{\jmath}}$$

plus an infinite tower of interaction terms. Because of a quirk of two dimensional theories, this theory is renormalizable despite having an infinite tower of interaction terms. On the other hand, systematically pursuing perturbative computations in a theory with infinitely many interaction terms is difficult, to say the least.

The idea behind gauged linear sigma models is to construct a comparatively easy quantum field theory, which renormalization-group flows to a nontrivial CFT. This has the advantage that the new QFT is much easier to analyze and think about. It also has the disadvantage that one has only indirect access to the CFT. Nevertheless, despite that disadvantage, gauged linear sigma models have proven to be a remarkably useful tool over the two decades since their introduction.

3. Review of gauged linear sigma models

The prototype for all gauged linear sigma models is the supersymmetric $\mathbb{CP}^n$ model in two dimensions. This is described by a (2,2) supersymmetric $U(1)$ gauge theory with $n+1$ chiral superfields each of charge $+1$.

This theory has a bosonic potential $V = D^2$, for

$$D = \sum_{i=1}^{n+1} |\phi_i|^2 - r,$$

where r is a constant known as the Fayet-Iliopoulos parameter.

In all gauged linear sigma models, we are primarily interested in the low energy behavior of the theory. Broadly speaking, we expect the low-energy behavior to be well-described by a nonlinear sigma model on the locus $\{V=0\}/G$ for G the gauge group (here, $G=U(1)$). In the present case, for $r>0$,

$$\begin{aligned}\{V=0\}/G &= \{D=0\}/U(1), \\ &= S^{2n+1}/U(1), \\ &= \mathbb{CP}^n,\end{aligned}$$

where in the last line, we have used the fact that the Hopf fibration over $\mathbb{P}^n$ has total space S^{2n+1}.

Now, the analysis above is rather naive, as it ignores quantum corrections. By working at large absolute values of r, we obtain a weakly-coupled theory, and supersymmetry constrains the possible quantum corrections.

However, even in such a limit, the resulting theory still flows under the renormalization group: quantum corrections will shrink the projective space, as it is positively curved. In the nonlinear sigma model, this is a result of the fact that the one-loop beta function is proportional to $R_{\mu\nu}$. In the GLSM, this is due to the fact that the one-loop renormalization of r is proportional to the sum of the charges of the scalars in the theory, in this case $n+1$.

Now, let us turn to a more interesting example: the GLSM for a hypersurface of degree d in $\mathbb{CP}^n$, defined by a homogeneous polynomial $G(\phi)$ of degree d, and denoted $\mathbb{P}^n[d]$.

As a first attempt, one might start with the $\mathbb{CP}^n$ model above, and try adding a superpotential $W=G(\phi)$. However, this superpotential is not gauge-invariant, so this cannot be correct.

The correct approach is to add an additional chiral superfield p of charge $-d$, and then take the superpotential $W=pG(\phi)$, which is now gauge-invariant. The p field appears to give an undesired extra direction, but further analysis reveals the p field drops out.

Let us work through this model more carefully, to understand how the desired low-energy physics emerges. First, note that the bosonic potential V is

$$V = |D|^2 + |\partial W|^2 = |D|^2 + |G|^2 + |pdG|^2,$$

where

$$D = \sum_i |\phi_i|^2 - d|p|^2 - r,$$

for r the Fayet-Iliopoulos parameter as above. The low-energy states are defined by $V=0$, which, because V is a sum of squares, implies

$$D=0,\ G=0,\ pdG=0.$$

For $r \gg 0$, in order for $D=0$, clearly the ϕ_i cannot all vanish. Furthermore, if we assume that the hypersurface $\{G=0\}$ is smooth, then there are no solutions to $dG=0$ along $\{G=0\}$, so the simultaneous conditions

$$G=0,\ pdG=0$$

imply that $p=0$. (A more detailed analysis shows that p is in fact massive.) Thus, the extra field we added (p) is removed, and we see the low-energy theory is described by fluctuations along $\{G=0\}\subset\mathbb{CP}^n$. Thus, for $r\gg 0$, the low-energy limit appears to be a nonlinear sigma model on the hypersurface $\{G=0\}\subset\mathbb{CP}^n$.

For completeness, now consider the limit $r \ll 0$. Here, $D = 0$ implies that $p \neq 0$, breaking the original $U(1)$ gauge symmetry to $\mathbb{Z}_d$ (since p has charge $-d$). For a smooth hypersurface, the simultaneous conditions

$$G = 0, \ dG = 0$$

imply that $\langle \phi_i \rangle = 0$, as they can only be solved for vanishing ϕ_i. However, although the ϕ_i have vanishing vev, it can be shown that they remain massless.

The $r \ll 0$ limit is interpreted as a $\mathbb{Z}_d$ orbifold of a Landau-Ginzburg model. This is essentially a ($\mathbb{Z}_d$ orbifold of a) nonlinear sigma model on $\mathbb{C}^{n+1}$ with superpotential $W = G(\phi)$. (Since the $U(1)$ is broken to $\mathbb{Z}_d$ and G is degree d, this superpotential is gauge-invariant in the low-energy theory.)

We interpret the Fayet-Iliopoulos parameter as describing a one-parameter family of theories, different theories in different limits. There exists a one-loop correction to r, proportional to the sum of the charges of the scalar fields. In the present case, that means

- If $d < n+1$, then r shrinks, consistent with the fact that $\mathbb{P}^n[d]$ is positively-curved.
- If $d > n+1$, then r grows, consistent with the fact that $\mathbb{P}^n[d]$ is negatively-curved.
- If $d = n+1$, then r is invariant under renormalization group flow. In this case, $\mathbb{P}^n[d]$ is Ricci-flat, a Calabi-Yau.

Historically this construction provided a key insight into the relationship between Landau-Ginzburg orbifolds and nonlinear sigma models on Calabi-Yau hypersurfaces. It was conjectured at one point in time that Landau-Ginzburg orbifolds might define the same CFT's as nonlinear sigma models on associated Calabi-Yau's; Witten's GLSM construction [1] explicitly demonstrated that the two CFT's were indeed related, but only by a marginal deformation (defined by r).

In this example, we have constructed a one-parameter family of theories that interpolate between a nonlinear sigma model on the space $\mathbb{P}^n[d]$ and a Landau-Ginzburg orbifold. In general, one can also construct families that interpolate between just nonlinear sigma models. A prototypical example is provided by the gauged linear sigma model with gauge group $U(1)$ and four chiral multiplets, of charges $+1$, $+1$, -1, -1. It is well-known that the two limits of the familiy describe nonlinear sigma models on the total spaces of rank two vector bundles on $\mathbb{CP}^1$, each a small resolution of the conifold singularity.

Let us review one more standard example, before going on to more recent constructions. Let us build a GLSM describing, in one limit, a nonlinear sigma model on a complete intersection of hypersurfaces

$$\{G_1 = 0\} \cap \{G_2 = 0\} \cap \cdots \cap \{G_r = 0\}.$$

If the G_a is a homogeneous polynomial of degree d_a, then this complete intersection is denoted $\mathbb{P}^n[d_1, \cdots, d_r]$.

Following the pattern above, the corresponding GLSM is given by a $U(1)$ gauge theory with $n+1$ chiral superfields of charge $+1$ (corresponding to homogeneous coordinates on $\mathbb{CP}^n$) and r chiral superfields p_a of charge $-d_a$, with all d_a assumed positive. This theory has a superpotential

$$W = \sum_a p_a G_a(\phi).$$

As before, the scalar potential in this theory is of the form

$$V = |D|^2 + |dW|^2 = |D|^2 + \sum_a |G_a(\phi)|^2 + \left| \sum_a p_a dG_a(\phi) \right|^2,$$

where

$$D = \sum_i |\phi_i|^2 - \sum_a d_a |p_a|^2 - r.$$

Let us analyze the space of low-energy fluctuations when $r \gg 0$. Demanding $D = 0$ implies that the ϕ_i cannot all vanish, as is appropriate for homogeneous coordinates on $\mathbb{P}^n$. The remaining constraints imposed by vanishing of the scalar potential, namely

$$G_a = 0, \ \sum_a p_a dG_a(\phi) = 0,$$

imply, for smooth cases, that the p_a all vanish. (As before, a more detailed analysis shows that the p_a are all massive.) The resulting low-energy theory is interpreted as a nonlinear sigma model on $\mathbb{P}^n[d_1, \cdots, d_r]$.

Now, let us describe the space of low-energy fluctuations when $r \ll 0$. Here, solving $D = 0$ implies that the p_a do not all vanish. The remaining constraints imposed by vanishing of the scalar potential, namely

$$G_a = 0, \ \sum_a p_a dG_a(\phi) = 0,$$

now imply that $\langle \phi_i \rangle = 0$. (However, with further work one can show that the ϕ_i remain massless.) This theory is interpreted as a Landau-Ginzburg model on the total space of a vector bundle, specifically as a Landau-Ginzburg model on

$$\mathrm{Tot}\left(\mathscr{O}(-1)^{n+1} \longrightarrow \mathbb{P}^{r-1}_{[d_1, \cdots, d_r]} \right),$$

with superpotential

$$W = \sum_a p_a G_a(\phi),$$

where the p_a are now interpreted as homogeneous coordinates on the weighted projective space $\mathbb{P}^{r-1}_{[d_1, \cdots, d_r]}$. This is sometimes known as a "hybrid Landau-Ginzburg model."

4. New developments

4.1 First pass at a prototypical example

So far we have reviewed standard analyses of GLSM's that have existed for over two decades now. In this section, we will turn to some newer tricks. Whereas older methods were entirely perturbative in nature, the newer ideas in this area will rely on more subtle tricks of quantum field theory.

As our first example, let us analyze the GLSM for a complete intersection of two quadrics in $\mathbb{CP}^3$, denoted $\mathbb{P}^3[2,2]$. This is a two-torus T^2, the simplest Calabi-Yau.

The corresponding GLSM is a $U(1)$ gauge theory with four chiral multiplets ϕ_i of charge $+1$ and two chiral multiplets p_a of charge -2, with superpotential

$$W = \sum_a p_a G_a(\phi).$$

Since the G_a's are degree-two homogeneous polynomials, the superpotential can alternatively be written as

$$W = \sum_{i,j} \phi_i \phi_j A^{ij}(p).$$

The analysis of the low-energy behavior of this theory at $r \gg 0$ is straightforward, and follows the same form we have described previously. At low-energies, this theory is well described by a nonlinear sigma model on $\mathbb{P}^3[2,2]$.

What is more interesting is the low-energy behavior at $r \ll 0$. Following the same analysis as before, the p_a cannot all vanish, so one is led to an interpretation as a Landau-Ginzburg model. However, unlike our previous examples, in this Landau-Ginzburg model the ϕ_i's appear to be massive, with mass matrix $A^{ij}(p)$. If the ϕ_i's are everywhere massive, then the superpotential is irrelevant, and the low-energy theory appears to be a nonlinear sigma model on $\mathbb{CP}^1$, the space of p's.

Such an interpretation is problematic, because at $r \gg 0$ we have a nonlinear sigma model on a Calabi-Yau, and because r does not receive any quantum corrections. If our one-parameter family is a nonlinear sigma model on a Calabi-Yau – a nontrivial CFT – in one limit, it should be a nontrivial CFT everywhere, but a nonlinear sigma model on $\mathbb{CP}^1$ flows to a trivial CFT, a contradiction. Instead, to be consistent, the $r \ll 0$ limit ought to be another CFT of the same central charge, which would necessarily be another two-torus.

4.2 Decomposition conjecture and corrected analysis

The correct analysis of the $r \ll 0$ limit is more subtle. Perhaps the most important subtlety is that, away from the locus $\{\det A = 0\}$ where some of the ϕ_i become massless, the only massless fields have nonminimal charges. Due to subtleties in two-dimensional quantum field theories [16, 57, 58, 59], if the only massless fields have charges ± 2, then physics sees a double cover, and so the correct interpretation of the $r \ll 0$ limit is as a nonlinear sigma model on a branched double cover of $\mathbb{CP}^1$, branched along the locus $\{\det A = 0\}$. Since A is a 4×4 matrix, $\det A$ is a degree four polynomial in p's, so the branch locus is degree four. Such a branched double cover is another two-torus, consistent with general expectations.

The essential point about nonminimal charges in this case is that the low-energy theory has a $\mathbb{Z}_2$ gauge symmetry which acts trivially on massless matter. A different example may help clarify why that makes a difference physically. Consider an orbifold $[X/D_4]$, where D_4 is the eight-element dihedral group that fits in a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1,$$

where the image of the left $\mathbb{Z}_2$ is identified with the center of D_4. Assume in the orbifold above that the $\mathbb{Z}_2$ subgroup of D_4 acts trivially on X. We shall demonstrate that the one-loop partition function of $[X/D_4]$, with a trivial $\mathbb{Z}_2$ action, is different from the one-loop partition function of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$.

To compute the one-loop orbifold partition function, we need a more explicit description of D_4. We can enumerate the elements as follows:

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\},$$

where $\{1,z\}$ is the center of D_4 (and the image of $\mathbb{Z}_2$), and so $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ can be represented as

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\},$$

where we have used the notation

$$\overline{g} = \{g, gz\}.$$

In this language, we can write the one-loop partition function in the form

$$Z(D_4) = \frac{1}{D_4} \sum_{g,h \in D_4 gh=hg} Z_{g,h},$$

where $Z_{g,h}$ denotes the contribution from a one-loop twisted sector with boundary conditions determined by g, h.

Now, as z acts trivially on X, $Z_{g,h} = Z_{\overline{g},\overline{h}}$ in the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Where the D_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds differ is in the counting of various contributions. Specifically, each $Z_{g,h}$ appears $2^2 = 4$ times, except for one-loop twisted sectors corresponding to the pairs $(\overline{a}, \overline{b})$, $(\overline{a}, \overline{ab})$, and $(\overline{b}, \overline{ab})$. These three pairs commute in $\mathbb{Z}_2 \times \mathbb{Z}_2$, but not in D_4, and so they are absent from the D_4 partition function.

Thus, we can write

$$Z(D_4) = \frac{|\mathbb{Z}_2|^2}{|D_4|} |\mathbb{Z}_2 \times \mathbb{Z}_2| \left(Z([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - \text{some sectors} \right).$$

Clearly, the D_4 orbifold partition function differs from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold partition function.

In fact, if we work a little harder, one can find a more precise relationship. The overall factor is given by

$$\frac{|\mathbb{Z}_2|^2}{|D_4|} |\mathbb{Z}_2 \times \mathbb{Z}_2| = 2,$$

and the missing sectors are precisely those which would be given a sign factor if one turned on discrete torsion [55]. Thus, the overall partition function can equivalently be obtained by adding two copies of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold partition function, one with and one without discrete torsion:

$$Z(D_4) = Z\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2] \text{ without d.t.}\right) + Z\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2] \text{ with d.t.}\right).$$

More generally, in two-dimensional (2,2) supersymmetric theories, it is now believed that gauging a trivially-acting finite group is equivalent to working with a disjoint union of effective orbifolds with B fields. This is known as the 'decomposition conjecture' [59], and can be phrased as follows:

$$\text{CFT}([X/H]) = \text{CFT}\left(\left[\frac{X \times \hat{G}}{K} \right] \right),$$

where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1.$$

G is a trivially-acting finite group, and $\hat{G}$ is the set of irreducible representations of G. (The K action on $X \times \hat{G}$ is described in [59].) In special cases (known technically as 'banded gerbes'), K acts trivially on $\hat{G}$, and the right-hand side of the decomposition conjecture becomes the CFT of a disjoint union of copies of $[X/K]$, with flat B fields determined by the image of a characteristic class under a map $Z(G) \to U(1)$ determined by an element of $\hat{G}$:

$$H^2\left([X/K], Z(G)\right) \longrightarrow H^2\left([X/K], U(1)\right).$$

In terms of the previous example, the flat B field above describes the choice of discrete torsion in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold [56].

There are a number of other equivalent ways to arrive at the same result. For one example, a nonlinear sigma model with a trivially-acting finite gauge group is an example of a sigma model on a mathematical object known as a 'gerbe.' In that language, from a path integral perspective, maps into a $\mathbb{Z}_k$ gerbe on a space X are equivalent to maps into X of degree divisible by k. One can then understand the decomposition conjecture above in terms of a projection operator inserted into a path integral. Schematically,

$$\begin{aligned} Z &= \int [D\phi] e^{iS} \left(\frac{1}{k} \sum_k \exp\left(ik \int \phi^* \omega \right) \right), \\ &= \frac{1}{k} \sum_k \int [D\phi] e^{iS} \exp\left(ik \int \phi^* \omega \right). \end{aligned}$$

The first line is the partition function of a nonlinear sigma model on X with a restriction on degrees of maps; the second line is the partition function of a nonlinear sigma model on a disjoint union of k copies of X, with variable B fields.

So far we have primarily focused on discrete gauge theories, but we can also understand the same effect in presentations as $U(1)$ gauge theories which are spontaneously broken to a finite subgroiup. This is discussed in [57, 58]. Briefly, on a compact worldsheet, to uniquely define the charged matter, one must specify which bundle the charged matter couples to, and this unambiguously distinguishes nonminimal charges from minimal charges. On a noncompact worldsheet, one way to distinguish such theories is to add massive minimally-charged fields, whose presence can be detected via the periodicity of the gauge theory θ angle.

4.3 Further examples and noncommutative resolutions

Now, let us return to the GLSM for $\mathbb{P}^3[2,2] = T^2$. In the limit $r \gg 0$, standard analyses imply that this is described at low energies by a nonlinear sigma model on $\mathbb{P}^3[2,2]$. In the opposite limit $r \ll 0$, we have argued that it is described by a nonlinear sigma model on a branched double cover of $\mathbb{CP}^1$, branched over a degree four locus, which is another T^2. Thus, we have a one-parameter family of elliptic curves, which is a consistent possibility.

Another simple example is the GLSM for $\mathbb{P}^5[2,2,2]$. Here, in the limit $r \gg 0$, standard analyses imply that this is described at low energies by a nonlinear sigma model on $\mathbb{P}^3[2,2,2]$, which is a K3 surface. In the limit $r \ll 0$, an analysis closely related to the one outlined above implies that we get a branched double cover of $\mathbb{CP}^2$, branched over a degree six locus, which is another K3. Thus, in this case we have a one-parameter family of K3's, which is a consistent possibility.

The next example in this pattern, $\mathbb{P}^7[2,2,2,2]$, is more interesting. If we analyze the corresponding GLSM at $r \gg 0$, then as expected we find a nonlinear sigma model on $\mathbb{P}^7[2,2,2,2]$. The limit $r \ll 0$ appears, naively, to describe a branched double cover of $\mathbb{CP}^3$, branched over a degree 8 locus, another Calabi-Yau.

However, this analysis is slightly too naive. One way to see that is to compare singularities. Mathematically, the branched double cover above is singular where

$$\{\det A = 0\} \text{ and } d(\det A) \,=\, 0.$$

By contrast, physically the GLSM exhibits singularities where there exists a vector v such that

$$A^{ij} v_j \,=\, 0 \text{ and } (dA^{ij}) v_j \,=\, 0.$$

The condition for a GLSM singularity implies the condition for a mathematics singularity, but not the converse: the GLSM can behave as if it sees a smooth manifold at places where the branched double cover is singular.

Thus, the description as a branched double cover is not the whole story. We believe that at places where the GLSM is smooth but the branched double cover is singular, physics is instead seeing a 'noncommutative resolution' of the singularity.

The reason for this is the D-branes in this theory. We can understand them locally over $\mathbb{P}^3$ as matrix factorizations of the quadratic superpotential

$$W \,=\, \sum_{i,j} \phi_i \phi_j A^{ij}(p).$$

It is known that matrix factorizations of such quadratic superpotentials form a module over a Clifford algebra defined by the matrix A^{ij} [60]. (Strictly speaking, because of the $\mathbb{Z}_2$ orbifold along the fibers, they form a module over the sheaf of even parts of the Clifford algebra.)

Mathematically, such matrix factorizations define what is known as a noncommutative resolution of the branched double cover [16, 22]. Noncommutative resolutions, in the pertinent sense, are defined by their sheaves, and our claim is ultimately just an unraveling of definitions. See [43] for a computation of Gromov-Witten invariants in such a noncommutative resolution, and [25] for an orthogonal analysis involving D-brane probes.

For physics, this was, to our knowledge, the first known CFT realizing a noncommutative resolution.

Another novel aspect of this example is that it violates old lore that all geometric phases of GLSM's should be related by birational transformations – this example does not have that property. Another example that also violates this rule was discussed in [11], relating a complete intersection of hypersurfaces in a Grassmannian to a Pfaffian variety.

Instead of being related by a birational transformation, these phases are instead related by a newer notion, namely Kuznetsov's homological projective duality [21, 22, 23]. Describing homological projective duality in detail is beyond the scope of these lectures, but we can observe briefly that not only the novel examples discussed above but also more traditional GLSM phases [17, 18, 19, 20] naturally fit into that framework, and so many now believe that all GLSM phases are related by homological projective duality.

Additional examples are straightforward to construct. We list a few samples below:

- The GLSM for $\mathbb{P}^{2g+1}[2,2]$ describes, in the $r \ll 0$ phase, a genus g curve given as a branched double cover of $\mathbb{CP}^1$, branched over a degree $2g+2$ locus, for $g \geq 1$.
- The GLSM for $\mathbb{P}^7[2,2,2]$ describes, in the $r \ll 0$ phase, a branched double cover of $\mathbb{CP}^2$, branched along a degree 8 locus.

A related set of examples was constructed in [24]. Two prototypical examples of their construction are as follows:

1. Consider a GLSM with gauge group $U(1) \times \mathbb{Z}_2$, with 4 chiral superfields p of charge -1 under the $U(1)$ and odd under the $\mathbb{Z}_2$, and 4 chiral superfields ϕ with charge $+1$ under the $U(1)$ and invariant under the $\mathbb{Z}_2$, with a superpotential
$$W = \sum_{ij} p_i p_j A^{ij}(\phi),$$
where the entries of the matrix A^{ij} are quadratic in ϕ's. Here the superpotential is quadratic in both p's and ϕ's, unlike our previous examples. For $r \gg 0$, the ϕ's are not all zero, and the p's are massive away from the locus $\{\det A = 0\}$, leaving massless fields invariant under a gauged $\mathbb{Z}_2$. As before, this is interpreted in terms of a branched double cover of $\mathbb{CP}^3$, branched over the degree 8 locus $\{\det A = 0\}$ (or, in general, a noncommutative resolution thereof), which is Calabi-Yau.
2. Consider a GLSM with gauge group $U(1) \times \mathbb{Z}_2$, with five chiral superfields ϕ of charge $+1$, invariant under the $\mathbb{Z}_2$, one chiral superfield p_1 of charge -2, invariant under the $\mathbb{Z}_2$, and three chiral superfields $p_{2,3,4}$ of charge -1, odd under the $\mathbb{Z}_2$, with superpotential
$$W = p_1 f_2(\phi) + \sum_{i,j=2,3,4} p_i p_j A^{ij}(\phi),$$
where $f_2(\phi)$ is degree two in ϕ's and the entries of the matrix A^{ij} are also of degree two in ϕ's. For $r \gg 0$, the ϕ's are not all zero, and the $p_{2,3,4}$ are generically massive. This is interpreted as a (noncommutative resolution of a) branched double cover of $\{f_2 = 0\} \subset \mathbb{CP}^4$, branched over the degree six locus $\{\det A = 0\}$.

The corresponding GLSM's without the gauged $\mathbb{Z}_2$ factor in the gauge group are discussed in [26]. In the two cases above:

1. For the first case, the gauged $\mathbb{Z}_2$ was essential for the interpretation as a branched double cover. Without it, away from the branch locus, one has only a single copy of $\mathbb{CP}^3$. We do expect that the result should be Ricci-flat, however, and there is a unique Ricci-flat metric possible, if we interpret this as a $\mathbb{CP}^3$ with a divisor of $\mathbb{Z}_2$ orbifolds along the locus $\{\det A = 0\}$. This is Ricci-flat but not Calabi-Yau, as the canonical bundle is two-torsion instead of trivial. This is also a $\mathbb{Z}_2$ quotient of the previous case, in which the $\mathbb{Z}_2$ acts by exchanging sheets of the cover. (Locally, the fact that the theory without a gauged $\mathbb{Z}_2$ is a $\mathbb{Z}_2$ orbifold of the theory with the gauged $\mathbb{Z}_2$ is a reflection of the fact that if one orbifolds an abelian orbifold by the quantum symmetry, the original theory is returned.)

2. For the second case, an analogous analysis implies that we get the hypersurface $\{f_2 = 0\} \subset \mathbb{CP}^4$, but with a divisor of $\mathbb{Z}_2$ orbifolds along the intersection with $\{\det A = 0\}$. Again, this is Ricci-flat, but not Calabi-Yau.

5. Conclusions

There has been a great deal of progress in understanding gauged linear sigma models over the last few years. In this lecture, after reviewing basic aspects of gauged linear sigma models, we outlined one particular set of insights recently obtained in the subject.

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