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Rotational Submanifolds in Pseudo-Euclidean Spaces

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We define rotational submanifolds in pseudo-euclidean spaces $\mathbb{R}^n_t$. We use the rotational immersion to classify all rotational submanifolds of $\mathbb{L}^n$ and we also generalize a result showing sufficient conditions for a riemannian submanifold of $\mathbb{R}^n_t$ be a rotational submanifold.

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1. Introduction

Rotational submanifolds play an important role at submanifolds theory of riemannian manifolds (see, for example, [1] and [2]). They also play an important role at the study of marginally trapped surfaces which, by their turn, are important to study black roles (see [3] and [4]).

There are lots of definitions of rotational submanifolds: rotational submanifolds in $\mathbb{R}^n$ (see [5]), rotational hypersurfaces in constant curvature spaces (see [1]), rotational hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ in (see [6]), and other definitions. But constant curvature spaces, $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ are submanifolds of pseudo-euclidean spaces, therefore, it is possible to use one definition which will serve at all theses cases, we just have to define rotational submanifolds in pseudo-euclidean spaces.

In order to define rotational submanifolds in pseudo-euclidean spaces, some notations are used. A pseudo-euclidean space $\mathbb{R}^n_t$, $t \leq n$, is the vector space $\mathbb{R}^n$ together with the inner product given by

$$\langle x,y\rangle := -\sum_{i=1}^{t} x_i y_i + \sum_{i=t+1}^{n} x_i y_i,$$

where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ and the symbol ":=" means "equal by definition". We are going to use the following definitions:

$$\begin{aligned}
&\|x\|^2 := \langle x,x\rangle \,;\\
&\mathbb{S}^n := \left\{ x \in \mathbb{R}^n \mid \|x\|^2 = 1 \right\};\\
&\mathbb{S}^n(p,r) := \left\{ x \in \mathbb{R}^n_t \mid \|x-p\|^2 = r^2 \right\};\\
&\mathbb{S}^n(p,-r) := \left\{ x \in \mathbb{R}^n_t \mid \|x-p\|^2 = -r^2 \right\};\\
&\mathbb{H}^n := \{ x \in \mathbb{S}^n(0,-1) \mid x_1 > 0\};\\
&\mathscr{L} := \left\{ x \in \mathbb{R}^n_t \mid \|x\|^2 = 0 \right\}, \text{ is the light cone;}\\
&\mathscr{L}^* := \left\{ x \in \mathbb{R}^n_t \mid \|x\|^2 = 0 \text{ and } x \neq 0 \right\}, \text{ is the light cone without the origin.}
\end{aligned}$$

Let $x \in \mathbb{R}^n_t$. We say that x is: spacelike, if $\|x\|^2 > 0$; timelike, if $\|x\|^2 < 0$; or lightlike, if $\|x\|^2 = 0$. Given $V \subset \mathbb{R}^n_t$ a vector subspace, we say that V is:

- spacelike, if every vector of V is spacelike;
- timelike, if there is a basis of V in which the inner product of two vectors of V can be written like
$$\langle v,w\rangle = -\sum_{i=1}^{s} v_i w_i + \sum_{i=s+1}^{m} v_i w_i,$$
where $s \leq t$ and $m \leq n$;
- lightlike, if the inner product in V is degenerated.

Let $\mathbb{R}^{n-q-1}$ a vector subspace of $\mathbb{R}^n_t$, with $1 \leq q \leq n-2$. Lets denote the group of all linear isometries of $\mathbb{R}^n_t$ by $\mathrm{O}_t(n)$ and by $\mathrm{O}(q+1)$ the subgroup of $\mathrm{O}_t(n)$ which fixes every point of $\mathbb{R}^{n-q-1}$.

Definition 1. *Let $\mathbb{R}^{n-q}$ be a vector subspace of $\mathbb{R}^n_t$ and $f\colon N^{m-q} \to \mathbb{R}^{n-q}$ be an immersion such that $\mathbb{R}^{n-q-1} \subset \mathbb{R}^{n-q}$ and $f(N) \cap \mathbb{R}^{n-q-1} = \varnothing$. The* ***rotational submanifold*** *with* ***axis*** *$\mathbb{R}^{n-q-1}$ on f is the union of the orbits of points of $f(N)$ under the action of the group $\mathrm{O}(q+1)$, ie., it is the set*

$$\{A(f(x)) \mid x \in N \text{ and } A \in \mathrm{O}(q+1)\}.$$

In the euclidean case ($\mathbb{R}^n_t = \mathbb{R}^n$), the above definition is the same given in [5]. A more general definition for the euclidean case can be found in [7].

Our first objective is to prove the following proposition:

Proposition 2. *Let $\mathbb{R}^{n-q-1} \subset \mathbb{R}^{n-q}$ be two vector subspaces of $\mathbb{R}^n_t$ and $f\colon N^{m-q} \to \mathbb{R}^{n-q}$ an immersion such that $f(N) \cap \mathbb{R}^{n-q-1} = \varnothing$. Let also M be a rotational submanifold on f, with axis $\mathbb{R}^{n-q-1}$.*

1. *Lets suppose that $\mathbb{R}^{n-q-1}$ has index s (ie. $\mathbb{R}^{n-q-1} = \mathbb{R}^{n-q-1}_s$), $\mathbb{R}^{q+1}_{t-s} := \left(\mathbb{R}^{n-q-1}_s\right)^{\perp}$ and $\pi\colon \mathbb{R}^n_t \to \mathbb{R}^{n-q-1}_s$ is the orthogonal projection of $\mathbb{R}^n_t = \mathbb{R}^{q+1}_{t-s} \ominus \mathbb{R}^{n-q-1}_s$ on $\mathbb{R}^{n-q-1}_s$.*

   (a) *If $\mathbb{R}^{n-q}$ has index s ($\mathbb{R}^{n-q} = \mathbb{R}^{n-q}_s$), lets consider $\mathbb{S}(0,1) \subset \mathbb{R}^{q+1}_{t-s}$ and $X_1 \in \mathbb{R}^{n-q}_s \cap \left(\mathbb{R}^{n-q-1}_s\right)^{\perp}$ a unit spacelike vector. In this case, we can define $\bar{M}$ and $g\colon N \times \mathbb{S}(0,1) \to \bar{M}$ by*

   $$\bar{M} := \{\, f_1(x)\xi + \pi(f(x)) \,|\, x \in N \text{ and } \xi \in \mathbb{S}(0,1)\} \quad \text{and} \quad g(x,\xi) := f_1(x)\xi + \pi(f(x)),$$

   where $f_1(x) := \langle f(x), X_1 \rangle$.

   (b) *If $\mathbb{R}^{n-q} = \mathbb{R}^{n-q}_{s+1}$, lets consider $\mathbb{S}(0,-1) \subset \mathbb{R}^{q+1}_{t-s}$ and $X_1 \in \mathbb{R}^{n-q}_{s+1} \cap \left(\mathbb{R}^{n-q-1}_s\right)^{\perp}$ a unit timelike vector. In this case, we can define $\bar{M}$ and $g\colon N \times \mathbb{S}(0,-1) \to \bar{M}$ by*

   $$\bar{M} := \{\, f_1(x)\xi + \pi(f(x)) \,|\, x \in N \text{ and } \xi \in \mathbb{S}(0,-1)\} \quad \text{and} \quad g(x,\xi) := f_1(x)\xi + \pi(f(x)),$$

   where $f_1(x) := -\langle f(x), X_1 \rangle$.

   (c) *If $\mathbb{R}^{n-q}$ is lightlike, lets consider $\mathscr{L}^* \subset \mathbb{R}^{q+1}_{t-s}$ and $X_1 \in \mathbb{R}^{n-q} \cap \left(\mathbb{R}^{n-q-1}_s\right)^{\perp}$ a lightlike vector. In this case, we can define $\bar{M}$ and $g\colon N \times \mathscr{L}^* \to \bar{M}$ by*

   $$\bar{M} := \{\, f_1(x)\xi + \pi(f(x)) \,|\, x \in N \text{ and } \xi \in \mathscr{L}^*\} \quad \text{and} \quad g(x,\xi) := f_1(x)\xi + \pi(f(x)),$$

   where $f_1(x)$ is the component of $f(x)$ in the X_1 direction, ie., $f(x) = f_1(x)X_1 + \pi(f(X))$.

2. *Let suppose that $\mathbb{R}^{n-q-1}$ is lightlike and there are non-degenerated vector subspaces $U, V \subset \mathbb{R}^n_t$ and lightlike vectors X_1 and X_2 such that $\langle X_1, X_2 \rangle = 1$, $\mathbb{R}^{n-q-1} = \operatorname{span}\{X_2\} \ominus U$ and $\mathbb{R}^n_t = \operatorname{span}\{X_1, X_2\} \ominus U \ominus V$. In this case, let $\pi\colon \operatorname{span}\{X_1\} \oplus V \oplus \mathbb{R}^{n-q-1} \to \mathbb{R}^{n-q-1}$ be the projection application.*

   (a) *If $\mathbb{R}^{n-q} = \operatorname{span}\{X_1, X_2\} \ominus U$, lets define $\bar{M}$ and $g\colon N \times V \to \mathbb{R}^n_t$ by*

   $$\bar{M} := \left\{ f_1(x)\left(X_1 + v - \frac{\|v\|^2}{2}X_2\right) + \pi(f(x)) \,\middle|\, x \in N \text{ and } v \in V \right\} \quad \text{and}$$
   $$g(x,v) := f_1(x)\left(X_1 + v - \frac{\|v\|^2}{2}X_2\right) + \pi(f(x)),$$

   where $f_1(x) = \langle f(x), X_2 \rangle$.

(b) *If* $\mathbb{R}^{n-q} = \operatorname{span}\{w, X_2\} \oplus U$*, where* $w \in V$ *is a unit vector, lets consider* $\varepsilon := \|w\|^2$ *and* $\mathbb{S}(0,\varepsilon) \subset V$ *and we can define* $\bar{M}$ *and* $g\colon N \times \mathbb{S}(0,\varepsilon) \times \mathbb{R} \to \mathbb{R}^n_t$ *by*

$$\bar{M} := \{ f_1(x)(\lambda X_2 + \xi) + \pi(f(x)) \mid x \in N,\ \xi \in \mathbb{S}(0,\varepsilon) \text{ and } \lambda \in \mathbb{R}\} \quad \text{and}$$
$$g(x,\xi,ł) := f_1(x)(\lambda X_2 + \xi) + \pi(f(x)),$$

where $f_1(x) = \varepsilon \langle f(x), w\rangle$.

In any of the above cases, $M = \bar{M}$*. Furthermore, in the cases (I.1), (I.2) and (II.1), g is an immersion. With the hypothesis that N is a riemannian manifold and f is an isometric immersion, g is also an immersion in the cases (I.3) and (II.2).*

This proposition studies some of the possible cases for rotational submanifolds in $\mathbb{R}^n_t$, but there are some other cases which were not studied, for example, the case in which $\mathbb{R}^{n-q-1} = \mathbb{R}^{n-q-1-\ell}_s \oplus \operatorname{span}\{v_1, \cdots, v_\ell\}$, where $v_1, \cdots, v_\ell$ are orthogonal lightlike vectors. Besides that, if $t = 1$, that is, $\mathbb{R}^n_t = \mathbb{L}^n$ is the Lorentz space, then Proposition 2 is enough.

Corollary 3. *Proposition 2 classifies all rotational submanifolds in* $\mathbb{L}^n$ *on an immersion f, according to the codomain of f and to the rotational axis.*

Once we have proved those results, we want to show another one but, first, we need some definitions.

Let M^m_s and N^n_t be two pseudo-riemannian manifolds and $f\colon M^m_s \to N^n_t$ an isometric immersion. Given a vector $\eta \in T^\perp_x M$, it's **conformal nullity** subspace is given by

$$E_\eta(x) := \{ X \in T_xM \mid \alpha(X,Y) = \langle X,Y\rangle \eta,\ \forall Y \in T_xM\}.$$

We say that $\eta \in \Gamma\left(T^\perp M\right)$ is a **principal normal** if $\dim E_\eta(x) \geq 1$, for all $x \in M$. If η is a principal normal, E_η has constant dimension and η is parallel in the normal connection of f along E_η, then η is called a **Dupin normal** of f. In this case, the number $\dim E_\eta$ is the **multiplicity** of η.

A distribution $\mathscr{D}$ in a riemannian manifold M^n is **umbilical** if there exists a vector field $\varphi \in \Gamma\left(\mathscr{D}^\perp\right)$ such that $\nabla^h_X Y = \langle X,Y\rangle \varphi$, for all X and all Y in $\Gamma(\mathscr{D})$, where $\nabla^h_X Y$ is the orthogonal projection of $\nabla_X Y$ on $\mathscr{D}^\perp$. The vector φ is called **mean curvature vector** of the umbilical distribution $\mathscr{D}$. If $\mathscr{D}$ is umbilical and it's mean curvature vector is null ($\varphi \equiv 0$), then $\mathscr{D}$ is called **totally geodesic**. $\mathscr{D}$ is called **spherical** if $\mathscr{D}$ is umbilical and $\nabla^h_X \varphi = 0$, for every $X \in \Gamma(\mathscr{D})$.

Our main result is the following theorem, which generalizes a similar theorem made in [5] for the euclidean case:

Theorem 1. *Let* M^m *be a riemannian manifold,* $f\colon M^m \to \mathbb{R}^n_t$ *an isometric immersion and* η *a Dupin normal of f with multiplicity q and such that* $\eta \neq 0$ *in every point of M. If* $E^\perp_\eta$ *is totally geodesic, then there exists a rotational immersion g such that* $f(M)$ *is a subset of the image of g. Furthermore, we have one of the following cases:*

1. *There is an orthogonal decomposition* $\mathbb{R}^n_t = \mathbb{R}^{q+1} \oplus \mathbb{R}^{m-q-1}_t$ *such that* $g\colon N^{m-q} \times \mathbb{S}^q \to \mathbb{R}^{q+1} \oplus \mathbb{R}^{n-q-1}_t$ *is given by*

$$g(x,y) = p + r(x)y + h(x),$$

where $p \in \mathbb{R}^n_t$ *is a fixed point,* $r(x) > 0$*,* $r(x)y \in \mathbb{R}^{q+1}$*,* $h(x) \in \mathbb{R}^{n-q-1}_t$ *and* $\mathbb{R}^{n-q-1}_t$ *is the rotational axis.*

2. *There is an orthogonal decomposition* $\mathbb{R}^n_t = \mathbb{L}^{q+1} \oplus \mathbb{R}^{n-q-1}_{t-1}$ *such that* $g\colon N^{m-q} \times \mathbb{S}(0,-1) \to \mathbb{L}^{q+1} \oplus \mathbb{R}^{n-q-1}_{t-1}$ *is given by*
$$g(x,y) = p + r(x)y + h(x),$$
where $p \in \mathbb{R}^n_t$ *is a fixed point,* $\mathbb{S}(0,-1) \subset \mathbb{L}^{q+1}$, $r(x) > 0$, $r(x)y \in \mathbb{L}^{q+1}$, $h(x) \in \mathbb{R}^{n-q-1}_{t-1}$ *and* $\mathbb{R}^{n-q-1}_{t-1}$ *is the rotational axis.*
3. *There are lightlike vectors* $e_1, e_2 \in \mathbb{R}^n_t$ *and an orthogonal decomposition* $\mathbb{R}^n_t = \operatorname{span}\{e_1,e_2\} \oplus \mathbb{R}^q \oplus \mathbb{R}^{n-q-2}_{t-2}$ *such that* $\langle e_1, e_2\rangle = 1$ *and* $g\colon N^{m-q} \times \mathbb{R}^q \to \mathbb{R}^n_t$ *is given by*
$$g(x,y) = q + g_1(x)e_1 + \left[g_2(x) - g_1(x)\frac{\|y\|^2}{2}\right] e_2 + g_1(x)y + g_3(x),$$
where $q \in \mathbb{R}^n_t$ *is a fixed point,* $g_1(x) > 0$, $g_3(x) \in \mathbb{R}^{n-q-2}_{t-s-2}$ *and* $\operatorname{span}\{e_2\} \oplus \mathbb{R}^{n-q-2}_{t-2}$ *is the rotational axis.*

In [9], this theorem is used to to show that some umbilical submanifolds of a product of two constat curvature spaces are rotational submanifols in $\mathbb{R}^N_t$.

2. Proof of Proposition 2 and Corollary 3

Let $\mathscr{L}^*$ be the light cone without the null vector (origin).

Proof of the cases (I) of the Proposition 2.

Let $M := \{A(f(x)) \mid x \in N \text{ and } A \in \mathrm{O}(q+1)\}$ be a rotational submanifold on f. We have to show that $M = \bar{M}$ and that g is an immersion.

(I.1): Let $\mathbb{R}^{n-q} = \mathbb{R}^{n-q}_s$. Since $\mathbb{R}^{n-q-1}_s$ is a vector subspace of $\mathbb{R}^{n-q}_s$, there exists a unit spacelike vector $X_1 \in \mathbb{R}^{n-q}_s \cap \left(\mathbb{R}^{n-q-1}_s\right)^\perp$. Thus, $f(x) = f_1(x)X_1 + \pi(f(x))$, where $f_1(x) := \langle f(x), X_1\rangle$.

<u>Affirmation 1</u>: $M \subset \bar{M}$.

If $A \in \mathrm{O}(q+1)$, then
$$A(f(x)) = A\big(f_1(x)X_1 + \pi(f(x)\big) = f_1(x)A(X_1) + A\big(\pi(f(x))\big).$$
But,
$$A\big(\pi(f(x))\big) = \pi(f(x)) \quad \text{and} \quad \langle A(X_1), Y\rangle = \langle A(X_1), A(Y)\rangle = \langle X_1, Y\rangle = 0,$$
for all $Y \in \mathbb{R}^{n-q-1}_s$, because A fixes the points of $\mathbb{R}^{n-q-1}_s$.

Thus $A(X_1) \in \mathbb{S}(0,1) \subset \mathbb{R}^{q+1}_{n-s} \perp \mathbb{R}^{n-q-1}_s$, since $A(X_1) \perp \mathbb{R}^{n-q-1}_s$ and $\|A(X_1)\|^2 = \|X_1\|^2 = 1$. Therefore $A(f(x)) = f_1(x)A(X_1) + \pi(f(x)) \in \{f_1(x)\xi + \pi(f(x)) \mid x \in N \text{ and } \xi \in \mathbb{S}(0,1)\}$. ✓

<u>Affirmation 2</u>: $\bar{M} \subset M$.

Let $x \in N$ and $\xi \in \mathbb{S}(0,1) \subset \mathbb{R}^{q+1}_{t-s} \perp \mathbb{R}^{n-q-1}_s$. Lets assume that $\{X_1, X_2, \cdots, X_{q+1}\}$ and $\{\xi, Y_2, \cdots, Y_{q+1}\}$ are two orthonormal basis of $\mathbb{R}^{q+1}_{t-s}$ such that $\|X_i\|^2 = \|Y_i\|^2$. If $\{X_{q+2}, \cdots, X_n\}$ is an orthonormal basis of $\mathbb{R}^{n-q-1}_s$, then we can define $A \in \mathrm{O}_t(n)$ by
$$A(X_i) = \begin{cases} \xi, & \text{if } i = 1; \\ Y_i, & \text{if } i = 2, \cdots, q+1; \\ X_i, & \text{if } i = q+2, \cdots, n. \end{cases}$$

It is clear that $A \in \mathrm{O}(q+1)$ and $f_1(x)\xi + \pi(f(x)) = f_1(x)A(X_1) + A(\pi(f(x))) = A(f(x))$. ✓

Affirmation 3: *g is an immersion.*

In deed, calculating $\mathrm{d}g(x,\xi)(v_1,v_2)$ we get

$$\mathrm{d}g(x,\xi)(v_1,v_2) = \langle \mathrm{d}f(x)v_1, X_1\rangle\,\xi + \langle f(x), X_1\rangle\, v_2 + \pi\big(\mathrm{d}f(x)v_1\big).$$

If $\mathrm{d}g(x,\xi)(v_1,v_2) = 0$, then $\langle \mathrm{d}f(x)v_1, X_1\rangle\,\xi = 0$, $\langle f(x), X_1\rangle\, v_2 = 0$ and $\pi(\mathrm{d}f(x)v_1) = 0$, since $v_2 \perp \xi$ and $\xi, v_2 \in \mathbb{R}^{q+1}_{t-s} \perp \mathbb{R}^{n-q-1}_s$. Thus

$$\begin{cases} \langle \mathrm{d}f(x)v_1, X_1\rangle = 0, & \text{cause } \xi \neq 0; \\ v_2 = 0, & \text{cause } f(x) \notin \mathbb{R}^{n-q-1}_s, \text{ ie., } \langle f(x), X_1\rangle \neq 0; \text{ and} \\ \pi(\mathrm{d}f(x)v_1) = 0. & \end{cases}$$

Thus

$$\langle \mathrm{d}f(x)v_1, X_1\rangle X_1 + \pi(\mathrm{d}f(x)v_1) = \mathrm{d}f(x)v_1 = 0 \Rightarrow (v_1,v_2) = (0,0).$$

Therefore g is an immersion. ✓ •

***(I.2)*:** The proof is analogous to the proof of the previous case. •

***(I.3)*:** Lets assume that $\mathbb{R}^{n-q}$ is lightlike (nondegenerate). Since $\mathbb{R}^{n-q-1}_s$ is a vector subspace of $\mathbb{R}^{n-q}$, there exists a lightlike vector $X_1 \in \mathbb{R}^{n-q} \cap \left(\mathbb{R}^{n-q-1}_s\right)^{\perp}$. Thus, $f(x) = f_1(x)X_1 + \pi(f(x))$.

Affirmation 1: $M \subset \bar{M}$.

Analogous to the Affirmation 1 of the case *(I.1)*. ✓

Affirmation 2: $\bar{M} \subset M$.

Let $x \in N$ and $\xi \in \mathscr{L}^* \subset \mathbb{R}^{q+1}_{t-s} = (\mathbb{R}^{n-q-1}_s)^{\perp}$ and lets consider $\{X_1, X_2, \cdots, X_{q+1}\}$ and $\{\xi, Y_2, \cdots, Y_{q+1}\}$ two basis of $\mathbb{R}^{q+1}_{t-s}$ such that

- X_1, X_2, ξ and Y_2 are lightlike;
- $\langle X_1, X_2\rangle = 1 = \langle \xi, Y_2\rangle$;
- $\{X_3, \cdots, X_{q+1}\}$ and $\{Y_3, \cdots, Y_{q+1}\}$ are orthonormal sets;
- $\{X_1, X_2\} \perp \{X_3, \cdots, X_{q+1}\}$ and $\{\xi, Y_2\} \perp \{Y_3, \cdots, Y_{q+1}\}$.

If $\{X_{q+2}, \cdots, X_n\}$ is an orthonormal basis of $\mathbb{R}^{n-q-1}_s$, then we can define $A \in \mathrm{O}_t(n)$ by

$$A(X_i) = \begin{cases} \xi, & \text{if } i = 1; \\ Y_i, & \text{if } i \in \{2, \cdots, q+1\}; \\ X_i, & \text{if } i \in \{q+2, \cdots, n\}. \end{cases}$$

Thus, $A \in \mathrm{O}(q+1)$ and $f_1(x)\xi + \pi(f(x)) = f_1(x)A(X_1) + A(\pi(f(x))) = A(f(x))$. ✓

Affirmation 3: *If N is a riemannian manifold and f is an isometric immersion, then g is also an immersion.*

In deed, calculating $\mathrm{d}g(x,\xi)(v_1,v_2)$ we get

$$\mathrm{d}g(x,\xi)(v_1,v_2) = \langle \mathrm{d}f(x)v_1, X_2\rangle\,\xi + \langle f(x),X_2\rangle\, v_2 + \pi\big(\mathrm{d}f(x)v_1\big),$$

where $X_2 \in \mathbb{R}^{q+1}_{t-s}$ is a lightlike vector such that $\langle X_1, X_2\rangle = 1$.

If $\mathrm{d}g(x)(v_1,v_2) = 0$, then $\langle \mathrm{d}f(x)v_1, X_2\rangle\,\xi + \langle f(x),X_2\rangle\, v_2 = 0$ and $\pi\,(\mathrm{d}f(x)v_1) = 0$, since $\xi, v_2 \in \mathbb{R}^{q+1}_{t-s} \perp \mathbb{R}^{n-q-1}_s$ and $\pi(f(x)) \in \mathbb{R}^{n-q-1}_s$.

Knowing that N is riemannian and f is an isometric immersion, we have that $\mathrm{d}f(x)v_1$ is null or it is a spacelike vector. But $\mathrm{d}f(x)v_1 = \langle \mathrm{d}f(x)v_1, X_2\rangle X_1 + \pi(\mathrm{d}f(x)v_1) = \langle \mathrm{d}f(x)v_1, X_2\rangle X_1$, ie., $\mathrm{d}f(x)v_1$ is not spacelike. Therefore $\mathrm{d}f(x)v_1 = 0$ and $v_1 = 0$.

Thus, $\mathrm{d}g(x,\xi)(v_1,v_2) = f_1(x)v_2 = 0$ and g is an immersion, cause $f(N)\cap\mathbb{R}^{n-q-1}_s = \varnothing$ and $f_1(x) \neq 0$. ✓ □

Remark 4. *In case (I.3), through the calculations of the differential* $\mathrm{d}g(x,\xi)$*, it is easily proved that g is an immersion if, and only if,* $f_*TN \cap \mathrm{span}\{X_1\} = \{0\} \Leftrightarrow \mathbb{R}^{n-q} \cap (\mathbb{R}^{n-q})^{\perp} \cap f_*(TN) = \{0\}$*. Therefore, instead of supposing that N is riemannian and f is an isometric immersion, we could suppose that* $f_*TN \cap \mathrm{span}\{X_1\} = \{0\}$*, without changing the thesis.*

We need more results in order to show case *(II)* of Proposition 2.

Let X_1 and X_2 be lightlike vectors of $\mathbb{R}^n_t$ such that $\langle X_1, X_2\rangle = 1$ and lets suppose that

$$\mathbb{R}^n_t = \mathrm{span}\{X_1, X_2\} \ominus U \ominus V,$$

where U and V are nondegenerate vector subspaces. Lets consider the lightlike vector subspace $W := \mathrm{span}\{X_2\} \ominus U \subset \mathbb{R}^n_t$, $\mathrm{O}(V)$ the group of linear isometries of V and $\mathrm{O}(V) \ltimes V$ the group of isometries of V. We can define the applications $\mathscr{I} : V \to \mathrm{span}\{X_1, X_2\} \ominus V$ and $\Phi\colon \mathrm{O}(V) \ltimes V \to \mathrm{O}_t(n)$ by

$$\mathscr{I}(x) := X_1 + x - \frac{\|x\|^2}{2} X_2 \quad \text{and} \tag{2.1}$$

$$\Phi(B,x)(v + v^{\perp}) := v^{\perp} - \left(\langle Bv, x\rangle + \frac{\langle X_2, v^{\perp}\rangle}{2}\|x\|^2 \right) X_2 + Bv + \left\langle X_2, v^{\perp}\right\rangle x, \tag{2.2}$$

for all $v + v^{\perp} \in V \ominus V^{\perp} = \mathbb{R}^n_t$.

In [8], it is proved the following lemma:

Lemma 5.
1. *$\mathscr{I} : V \to \mathscr{I}(V)$ is an isometry.*
2. *$\Phi\colon \mathrm{O}(V) \ltimes V \to \mathscr{W}$ is a group isomorphism, where $\mathscr{W}$ is the subgroup of $\mathrm{O}_t(n)$ which fixes the points of W.*
3. *$\mathscr{W}$ is the isometries group of $\mathscr{I}(V) = \left\{ X_1 + x - \frac{\|x\|^2}{2} X_2 \,\middle|\, x \in V \right\}$.*

Proof of the case (II) of Proposition 2.

Lets suppose that $\mathbb{R}^{n-q-1}$ is lightlike and that there exist a nondegenerate vector subspace $U \subset \mathbb{R}^n_t$ and a lightlike vector $X_2 \in \mathbb{R}^n_t$ such that $\mathbb{R}^{n-q-1} = \mathrm{span}\{X_2\} \ominus U$. In this case, there exist a lightlike vector $X_1 \in \mathbb{R}^n_t$ and a nondegenerate subspace $V \subset \mathbb{R}^n_t$ such that

$$\mathbb{R}^n_t = \mathrm{span}\{X_1, X_2\} \ominus U \ominus V, \quad \langle X_1, X_2\rangle = 1 \quad \text{and} \quad \mathbb{R}^{n-q} = \mathrm{span}\{w, X_2\} \oplus U,$$

where $w \in V$, or $w = X_1$.

If $A \in \mathrm{O}(q+1)$ and $x \in N$, then

$$A\big(f(x)\big) = A\left(f_1(x)w + \pi\big(f(x)\big)\right) = f_1(x)A(w) + \pi(f(x)).$$

By Lemma 5, there exist an isometry B of V and a vector $v \in V$ such that $A = \Phi(B,v)$.

***(II.1)*:** Lets suppose that $\mathbb{R}^{n-q} = \mathbb{R}^{n-q}_s = \mathrm{span}\{X_1,X_2\} \ominus U$. In this case, $f_1(x) = \langle f(x), X_2\rangle$ and we can write $f(x) = f_1(x)X_1 + \pi(f(x))$. Thus,

$$A\big(f(x)\big) = f_1(x)A(X_1) + \pi(f(x)).$$

By the other side,

$$A(X_1) = \Phi(B,v)(X_1) \overset{(2.2)}{=} X_1 - \frac{\|v\|^2}{2}X_2 + v \;\Rightarrow\; A\big(f(x)\big) = f_1(x)\left(X_1 - \frac{\|v\|^2}{2}X_2 + v\right) + \pi(f(x)).$$

Thus, $M \subset \bar{M}$.

Let $f_1(x)\left(X_1 - \frac{\|v\|^2}{2}X_2 + v\right) + \pi(f(x)) \in \bar{M}$. Given $B \in \mathrm{O}(V)$, $\Phi(B,v) \in \mathrm{O}(q+1)$, by Lemma 5. Furthermore, $\Phi(B,v)(X_1) = X_1 - \frac{\|v\|^2}{2}X_2 + v$, thus

$$f_1(x)\left(X_1 - \frac{\|v\|^2}{2}X_2 + v\right) + \pi(f(x)) = f_1(x)\Phi(B,v)(X_1) + \pi(f(x)) = \Phi(B,v)(f(x)) \in M.$$

Therefore, $M = \bar{M}$.

Calculating $\mathrm{d}g(x,v)$ we get

$$\mathrm{d}g(x,v)(v_1,v_2) = \langle \mathrm{d}f(x)v_1, X_2\rangle X_1 - \left(\langle \mathrm{d}f(x)v_1, X_2\rangle \frac{\|v\|^2}{2} + f_1(x)\langle v, v_2\rangle\right) X_2 + \pi(\mathrm{d}f(x)v_1) + \\ + \langle \mathrm{d}f(x)v_1, X_2\rangle v + f_1(x)v_2.$$

If $\mathrm{d}g(x,v)(v_1,v_2) = 0$, then

$$\begin{cases} \langle \mathrm{d}f(x)v_1, X_2\rangle X_1 = 0 \Rightarrow \langle \mathrm{d}f(x)v_1, X_2\rangle = 0, \\ \langle \mathrm{d}f(x)v_1, X_2\rangle v + f_1(x)v_2 = 0 \Rightarrow f_1(x)v_2 = 0 \Rightarrow v_2 = 0, \\ -\left(f_1(x)\langle v, v_2\rangle + \langle \mathrm{d}f(x)v_1, X_2\rangle \frac{\|v\|^2}{2}\right) X_2 + \pi(\mathrm{d}f(x)v_1) = 0 \Rightarrow \pi(\mathrm{d}f(x)v_1) = 0, \end{cases}$$

since $v, v_2 \in V \perp \mathbb{R}^{n-q}$, $\mathbb{R}^{n-q} = \mathrm{span}\{X_1,X_2\} \ominus U$ and $\pi(f(x)) \in \mathbb{R}^{n-q-1} = \mathrm{span}\{X_2\} \ominus U$. Therefore f is an immersion. $\bullet$

***(II.2)*:** Lets suppose that $\mathbb{R}^{n-q} = \mathrm{span}\{w\} \ominus \mathbb{R}^{n-q-1} = \mathrm{span}\{w, X_2\} \ominus U$, for some unit vector $w \in V$. In this case, $f_1 = \varepsilon \langle f(x), w\rangle$, where $\varepsilon = \|w\|^2$. Thus,

$$A\big(f(x)\big) = f_1(x)\Phi(B,v)(w) + \pi(f(x)) \overset{(2.2)}{=} f_1(x)(-\langle Bw, v\rangle X_2 + Bw) + \pi(f(x)).$$

Calling $\lambda := -\langle Bw, v\rangle$, we have that $M \subset \bar{M}$, since $\|Bw\|^2 = \|w\|^2$. Lets consider $f_1(x)(\lambda X_2 + \xi) + \pi(f(x)) \in \bar{M}$, $B \in \mathrm{O}(V)$ and $v \in V$ such that $Bw = \xi$ and $\langle \xi, v\rangle = -\lambda$, in this way

$$f_1(x)(\lambda X_2 + \xi) + \pi(f(x)) = f_1(x)(-\langle Bw, v\rangle X_2 + Bw) + \pi(f(x)) = \\ = f_1(x)\Phi(B,v)(w) + \pi(f(x)) = \Phi(B,v)(f(x)).$$

Therefore $f_1(x)(\lambda X_2+\xi)+\pi(f(x))\in M$.

Calculating $\mathrm{d}g(x,\xi,\lambda)$ we get

$$\mathrm{d}g(x,\xi,\lambda)(v_1,v_2,r)=[\varepsilon\langle \mathrm{d}f(x)v_1,w\rangle\,\xi+f_1(x)v_2]+[\varepsilon\langle \mathrm{d}f(x)v_1,w\rangle\,\lambda+f_1(x)r]\,X_2+ \\ +\pi(\mathrm{d}f(x)v_1).$$

If $\mathrm{d}g(x,\xi,\lambda)(v_1,v_2,r)=0$, then

$$\begin{cases}\varepsilon\langle \mathrm{d}f(x)v_1,w\rangle\,\xi+f_1(x)v_2=0,\\ [\varepsilon\langle \mathrm{d}f(x)v_1,w\rangle\,\lambda+f_1(x)r]\,X_2+\pi(\mathrm{d}f(x)v_1)=0,\end{cases}$$

since $\xi,v_2\in V$ and $X_2,\pi(\mathrm{d}f(x)v_1)\in V^\perp$.

In this way, $\langle \mathrm{d}f(x)v_1,w\rangle=0$ and $v_2=0$, since $\xi\in\mathbb{S}(0,\varepsilon)$, $v_2\perp\mathbb{S}(0,\varepsilon)$ and $f(x)\notin\mathbb{R}^{n-q-1}$, thus $f_1(x)rX_2+\pi(\mathrm{d}f(x)v_1)=0$. If f is an isometric immersion and N is riemannian, then g is an immersion. • □

Remark 6. *Using the calculations above for the case (II.2) of Proposition 2,*

$$\mathrm{d}g(x,\xi,\lambda)(v_1,v_2,r)=0 \ \Leftrightarrow\ \begin{cases}\langle \mathrm{d}f(x)v_1,w\rangle=0,\\ v_2=0,\\ f_1(x)rX_2+\pi(\mathrm{d}f(x)v_1)=0.\end{cases}$$

Therefore, g is an immersion if, and only if, $f_*(TN)\cap\operatorname{span}\{X_2\}=\{0\}$ $\Leftrightarrow$ $\mathbb{R}^{n-q}\cap(\mathbb{R}^{n-q})^\perp\cap f_*(TN)=\{0\}$.

Definition 7. *The immersion g given at Proposition 2 is called* ***rotational immersion*** *of the rotational submanifold M.*

Proof of Corollary 3. Let $f\colon N^{m-q}\to\mathbb{R}^{n-q}\subset\mathbb{L}^n$ be an immersion and M a rotational submanifold on f with axis $\mathbb{R}^{n-q-1}\subset\mathbb{R}^{n-q}$. The only possibilities we have for $\mathbb{R}^{n-q-1}$ and $\mathbb{R}^{n-q}$ are:

1. $\mathbb{R}^{n-q-1}$ and $\mathbb{R}^{n-q}$ are both spacelike or both timelike, ie., both have the same index (equals to ±1);
2. $\mathbb{R}^{n-q-1}$ is spacelike and $\mathbb{R}^{n-q}$ is timelike, ie., $\mathbb{R}^{n-q-1}$ has index 0 and $\mathbb{R}^{n-q}$ has index 1;
3. $\mathbb{R}^{n-q-1}$ is spacelike and $\mathbb{R}^{n-q}$ is lightlike;
4. $\mathbb{R}^{n-q-1}$ is lightlike and $\mathbb{R}^{n-q}$ is timelike;
5. $\mathbb{R}^{n-q-1}$ and $\mathbb{R}^{n-q}$ are both lightlike.

But all cases above were studied by Proposition 2. □

Remarks 8. *By observations 4 and 6, if M is a rotational submanifold in $\mathbb{L}^n$ on $f\colon N\to\mathbb{R}^{n-q}$ and $\mathbb{R}^{n-q}$ is lightlike, then g is an immersion if, and only if, $\mathbb{R}^{n-q}\cap(\mathbb{R}^{n-q})^\perp\cap\varphi_*(TN)=\{0\}$, ie., g is an immersion if, and only if, N is a riemannian manifold with the metric induced by f.*

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3. Proof of Theorem 1

In order to prove Theorem 1, we need some additional results. The euclidean versions of these results can be found in [5].

Lemma 9. *Let $f\colon M^m \to \mathbb{R}^n_t$ be an isometric immersion and η a principal normal of f. Then, for all $X \in E_\eta(x)$ and all $\xi, \zeta \in T_x^\perp M$ such that $\xi \perp \eta$ and $\langle \zeta, \eta \rangle = 1$, the following formulas are true:*

$$A_\eta X = \|\eta\|^2 X, \quad A_\xi X = 0 \quad e \quad A_\zeta X = X. \tag{3.1}$$

Let $\mathscr{D}$ be a distribution in M such that $\mathscr{D}(x) \subset E_\eta(x)$, for all $x \in M$.

1. *If η is parallel in the normal connexion of f along $\mathscr{D}$, then $\nabla\|\eta\|^2 \in \Gamma\left(\mathscr{D}^\perp\right)$, where $\nabla\|\eta\|^2$ is the gradient vector of $\|\eta\|^2$. Furthermore, the following formulas are true:*

$$\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) \nabla_X Y = \frac{\langle X, Y\rangle}{2} \nabla\|\eta\|^2, \tag{3.2}$$

$$\left\langle A_\xi \nabla_X Y, Z\right\rangle = \langle X, Y\rangle \left\langle \nabla^\perp_Z \xi, \eta\right\rangle, \tag{3.3}$$

$$\left\langle (\operatorname{Id} - A_\zeta) \nabla_X Y, Z\right\rangle = -\langle X, Y\rangle \left\langle \nabla^\perp_Z \zeta, \eta\right\rangle, \tag{3.4}$$

   for all $X, Y \in \Gamma(\mathscr{D})$, all $Z \in \Gamma\left(\mathscr{D}^\perp\right)$ an all $\xi, \zeta \in \Gamma\left(T^\perp M\right)$ such that $\xi \perp \eta$ and $\langle \zeta, \eta\rangle = 1$.

2. *If $\mathscr{D}$ is an umbilical distribution and φ is its mean curvature vector, then*

$$\tilde{\nabla}_X f_* Y = f_* \nabla^v_X Y + \langle X, Y\rangle \sigma, \ \forall X, Y \in \Gamma(\mathscr{D}), \tag{3.5}$$

   where $\sigma := f_*\varphi + \eta$ and $\nabla^v_X Y$ is the orthogonal projection of $\nabla_X Y$ on $\mathscr{D}$.

3. *With the same hypothesis of (I) and (II),*

$$\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) \varphi = \frac{1}{2} \nabla\left(\|\eta\|^2\right), \tag{3.6}$$

$$\left\langle A_\xi \varphi, Z\right\rangle = \left\langle \nabla^\perp_Z \xi, \eta\right\rangle, \tag{3.7}$$

$$\left\langle (\operatorname{Id} - A_\zeta)\varphi, Z\right\rangle = -\left\langle \nabla^\perp_Z \zeta, \eta\right\rangle, \tag{3.8}$$

$$\left\langle \nabla_X \varphi, \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) Z\right\rangle = 0, \tag{3.9}$$

$$\left\langle \nabla_X \varphi, A_\xi Z\right\rangle = 0, \tag{3.10}$$

$$\left\langle \nabla_X \varphi, (\operatorname{Id} - A_\zeta) Z\right\rangle = 0, \tag{3.11}$$

   for all $X \in \Gamma(\mathscr{D})$, all $Z \in \Gamma\left(\mathscr{U}^\perp\right)$ and all $\xi, \zeta \in \Gamma\left(T^\perp M\right)$ such that $\xi \perp \eta$ e $\langle \zeta, \eta\rangle = 1$.

Proof. Let $X \in E_\eta(x)$, $Y \in T_x M$ and $\xi, \zeta \in T_x^\perp M$ such that $\xi \perp \eta$ and $\langle \zeta, \eta\rangle = 1$. Then

$$\begin{aligned} \langle A_\eta X, Y\rangle &= \langle \alpha(X, Y), \eta\rangle = \langle \langle X, Y\rangle \eta, \eta\rangle = \|\eta\|^2 \langle X, Y\rangle, \\ \left\langle A_\xi X, Y\right\rangle &= \langle \alpha(X, Y), \xi\rangle = \langle X, Y\rangle \langle \eta, \xi\rangle = 0, \\ \left\langle A_\zeta X, Y\right\rangle &= \langle \alpha(X, Y), \zeta\rangle = \langle X, Y\rangle \langle \eta, \zeta\rangle = \langle X, Y\rangle. \end{aligned}$$

Therefore $A_\eta X = \|\eta\|^2 X$, $A_\xi X = 0$ e $A_\zeta X = X$. •

Let $X,Y \in \Gamma(\mathscr{D})$, $Z \in \Gamma\left(\mathscr{D}^{\perp}\right)$ and $\xi,\zeta \in \Gamma\left(T_f^{\perp}M\right)$ such that $\xi \perp \eta$ e $\langle \xi,\zeta\rangle = 1$.

***(I)*:** Knowing that η is parallel in the normal connection of f along $\mathscr{D}$, then

$$X\left(\|\eta\|^2\right) = 0 \Rightarrow \left\langle X, \nabla\|\eta\|^2\right\rangle = 0.$$

Therefore $\nabla\|\eta\|^2 \in \Gamma\left(\mathscr{D}^{\perp}\right)$.

Using the Codazzi Equation and equation (3.1), and after some computations, we get that

$$\nabla_X A_\eta Z - A_\eta \nabla_X Z = Z\left(\|\eta\|^2\right) X + \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\nabla_Z X - A_{\nabla_Z^{\perp}\eta} X.$$

Taking the inner product of both sides of the above equality by Y, and after some computations, we obtain

$$\left\langle Z, \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\nabla_X Y\right\rangle = \frac{\langle X,Y\rangle}{2}\left\langle \nabla\|\eta\|^2, Z\right\rangle. \tag{3.12}$$

We know that, if $K \in \mathscr{D}$, then $\left\langle \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\nabla_X Y, K\right\rangle = \left\langle \nabla_X Y, \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) K\right\rangle = 0$, that is, the only component of $\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\nabla_X Y$ is in $\mathscr{D}^{\perp}$. Therefore, equation (3.2) follows from equation (3.12).

We can derive Equation (3.3) making similar computations from Codazzi Equation for A_ξ, X and Z and taking the inner product with Y. Equation (3.4) is similar, but we must use X, A_ζ and Z at Codazzi Equation.

***(II)*:** If $\mathscr{D}$ is an umbilical distribution and φ is its mean curvature vector, then

$$\begin{aligned}\tilde{\nabla}_X f_* Y &= f_* \nabla_X Y + \alpha(X,Y) = f_* \nabla_X^v Y + f_* \nabla_X^h Y + \langle X,Y\rangle \eta = \\ &= f_* \nabla_X^v Y + \langle X,Y\rangle f_* \varphi + \langle X,Y\rangle \eta = f_* \nabla_X^v Y + \langle X,Y\rangle \sigma. \bullet\end{aligned}$$

***(III)*:** If $\mathscr{D}$ is an umbilical distribution and φ is its mean curvature vector, then $\nabla_X X = \nabla_X^v X + \nabla_X^h X$ e $\nabla_X^h X = \varphi$, where $\nabla_X^v X$ and $\nabla_X^h X$ are the orthogonal projections of $\nabla_X X$ on $\mathscr{D}$ and on $\mathscr{D}^{\perp}$, respectively. Thus,

$$\begin{aligned}&\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\varphi = \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\nabla_X^h X \overset{(3.1)}{=} \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\left(\nabla_X^v X + \nabla_X^h X\right) = \\ &= \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right)\nabla_X X \overset{(3.2)}{=} \frac{1}{2}\nabla\|\eta\|^2\end{aligned}$$

Therefore equation (3.6) is true.

The equations (3.7) and (3.8) follow, respectively, from equations (3.3) and (3.4), using equation (3.1).

Using (3.6), we can compute that

$$\left\langle \nabla_X \varphi, \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) Z\right\rangle = \frac{1}{2} X\left\langle \nabla\|\eta\|^2, Z\right\rangle - \left\langle \varphi, \nabla_X\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) Z\right\rangle. \tag{3.13}$$

Using Codazzi Equation for A_η, X and Z, using equation (3.6), and after some computations, we obtain

$$\left\langle \nabla_X\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) Z, \varphi\right\rangle = \frac{1}{2} XZ\left(\|\eta\|^2\right) = \frac{1}{2} X\left\langle Z, \nabla\|\eta\|^2\right\rangle.$$

Thus we get the equation (3.9) replacing the last equation in (3.13).

We know that η is parallel in the normal connection of f along $\mathscr{D}$ and $\xi \perp \eta$, thus $\left\langle \nabla_X^\perp \xi, \eta \right\rangle = -\left\langle \xi, \nabla_X^\perp \eta \right\rangle = 0$, that is, $\nabla_X^\perp \xi \perp \eta$. In this way, using the Codazzi Equation for A_ξ, X and Z, using equation (3.7), and after some computations, we obtain

$$\left\langle A_\xi Z, \nabla_X \varphi \right\rangle = \left\langle \mathscr{R}^\perp(X,Z)\xi, \eta \right\rangle.$$

By the other side, by Ricci Equation,

$$\left\langle \mathscr{R}^\perp(X,Z)\xi, \eta \right\rangle = \left\langle \cancel{\tilde{\mathscr{R}}(X,Z)\xi}, \eta \right\rangle - \left\langle \cancel{[A_\xi, A_\eta]} X, Z \right\rangle = 0.$$

Therefore, the equation (3.10) is true.

Similarly, equation (3.11) is obtained using the Codazzi equation for A_ζ, X and Z, equations (3.7) and (3.8) and the Ricci Equation for X, Z, ζ and η. • □

Corollary 10. *Let $f\colon M^m \to \mathbb{R}^n_t$ be an isometric immersion. If η is a non null Dupin normal of f, E_η is an umbilical distribution and φ is the mean curvature vector of E_η, then E_η is a spherical distribution and the equations of Lemma 9 are true.*

Proof. Taking $\mathscr{D} := E_\eta$, the formulas of Lemma 9 are true. To show that E_η is spherical, we will show that $\nabla_X \varphi(x) \in E_\eta(x)$, for all $x \in M$ and all $X \in E_\eta(x)$. But this is equivalent to show that

$$\left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi(x) = 0,$$

for all $x \in M$ and all $\psi \in T_x^\perp M$.

Let $x \in M$ and $\psi \in T_x^\perp M$.

<u>If $\eta(x)$ is timelike or spacelike.</u>

In this case, $\|\eta(x)\|^2 \neq 0$ and

$$\begin{aligned} A_\psi - \langle \psi, \eta \rangle \operatorname{Id} &= A_{\psi - \langle \psi, \eta \rangle \frac{\eta}{\|\eta\|^2}} + \langle \psi, \eta \rangle A_{\frac{\eta}{\|\eta\|^2}} - \langle \psi, \eta \rangle \operatorname{Id} = \\ &= A_{\psi - \langle \psi, \eta \rangle \frac{\eta}{\|\eta\|^2}} + \langle \psi, \eta \rangle \left(A_{\frac{\eta}{\|\eta\|^2}} - \operatorname{Id} \right) = A_\xi + \langle \psi, \eta \rangle \left(A_{\frac{\eta}{\|\eta\|^2}} - \operatorname{Id} \right), \end{aligned}$$

where $\xi := \psi - \langle \psi, \eta \rangle \frac{\eta}{\|\eta\|^2} \perp \eta$. If $Z \in E_\eta^\perp(x)$, then

$$\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi, Z \right\rangle = \left\langle A_\xi \nabla_X \varphi, Z \right\rangle + \langle \psi, \eta \rangle \left\langle \left(A_{\frac{\eta}{\|\eta\|^2}} - \operatorname{Id} \right) \nabla_X \varphi, Z \right\rangle.$$

By equations (3.9) e (3.10),

$$\left\langle \nabla_X \varphi, A_\xi Z \right\rangle = 0 = \left\langle \nabla_X \varphi, \left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) Z \right\rangle.$$

Therefore $\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi, Z \right\rangle = 0$. It remains to prove that $\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi(x), Y \right\rangle = 0$, for all $Y \in E_\eta(x)$. But

$$\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi(x), Y \right\rangle = \left\langle \nabla_X \varphi, A_\xi Y \right\rangle + \langle \psi, \eta \rangle \left\langle \nabla_X \varphi, \left(A_{\frac{\eta}{\|\eta\|^2}} - \operatorname{Id} \right) Y \right\rangle \overset{(3.1)}{=} 0.$$

If $\eta(x)$ is non null and lightlike.

In this case, there exists a lightlike vector $\zeta \in T_x^\perp M$ such that $\langle \eta, \zeta \rangle = 1$. Thus,

$$A_\psi - \langle \psi, \eta \rangle \operatorname{Id} = A_{\psi - \langle \psi, \eta \rangle \zeta} + \langle \psi, \eta \rangle A_\zeta - \langle \psi, \eta \rangle \operatorname{Id} = A_\xi + \langle \psi, \eta \rangle (A_\zeta - \operatorname{Id}),$$

where $\xi := \psi - \langle \psi, \eta \rangle \zeta \perp \eta$.

If $Z \in E_\eta^\perp(x)$,

$$\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi, Z \right\rangle = \left\langle A_\xi \nabla_X \varphi, Z \right\rangle + \langle \psi, \eta \rangle \left\langle (A_\zeta - \operatorname{Id}) \nabla_X \varphi(x), Z \right\rangle.$$

By the equalities (3.10) e (3.11),

$$\left\langle \nabla_X \varphi, A_\xi Z \right\rangle = 0 = \left\langle \nabla_X \varphi, (\operatorname{Id} - A_\zeta) Z \right\rangle.$$

Therefore $\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi, Z \right\rangle = 0$.

By the other side, if $Y \in E_\eta(x)$,

$$\left\langle \left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X \varphi, Y \right\rangle = \left\langle \nabla_X \varphi, A_\xi Y \right\rangle + \langle \psi, \eta \rangle \left\langle \nabla_X \varphi, (A_\zeta - \operatorname{Id}) Y \right\rangle \overset{(3.1)}{=} 0.$$

□

Proposition 11. *Let M^m be a riemannian manifold, $f\colon M^m \to \mathbb{R}_t^n$ an isometric immersion and η its non null principal normal.*

1. *If $\dim E_\eta$ is constant and $\dim E_\eta \geq 2$, then η is parallel in the normal connexion of f along E_η, ie., η is a Dupin normal.*
2. *If $\mathscr{D} \subset E_\eta$ is a spherical distribution in M whose leafs are open subsets of*
   - (a) *q-dimensional ellipsoids given by the intersection $\mathbb{S}(c,r) \cap (c+L) \subset \mathbb{R}_t^n$, where L is a spacelike $(q+1)$-dimensional vector of $\mathbb{R}_t^n$;*
   - (b) *or q-dimensional hyperboloids given by the intersection $\mathbb{S}(c,-r) \cap (c+L) \subset \mathbb{R}_t^n$, where L is a timelike $(q+1)$-dimensional vector of $\mathbb{R}_t^n$;*
   - (c) *or q-dimensional paraboloids given by $[\mathscr{L}_* \cap (c+L)] + d \subset \mathbb{R}_t^n$, where $L = \operatorname{span}\{w\} \oplus V$ is a lightlike $(q+1)$-dimensional vector of $\mathbb{R}_t^n$ (with V spacelike and w lightlike), $c \perp V$ is lightlike and $\langle c, w \rangle \neq 0$;*

   then η is parallel in the normal connexion of f along $\mathscr{D}$.
3. *If η Dupin normal with multiplicity q, then E_η is an spherical distribution in M^m.*
   In this case, let $x \in M$, N be a leaf of E_η with $x \in N$ and $\sigma := f_* \varphi + \eta$, where φ is the mean curvature vector of E_η.
   - (a) *If $\sigma(x)$ is spacelike, then $f(N)$ is an open subset of a q-dimensional ellipsoid in $\mathbb{R}_t^n$ given by the intersection $\mathbb{S}(c,r) \cap (c+L)$, where L is a spacelike $(q+1)$-dimensional subspace of $\mathbb{R}_t^n$.*
   - (b) *If $\sigma(x)$ is timelike, then $f(N)$ is an open subset of a q-dimensional hyperboloid in $\mathbb{R}_t^n$ given by the intersection $\mathbb{S}(c,-r) \cap (c+L)$, where L is a timelike $(q+1)$-dimensional subspace of $\mathbb{R}_t^n$.*

(c) *If $\sigma(x)$ is lightlike and non null, then $f(N)$ is an open subset of a q-dimensional paraboloid in $\mathbb{R}^n_t$ given by $c+\left\{ v+\frac{\|v\|^2}{2}w \,\middle|\, v\in V(x)\right\}$, where $V\subset\mathbb{R}^n_t$ is a spacelike q-dimensional vector subspace and $w\perp V$ is lightlike.*

Remarks 12. *Through the proof made ahead, at the items (III.1) and (III.2) of Proposition 11,*

$$c=f(x)+\frac{\sigma(x)}{\|\sigma(x)\|^2},\quad r=\frac{1}{\sqrt{|\|\sigma(x)\|^2|}}\quad \text{and}\quad L(x)=f_*E_\eta(x)\ominus\operatorname{span}\{\sigma(x)\}$$

are constant in each leaf of E_η.

At the item (III.3), the paraboloids containing the leafs of E_η are given by

$$p(x)+(-\tilde{\sigma}(x)+L)\cap\mathscr{L}=p(x)-\tilde{\sigma}(x)+\left\{ v+\frac{\|v\|^2}{2}\sigma(x)\,\middle|\, v\in V(x)\right\},$$

$$\xi(x):=-\sum_{i=1}^{q}\langle \mathrm{d}f(x)e_i,\tilde{\sigma}(x)\rangle\,\mathrm{d}f(x)e_i+\frac{1}{2}\sum_{i=1}^{q}\langle \mathrm{d}f(x)e_i,\tilde{\sigma}(x)\rangle^2\,\sigma(x)+\tilde{\sigma}(x).$$

Proof of Proposition 11.

Let X^v and X^h be the orthogonal projections of $X\in\Gamma(TM)$ on $\mathscr{D}$ and $\mathscr{D}^\perp$, respectively. Likewise, let $\nabla^v_X Y$ and $\nabla^h_X Y$ be the orthogonal projections of $\nabla_X Y$ on $\mathscr{D}$ and $\mathscr{D}^\perp$, respectively.

***(I)*:** Let $\mathscr{D}:=E_\eta$, $X,Y\in\Gamma(E_\eta)$ and $\xi,\zeta\in\Gamma\left(T^\perp M\right)$ such that $\xi\perp\eta$ e $\langle\zeta,\eta\rangle=1$. By Codazzi Equation for A_ξ, X and Y and using (3.1), we get

$$A_\xi\nabla_X Y+A_{\nabla^\perp_X\xi}Y=A_\xi\nabla_Y X+A_{\nabla^\perp_Y\xi}X.$$

We suppose that $X\perp Y$ and that $\|Y\|^2=1$, since $\dim E_\eta\geq 2$. Thus, taking the inner product with Y of both sides of the above equation, using (3.1) and after some calculations, we can get that $\left\langle\nabla^\perp_X\eta,\xi\right\rangle=0$.

Similarly, by Codazzi Equation for A_ζ, X and Y, and taking the inner product with Y, we can compute that $\left\langle\nabla^\perp_X\eta,\zeta\right\rangle=0$.

We conclude that $\nabla^\perp_X\eta=0$, cause $\left\langle\nabla^\perp_X\eta,\xi\right\rangle=0$ and $\left\langle\nabla^\perp_X\eta,\zeta\right\rangle=0$, for all $\xi,\zeta\in\Gamma\left(T^\perp M\right)$ such that $\xi\perp\eta$ and $\langle\zeta,\eta\rangle=1$. •

***(II.1)* and *(II.2)*:** Lets suppose that the leafs of $\mathscr{D}$ are open subsets of q-dimensional ellipsoids or hyperboloids given by $\mathbb{S}(c,\varepsilon r)\cap(c+L)\subset\mathbb{R}^n_t$, where

a) or $\varepsilon=1$ and L is an $(q+1)$-dimensional spacelike subspace of $\mathbb{R}^n_t$, if L is spacelike;
b) or $\varepsilon=-1$ and L is and $(q+1)$-dimensional timelike subspace of $\mathbb{R}^n_t$, if L is timelike.

Let $N\subset M$ be a leaf (integral submanifold) of $\mathscr{D}$. Thus, $f(N)\subset\mathbb{S}(c,\varepsilon r)\cap(c+L)\subset\mathbb{R}^n_t$, for some $c\in\mathbb{R}^n_t$, $r>0$ and some $(q+1)$-dimensional spacelike or timelike vector subspace $L^{q+1}\subset\mathbb{R}^n_t$. Lets define the field $\sigma\colon N\to\mathbb{R}^n_t$ by $\sigma(x):=-\varepsilon\frac{f(x)-c}{r^2}$ and let $X\in\Gamma(\mathscr{D})$, in this way

$$\|\sigma\|^2=\frac{\varepsilon^2}{r^4}\|f(x)-c\|^2=\frac{\varepsilon^3r^2}{r^4}=\frac{\varepsilon}{r^2}\quad\text{and}$$
$$\langle\sigma,f_*X\rangle=-\varepsilon\left\langle\frac{f(x)-c}{r^2},f_*X\right\rangle=-\varepsilon r^2\left\langle-\varepsilon\frac{f(x)-c}{r^2},-\varepsilon\frac{f_*X}{r^2}\right\rangle=-r^2\varepsilon\langle\sigma,\sigma_*X\rangle=0,$$

that is, σ is normal to N and $\|\sigma\|^2 = \frac{\varepsilon}{r^2}$ is constant in N.

Knowing that $\mathscr{D} \subset E_\eta$ and that $\mathscr{D}$ is a spherical distribution, we can get that

$$\tilde{\nabla}_X f_* Y = f_* \nabla^v_X Y + \langle X, Y \rangle (f_* \varphi + \eta).$$

By the other side, $c + L$ is totally geodesic if $\mathbb{R}^n_t$, $f(N) \subset \mathbb{S}(c, \varepsilon r) \cap (c+L) \subset \mathbb{R}^n_t$ and ∇^v is the Levi-Civita connection of N, then

$$\tilde{\nabla}_X f_* Y = f_* \nabla^v_X Y - \langle X, Y \rangle \varepsilon \frac{f-c}{r^2} = f_* \nabla^v_X Y + \langle X, Y \rangle \sigma$$

Comparing the last two equations, we get that $\sigma = f_*\varphi + \eta$ and $\eta = \sigma - f_*\varphi$. Thus,

$$\begin{aligned} \tilde{\nabla}_X \eta &= \tilde{\nabla}_X \sigma - \tilde{\nabla}_X f_* \varphi = -\tilde{\nabla}_X \varepsilon \frac{f-c}{r^2} - f_* \nabla_X \varphi - \cancel{\alpha(X,\varphi)} = \\ &= -\frac{\varepsilon}{r^2} f_* X - f_* \nabla_X \varphi, \text{ cause } X \in \mathscr{D} \subset E_\eta. \end{aligned}$$

Therefore $\nabla^\perp_X \eta = 0$. •

***(II.3)*:** Lets suppose that the leafs of $\mathscr{D}$ are open subsets of q-dimensional paraboloids given by $[\mathscr{L} \cap (L+c)] + d \subset \mathbb{R}^n_t$, where $L = \text{span}\{w\} \oplus V$ is a $(q+1)$-dimensional lightlike vector subspace of $\mathbb{R}^n_t$ (with V spacelike and w lightlike), $c \perp V$ is lightlike and $\langle c, w \rangle \neq 0$.

Let N be a leaf of $\mathscr{D}$. But $[\mathscr{L} \cap (L+c)] + d \subset \text{span}\{c,w\} \oplus V + d \subset \mathbb{R}^n_t$ and $\text{span}\{c,w\} \oplus V + d$ is totally geodesic in $\mathbb{R}^n_t$, thus we can consider $f|_N : N \to \text{span}\{c,w\} \oplus V + d$.

But $f - d \in \mathscr{L}$, thus $f - d$ is field normal to N. Let $\{w, X_1, \cdots, X_q\}$ be a basis of L such that $\{X_1, \cdots, X_q\}$ is a orthonormal basis of V. In this way, $\text{span}\{c,w\} \oplus V = L + \text{span}\{c\} = \text{span}\{w, \tilde{w}, X_1, \cdots, X_q\}$, where $\{w, \tilde{w}\}$ is a pseudo-orthonormal basis of $\text{span}\{w, c\}$. We can suppose that $c = b\tilde{w}$.

We will show that $\langle f - d, \frac{w}{b} \rangle = 1$. Indeed, $f(x) - d \in L + c$, thus

$$f(x) - d = a(x) w + b\tilde{w} + \sum_{i=1}^{q} x_i(x) X_i \;\Rightarrow\; \left\langle f - d, \frac{w}{b} \right\rangle = 1,$$

and thus $w \perp N$.

But $f - d$ and $\frac{w}{b}$ are orthogonal to N and $f(N) \subset \text{span}\{c,w\} \oplus V + d$, then

$$\begin{aligned} \alpha_{f|_N}(X,Y) &= \langle \alpha_{f|_N}(X,Y), f-d \rangle \frac{w}{b} + \left\langle \alpha_{f|_N}(X,Y), \frac{w}{b} \right\rangle (f-d) = \\ &= \langle A_{f-d} X, Y \rangle \frac{w}{b} + \left\langle A_{\frac{w}{b}} X, Y \right\rangle (f-d). \end{aligned}$$

By the other side, $\tilde{\nabla}_X \frac{w}{b} = 0$ and $\tilde{\nabla}_X (f-d) = f_* X$. Therefore $\alpha_{f|_N}(X,Y) = -\langle X, Y \rangle \frac{w}{b}$.

By the same calculations made at the cases *(II.1)* and *(II.2)*, we get that $\tilde{\nabla}_X f_* Y = f_* \nabla^v_X Y + \langle X, Y \rangle (f_* \varphi + \eta)$. Thus

$$\begin{aligned} &-\frac{w}{b} = f_* \varphi + \eta \;\Rightarrow\; \eta = -\frac{w}{b} - f_* \varphi \;\Rightarrow \\ &\Rightarrow\; \tilde{\nabla}_X \eta = -\tilde{\nabla}_X (f_* \varphi) = -f_* \nabla_X \varphi - \cancel{\langle X, \varphi \rangle} \eta = -f_* \nabla_X \varphi. \end{aligned}$$

Therefore $\nabla^\perp_X \eta = 0$, for all $X \in \mathscr{D}$. •

***(III)*:** If $\mathscr{D} := E_\eta$, then, by Lemma 9, the equations (3.1) to (3.4) hold.

Affirmation 1: *If* $X, Y \in \Gamma(E_\eta)$ *and* $X \perp Y$, *then* $\nabla_X Y \in \Gamma(E_\eta)$.

If $Z \in \Gamma\left(E_\eta^\perp\right)$, $\xi, \zeta \in \Gamma\left(T^\perp M\right)$, $\xi \perp \eta$ and $\langle \zeta, \eta \rangle = 1$, then

$$\begin{aligned} &\left(\|\eta\|^2 \operatorname{Id} - A_\eta\right) \nabla_X Y \overset{(3.2)}{=} \frac{\cancel{\langle X, Y \rangle}}{2} \nabla \|\eta\|^2 = 0 \Rightarrow \|\eta\|^2 \nabla_X Y = A_\eta \nabla_X Y; \qquad (3.14)\\ &\left\langle A_\xi \nabla_X Y, Z \right\rangle \overset{(3.3)}{=} \cancel{\langle X, Y \rangle} \langle \nabla_Z \xi, \eta \rangle = 0 \Rightarrow A_\xi \nabla_X Y \in \Gamma(E_\eta);\\ &\left\langle \left(\operatorname{Id} - A_\zeta\right) \nabla_X Y, Z \right\rangle \overset{(3.4)}{=} -\cancel{\langle X, Y \rangle} \left\langle \nabla_Z^\perp \zeta, \eta \right\rangle = 0 \Rightarrow (\operatorname{Id} - A_\zeta) \nabla_X Y \in \Gamma(E_\eta). \end{aligned}$$

By the other side, if $W \in E_\eta$, then

$$\begin{cases} \left\langle A_\xi \nabla_X Y, W \right\rangle = \left\langle \nabla_X Y, A_\xi W \right\rangle \overset{(3.1)}{=} 0;\\ \left\langle (\operatorname{Id} - A_\zeta) \nabla_X Y, W \right\rangle = \left\langle \nabla_X Y, (\operatorname{Id} - A_\zeta) W \right\rangle \overset{(3.1)}{=} 0. \end{cases}$$

Therefore

$$A_\xi \nabla_X Y = 0 \quad \text{e} \quad (\operatorname{Id} - A_\zeta) \nabla_X Y = 0, \qquad (3.15)$$

for all $\xi, \zeta \in \Gamma\left(T^\perp M\right)$ such that $\xi \perp \eta$ and $\langle \zeta, \eta \rangle = 1$.

Let $x \in M$ be a point and $\psi \in T_x^\perp M$ be a normal vector. If $\eta(x)$ is timelike or spacelike, then $\|\eta(x)\|^2 \neq 0$, thus

$$\left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X Y = A_{\psi - \langle \psi, \eta \rangle \frac{\eta}{\|\eta\|^2}} \nabla_X Y - \langle \psi, \eta \rangle \left(\operatorname{Id} - A_{\frac{\eta}{\|\eta\|^2}}\right) \nabla_X Y \overset{(3.14),(3.15)}{=} 0.$$

If $\eta(x)$ is lightlike, then there exists a lightlike vector $\zeta \in T_x^\perp M$ such that $\langle \eta(x), \zeta \rangle = 1$. In this case,

$$\left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X Y = A_{\psi - \langle \psi, \eta \rangle \zeta} \nabla_X Y - \langle \psi, \eta \rangle \left(\operatorname{Id} - A_\zeta\right) \nabla_X Y \overset{(3.14),(3.15)}{=} 0.$$

But $\left(A_\psi - \langle \psi, \eta \rangle \operatorname{Id}\right) \nabla_X Y = 0$, for all ψ, is equivalent to $\nabla_X Y \in E_\eta$. ✓

Affirmation 2: E_η *is umbilical.*

We have to show that there exists $\varphi \in \Gamma\left(E_\eta^\perp\right)$ such that $\nabla_X^h Y = \langle X, Y \rangle \varphi$, for any pair of vector fields $X, Y \in \gamma(E_\eta)$. But the application $(X, Y) \mapsto \nabla_X^h Y$ is bilinear in E_η because, for any $Z \in \Gamma\left(E_\eta^\perp\right)$, $\left\langle \nabla_X^h Y, Z \right\rangle = -\langle Y, \nabla_X Z \rangle$. Besides that, Affirmation 1 stands that $X \perp Y \Rightarrow \nabla_X^h Y = 0$. Then, a known Lemma stands that there exists φ such that $\nabla_X^h Y = \langle X, Y \rangle \varphi$ (see, for example, Lemma A.9 in [9]).

If we take a unit differentiable vector field $X \in E_\eta$, then $\varphi = \nabla_X^h X$. Therefore φ is differentiable. ✓

Affirmation 3: E_η *is spherical and the equations from Lemma 9 hold.*

Just see Corollary 10. ✓

Let $N \subset M$ be a leaf of E_η passing through x. Equation (3.5) stands that $f|_N : N \to \mathbb{R}_t^n$ is an umbilical isometric immersion and that σ is its mean curvature vector. Therefore, knowing the classifications of umbilical immersions in $\mathbb{R}_t^n$, we have that Remarks 12 hold and that

- or $f(N) \subset \mathbb{S}\left(c(x); \frac{1}{\|\sigma(x)\|}\right) \cap (c(x) + L(x))$, if $\sigma(x)$ is spacelike;

- or $f(N) \subset \mathbb{S}\left(c(x); -\frac{1}{\|\sigma(x)\|}\right) \cap (c(x) + L(x))$, if $\sigma(x)$ is timelike;
- or $f(N) \subset p(x) + (-\tilde{\sigma}(x) + L(x)) \cap \mathscr{L} = p(x) - \tilde{\sigma}(x) + \left\{v + \frac{\|v\|^2}{2}\sigma(x) : v \in V(x)\right\}$, if $\sigma(x)$ is lightlike.

For more details about umbilical immersions of a riemannian manifold in $\mathbb{R}^n_t$, see Chapter 1 of [9]. □

The following definition was given at [5]

Definition 13. *Let $\mathscr{D}$ be an umbilical distribution in an riemannian manifold M. The* ***splitting tensor*** *C of $\mathscr{D}$ is given by $C_X Z := -\nabla^h_Z X$, for all $X \in \Gamma(\mathscr{D})$ and all $Z \in \Gamma\left(\mathscr{D}^\perp\right)$.*

Remarks 14. *Given an orthonormal frame $\{w_1, \cdots, w_k\}$ of $\mathscr{D}^\perp$, it follows that*

$$C_X Z = -\nabla^h_Z X = -\sum_{i=1}^{k} \left\langle \nabla^h_Z X, w_i \right\rangle w_i = \sum_{i=1}^{k} \langle X, \nabla_Z w_i \rangle w_i.$$

Therefore $C_{f \cdot X} g \cdot Z = f \cdot g \cdot C_X Z$, for any pair of differentiable applications $f, g \colon M \to \mathbb{R}$, every $X \in \Gamma(\mathscr{D})$ and every $Z \in \Gamma\left(\mathscr{D}^\perp\right)$. Therefore C is a tensor.

Lemma 15. *Let $\mathscr{D}$ be an umbilical distribution in M and φ its mean curvature vector. If $X, Y \in \mathscr{D}$ and $W, Z \in \mathscr{D}^\perp$, then:*

$$\left(\nabla^h_X C_Y\right) W = C_Y C_X W + C_{\nabla^v_X Y} W - \mathscr{R}^h(X,W)Y + \langle X, Y \rangle \left(\langle W, \varphi \rangle - \nabla^h_W \varphi\right), \tag{3.16}$$

$$\left(\nabla^h_W C_X\right) Z - \left(\nabla^h_Z C_X\right) W = C_{\nabla^v_W X} Z - C_{\nabla^v_Z X} W - \mathscr{R}^h(W,Z)X - \langle [W,Z], X \rangle \varphi, \tag{3.17}$$

where $\mathscr{R}^h(X,W)Y$ is the orthogonal projection of $\mathscr{R}(X,W)Y$ on $\mathscr{D}^\perp$.

Se $\mathscr{D} \subset E_\eta$, then

$$\left(\nabla^h_X C_Y\right) W = C_Y C_X W + C_{\nabla^v_X Y} W + \langle X, Y \rangle \left(A_\eta W + \langle W, \varphi \rangle \varphi - \nabla^h_W \varphi\right), \tag{3.18}$$

$$\left(\nabla^h_W C_X\right) Z - \left(\nabla^h_Z C_X\right) W = C_{\nabla^v_W X} Z - C_{\nabla^v_Z X} W - \langle [W,Z], X \rangle \varphi. \tag{3.19}$$

If η is a principal normal of $f \colon M \to \mathbb{R}N$, $\mathscr{D} \subset E_\eta$ and $\mathscr{D}^\perp$ is a totally geodesic distribution, then

$$\nabla^h_W \varphi = A_\eta W + \langle W, \varphi \rangle \varphi. \tag{3.20}$$

Proof. See Lemma 9 of [5], where it was first proved, or Lemma 2.15 of [9]. □

Now we can prove Theorem 1.

Poof of Theorem 1.

Taking $\mathscr{D}(x) = E_\eta(x)$, the items *(I)* of Lemma 9 and *(III)* of Proposition 11 stands that $\nabla\left(\|\eta\|^2\right) \in E^\perp_\eta$ and that E_η is an spherical distribution. Let φ be the mean curvature vector of E_η and $\sigma := f_* \varphi + \eta$.

We will prove the following equation:

$$\tilde{\nabla}_Z \sigma = \langle Z, \varphi \rangle \sigma, \ \forall Z \in E^\perp_\eta. \tag{3.21}$$

By Lemmas 9 and 15, we have that

$$\left\langle \nabla^{\perp}_{Z}\eta, \xi \right\rangle = -\left\langle \alpha\left(Z,\varphi\right), \xi \right\rangle \quad \text{and} \quad \nabla^{h}_{Z}\varphi = A_{\eta}Z + \left\langle Z,\varphi\right\rangle \varphi,$$

for all $Z \in E^{\perp}_{\eta}$ and all $\xi \perp \eta$.

By (3.6), $\left(\|\eta\|^2\,\mathrm{Id} - A_{\eta}\right)\varphi = \frac{1}{2}\nabla\|\eta\|^2$, thus

$$\|\eta\|^2 \left\langle \varphi, Z\right\rangle - \left\langle A_{\eta}Z, \varphi\right\rangle = \frac{1}{2}Z\left(\|\eta\|^2\right),\ \forall Z \in E^{\perp}_{\eta}. \tag{3.22}$$

In this way, using that $E^{\perp}_{\eta}$ is totally geodesic, we can compute

$$\tilde{\nabla}_{Z}\sigma = \left\langle Z,\varphi\right\rangle f_{*}\varphi + \alpha\left(\varphi, Z\right) + \nabla^{\perp}_{Z}\eta.$$

Thus,

$$\left\langle \tilde{\nabla}_{Z}\sigma, \xi\right\rangle = \left\langle \alpha\left(\varphi,Z\right),\xi\right\rangle + \left\langle \nabla^{\perp}_{Z}\eta, \xi\right\rangle = 0,\ \forall \xi \perp \eta \text{ in } T^{\perp}_{f}M.$$

If η is spacelike or timelike (at some point), then

$$\begin{aligned}\tilde{\nabla}_{Z}\sigma &= \left\langle Z,\varphi\right\rangle f_{*}\varphi + \left\langle \alpha\left(Z,\varphi\right) + \nabla^{\perp}_{Z}\eta, \eta\right\rangle \frac{\eta}{\|\eta\|^2} = \\ &= \left\langle Z,\varphi\right\rangle f_{*}\varphi + \left[\left\langle A_{\eta}Z,\varphi\right\rangle + \frac{1}{2}Z\left(\|\eta\|^2\right)\right]\frac{\eta}{\|\eta\|^2} = \\ &\overset{(3.22)}{=} \left\langle Z,\varphi\right\rangle f_{*}\varphi + \|\eta\|^2\left\langle \varphi, Z\right\rangle \frac{\eta}{\|\eta\|^2} = \left\langle Z,\varphi\right\rangle\left(f_{*}\varphi + \eta\right) = \left\langle Z,\varphi\right\rangle \sigma.\end{aligned}$$

Lets suppose that η is lightlike at $x \in M$. In this case, there exists a lightlike vector $\zeta \in T^{\perp}_{x}M$ such that $\left\langle \eta(x), \zeta\right\rangle = 1$. Thus, at x, the following equations hold:

$$\begin{aligned}\tilde{\nabla}_{Z}\sigma &= \left\langle Z,\varphi\right\rangle f_{*}\varphi + \left\langle \alpha\left(\varphi,Z\right) + \nabla^{\perp}_{Z}\eta, \zeta\right\rangle \eta = \\ &= \left\langle Z,\varphi\right\rangle f_{*}\varphi + \left[\left\langle A_{\zeta}\varphi, Z\right\rangle - \left\langle \eta, \nabla^{\perp}_{Z}\zeta\right\rangle\right]\eta = \\ &\overset{(3.8)}{=} \left\langle Z,\varphi\right\rangle\left[f_{*}\varphi + \eta\right] = \left\langle \varphi, Z\right\rangle \sigma.\end{aligned}$$

Therefore equation (3.21) holds.

<u>Affirmation 1</u>: *$\tilde{\nabla}_{Z}f_{*}X = f_{*}\nabla^{v}_{Z}X$, for all $X \in E_{\eta}$ and all $Z \in E^{\perp}_{\eta}$.*

If $X \in \Gamma\left(E_{\eta}\right)$ and $Z, W \in \Gamma\left(E^{\perp}_{\eta}\right)$, then $\left\langle \nabla_{Z}X, W\right\rangle = -\left\langle X, \nabla_{Z}W\right\rangle = -\left\langle X, \nabla^{v}_{Z}W\right\rangle = 0$, since $E^{\perp}_{\eta}$ is totally geodesic. Thus, $\tilde{\nabla}_{Z}f_{*}X = f_{*}\nabla_{Z}X + \cancel{\alpha\left(Z,X\right)} = f_{*}\nabla^{v}_{Z}X$. ✓

<u>Affirmation 2</u>: *The distribution $L := f_{*}E_{\eta} \oplus [\sigma]$ is parallel in $\mathbb{R}^{n}_{t}$ along M, that is, $L = f_{*}E_{\eta} \oplus [\sigma]$ is a constant vector subspace of $\mathbb{R}^{n}_{t}$.*

Indeed, if $X \in E_{\eta}$ and $f_{*}Y + \beta\sigma \in f_{*}E_{\eta} \oplus [\sigma]$, then, using that E_{η} is spherical and after some computations, we obtain

$$\tilde{\nabla}_{X}\left(f_{*}Y + \beta\sigma\right) = f_{*}\left[\nabla^{v}_{X}Y - \beta\left(\|\varphi\|^2 + \|\eta\|^2\right)X\right] + \left[\left\langle X,Y\right\rangle + X(\beta)\right]\sigma$$

By the other side, using (3.21) and Affirmation 1, we get that

$$\tilde{\nabla}_{Z}\left(f_{*}Y + \beta\sigma\right) = f_{*}\nabla^{v}_{Z}Y + \left[Z(\beta) + \beta\left\langle Z,\varphi\right\rangle\right]\sigma.$$

Therefore L is parallel in $\mathbb{R}^n_t$ along M. ✓

We know that L is constant and f_*E_η is spacelike, thus L and σ are spacelike at all points of M, or L and σ are timelike at all points of M, or L and σ are lightlike at all points of M.

Case 1: Lets suppose that σ is spacelike.

In this case, using item *(III.1)* of Proposition 11 and Remarks 12, it follows that the leafs of E_η are q-dimensional ellipsoids in $\mathbb{R}^n_t$ given by the intersection $\mathbb{S}\left(c(x);\frac{1}{\|\sigma(x)\|}\right)\cap(c(x)+L)$, where $\|\sigma(x)\|^2$ e $c(x)=f(x)+\frac{\sigma(x)}{\|\sigma(x)\|^2}$ are constant in each leaf of E_η.

We stand that $c_*TM\perp L$. Indeed, c is constant in the leafs of E_η, thus $c_*X=0$, for all $X\in E_\eta$. If $Z\in E_\eta^\perp$, then, using (3.21), we get that

$$c_*Z = f_*Z - \frac{\langle Z,\varphi\rangle}{\|\sigma\|^2}\sigma.$$

Thus, $\langle c_*Z, f_*X\rangle = 0$ and $\langle c_*Z,\sigma\rangle = \langle f_*Z,\sigma\rangle - \langle Z,\varphi\rangle = \langle Z,\varphi\rangle - \langle Z,\varphi\rangle = 0$. Therefore $c_*TM\perp L$.

Lets consider the manifold $N^{m-q} := M/\!\sim$, where $\sim$ is the equivalence relation given by

$$x\sim y \equiv x \text{ and } y \text{ are at the same leaf of distribution } E_\eta.$$

We know that $c(x)=f(x)+\frac{\sigma(x)}{\|\sigma\|^2}$ and $\|\sigma(x)\|^2$ are constant in each leaf of E_η, thus we can define the applications $\bar{c}\colon N\to\mathbb{R}^n_t$ and $r\colon N\to\mathbb{R}$ by $\bar{c}(\bar{x}) := c(x)$ e $r(\bar{x}) := \frac{1}{\|\sigma(x)\|}$, where $\bar{x}$ is the equivalence class of x.

Let $\Pi\colon\mathbb{R}^n_t\to L$ be the orthogonal projection. Thus, $\Pi\circ c$ and $\Pi\circ\bar{c}$ are constant in M and N respectively, cause $c_*TM\perp L$. In this way,

$$f(x) = c(x) - \frac{\sigma(x)}{\|\sigma\|^2} = p + h(\bar{x}) - r(\bar{x})\frac{\sigma(x)}{\|\sigma(x)\|},$$

where $p := \Pi(c(x))$ and $h(\bar{x})$ is the orthogonal projection of $\bar{c}(\bar{x})$ on $L^\perp$.

Therefore $f(M)$ is an open subset of the rotational submanifold with axis $L^\perp$ on the immersion $\bar{f}\colon N\to L^\perp\oplus\operatorname{span}\{\xi\}$, where $\bar{f}(\bar{x}) := \bar{h}(\bar{x}) + \bar{r}(\bar{x})\xi$ and $\xi\in\mathbb{S}(0,1)\subset L$ is a fixed vector. It's rotational parametrization $g\colon N\times\mathbb{S}(0,1)\to\mathbb{R}^n_t$ is given by $g(\bar{x},y) := p + h(\bar{x}) + r(\bar{x})y$. •

Case 2: Lets suppose that σ is timelike.

This case is analogous to the first case. We can prove that $f(M)$ is an open subset of the rotational submanifold with axis $L^\perp$ on the immersion $\bar{f}\colon N\to L^\perp\oplus\operatorname{span}\{\xi\}$, where $\bar{f}(\bar{x}) := \bar{h}(\bar{x}) + \bar{r}(\bar{x})\xi$, $\xi\in\mathbb{S}(0,-1)\subset L$ is a fixed vector, $N := M/\!\sim$ and $\sim$ is the equivalence relation given at Case 1. The rotational parametrization is $g\colon N\times\mathbb{S}(0,-1)\to\mathbb{R}^n_t$, given by $g(\bar{x},y) := p + h(\bar{x}) + r(\bar{x})y$, $\mathbb{S}(0,-1)\subset L$. •

Case 3: Lets suppose that σ is lightlike.

In this case, $L = E_\eta\oplus\operatorname{span}\{\sigma\}$ is a lightlike subspace subspace of $\mathbb{R}^n_t$.

Affirmation 4: *If $x_0\in M$ and $\sigma_0=\sigma(x_0)$, then $\sigma(x)=\frac{1}{r(x)}\sigma_0$, for some differentiable function $r\colon M\to\mathbb{R}$.*

If $x_0\in M$ and $\{X_1,\cdots,X_q\}$ is an orthonormal basis of $E_\eta(x_0)$, then $L=\operatorname{span}\{X_1,\cdots,X_q,\sigma(x_0)\}$, cause L is constant. Thus, $\sigma(x)=a_1(x)X_1+\cdots+a_m(x)X_m+\frac{1}{r(x)}\sigma_0$ and $0=\|\sigma(x)\|^2=\sum_{i=1}^m a_i^2(x)$. It follows that $a_1(x)=\cdots=a_m(x)=0$ and $\sigma(x)=\frac{1}{r(x)}\sigma_0$.

Let $V \subset L$ be a spacelike vector subspace and $\tilde{\sigma}_0$ be a lightlike vector such that $\tilde{\sigma}_0 \perp V$ and $\langle \sigma_0, \tilde{\sigma}_0 \rangle = 1$. Thus, $\frac{1}{r(x)} = \langle \sigma(x), \tilde{\sigma}_0 \rangle$ is differentiable. ✓

Lets define $\tilde{\sigma}(x) := r(x)\tilde{\sigma}_0$. Thus, $\tilde{\sigma}$ is a lightlike differentiable field such that $\tilde{\sigma} \perp V$ and $\langle \sigma, \tilde{\sigma} \rangle = 1$. Besides that, $\mathbb{R}^n_t = \operatorname{span}\{\sigma, \tilde{\sigma}\} \oplus U \oplus V = \operatorname{span}\{\sigma_0, \tilde{\sigma}_0\} \oplus U \oplus V$, where $U = (\operatorname{span}\{\sigma, \tilde{\sigma}\} \oplus V)^\perp$ is a nondegenerated vector subspace of $L^\perp \subset \mathbb{R}^n_t$.

Lets consider

$$\xi(x) := -\sum_{i=1}^{q} \langle v_i(x), \tilde{\sigma}(x) \rangle v_i(x) + \frac{1}{2}\sum_{i=1}^{q} \langle v_i(x), \tilde{\sigma}(x) \rangle^2 \sigma(x) + \tilde{\sigma}(x),$$

where $v_i(x) = f_* e_i(x)$ e $\{e_1(x), \cdots, e_q(x)\}$ is an orthonormal basis of $E_\eta(x)$. It can be shown that ξ is a lightlike differentiable field such that, $\xi \perp E_\eta$, $\xi \in L \oplus \operatorname{span}\{\tilde{\sigma}\} = L \oplus \operatorname{span}\{\tilde{\sigma}_0\}$ and $\langle \xi, \sigma \rangle = 1$ (see the arguments at Lemma 1.2 of [9]).

By item *(III.3)* of Proposition 11 and by Remarks 12,

$$f(x) \in p(x) + (-\tilde{\sigma}(x) + L) \cap \mathscr{L} = p(x) - \tilde{\sigma}(x) + \left\{ v + \frac{\|v\|^2}{2}\sigma(x) \,\middle|\, v \in V \right\},$$

where $p(x) = f(x) + \xi(x)$ is constant in each leaf of E_η.

Let $P\colon \mathbb{R}^n_t \to V$ be the orthogonal projection and $v(x) = P(f(x) - p(x))$. Thus, $f(x) - p(x) \in \operatorname{span}\{\tilde{\sigma}, \sigma\} \oplus V$ and

$$f(x) = p(x) - \tilde{\sigma}(x) + v(x) + \frac{\|v(x)\|^2}{2}\sigma(x) = p(x) + r(x)\left(-\tilde{\sigma}_0 + w(x) + \frac{\|w(x)\|^2}{2}\sigma_0\right),$$

where $w(x) := \frac{v(x)}{r(x)}$.

<u>Affirmation 5</u>: $\{v_* e_1, \cdots, v_* e_q\}$ *is an orthonormal basis of* V.

If $X \in \Gamma(E_\eta)$, then, using that E_η is spherical and η is a Dupin normal, we can get that

$$\tilde{\nabla}_X \sigma = -\|\sigma\|^2 f_* X = 0.$$

Thus, σ, $\tilde{\sigma}$ and r are constant in the leafs of E_η. But p is also constant in the leafs of E_η, therefore $f_* e_i = v_* e_i + \langle v, v_* e_i \rangle \sigma$ $\langle v_* e_i, v_* e_j \rangle = \langle f_* e_i, f_* e_j \rangle$ and $\{v_* e_1, \cdots, v_* e_q\}$ is an orthonormal basis of V. ✓

<u>Affirmation 6</u>: $\tilde{\nabla}_Z \tilde{\sigma} = -\langle Z, \varphi \rangle \tilde{\sigma}$, *for all* $Z \in E_\eta^\perp$.

By (3.21), $\langle Z, \varphi \rangle \sigma = \tilde{\nabla}_Z \sigma = \tilde{\nabla}_Z \frac{\sigma_0}{r} = -\frac{Z(r)}{r^2}\sigma_0 = -\frac{Z(r)}{r}\sigma$. Thus, $\varphi = -\frac{\nabla r}{r}$ and $\nabla r = -r\varphi$.

Therefore, $\tilde{\nabla}_Z \tilde{\sigma} = \tilde{\nabla}_Z r\tilde{\sigma}_0 = Z(r)\tilde{\sigma}_0 = \langle Z, \nabla r \rangle \tilde{\sigma}_0 = \langle Z, -r\varphi \rangle \tilde{\sigma}_0 = -\langle Z, \varphi \rangle \tilde{\sigma}$. ✓

We know that $V \subset L$ is a fixed subspace, thus $V \oplus \operatorname{span}\{\tilde{\sigma}_0\} = V \oplus \operatorname{span}\{\tilde{\sigma}\}$ is also a constant subspace. If $\Pi\colon (\operatorname{span}\{\tilde{\sigma}\} \oplus V) \oplus (\operatorname{span}\{\sigma\} \oplus U) \to \operatorname{span}\{\tilde{\sigma}\} \oplus V$ is the projection, then $\mathrm{d}(\Pi \circ p)(x)X = 0$, for any $X \in E_\eta$, because p is constant in the leafs of E_η.

If $Z \in E_\eta^\perp$, then

$$\mathrm{d}(\Pi \circ p)(x)Z = \Pi\left(\tilde{\nabla}_Z p(x)\right) = \Pi\left[\tilde{\nabla}_Z (f + \xi)(x)\right].$$

But, using Affirmation 2 and after some computations, we get that

$$\tilde{\nabla}_Z(f+\xi) = f_*Z - \sum_{i=1}^{q} [(\langle f_*\nabla^v_Z e_i, \tilde{\sigma}\rangle - \langle Z,\varphi\rangle \langle f_*e_i,\tilde{\sigma}\rangle) f_*e_i + \langle f_*e_i,\tilde{\sigma}\rangle f_*\nabla^v_Z e_i] + \\ + \sum_{i=1}^{q} \langle f_*e_i,\tilde{\sigma}\rangle \langle f_*\nabla^v_Z e_i,\tilde{\sigma}\rangle \sigma - \langle Z,\varphi\rangle \tilde{\sigma}$$

By the other side, if $X \in \Gamma(E_\eta)$, then $f_*X = v_*X + \langle v, v_*X\rangle \sigma$ e $\langle f_*X, \tilde{\sigma}\rangle = \langle v, v_*X\rangle$. Thus

$$\tilde{\nabla}_Z(f+\xi) = f_*Z - \sum_{i=1}^{q} [(\langle v, v_*\nabla^v_Z e_i\rangle - \langle Z,\varphi\rangle \langle v, v_*e_i\rangle) f_*e_i + \langle v, v_*e_i\rangle f_*\nabla^v_Z e_i] + \\ + \sum_{i=1}^{q} \langle v, v_*e_i\rangle \langle v, v_*\nabla^v_Z e_i\rangle \sigma - \langle Z,\varphi\rangle \tilde{\sigma}. \quad (3.23)$$

Besides that, we can easily compute that

$$\Pi(\tilde{\sigma}) = \tilde{\sigma}; \quad \Pi(\sigma) = 0; \quad \Pi(f_*X) = v_*X; \quad \Pi(f_*Z) = -\langle Z,\varphi\rangle v + \langle Z,\varphi\rangle \tilde{\sigma};$$

Therefore, after some calculations, we conclude that $\Pi\left[\tilde{\nabla}_Z(f+\xi)(x)\right] = 0$, that is, $q = \Pi(p(x))$ is constant.

Let $N := M/\sim$, where $\sim$ is the equivalence relation of Case 1, and $\pi\colon \mathbb{R}^n_t \to \mathrm{span}\{\sigma\} \oplus U$ is given by $\pi := \mathrm{Id} - \Pi$. Thus,

$$f(x) = q + \pi(p(x)) - \tilde{\sigma}(x) + v(x) + \frac{\|v(x)\|^2}{2}\sigma(x) = q + h(\bar{x}) + \bar{r}(\bar{x})\left(-\tilde{\sigma}_0 + w(x) + \frac{\|w(x)\|^2}{2}\sigma_0\right),$$

where $h\colon N \to \mathrm{span}\{\tilde{\sigma}_0\} \oplus U$ and $r\colon N \to \mathbb{R}$ are given by $h(\bar{x}) = \pi(q(x))$ and $\bar{r}(\bar{x}) = r(x)$.

Therefore, $f(M)$ is an open subset of the rotational submanifold with axis $\mathrm{span}\{\sigma_0\} \oplus U$ on $\bar{f}\colon N \to \mathrm{span}\{\tilde{\sigma}_0, \sigma_0\} \oplus U$, where $\bar{f}(\bar{x}) := h(\bar{x}) - \bar{r}(\bar{x})\tilde{\sigma}_0$. The rotational parametrization $g\colon N \times V \to \mathbb{R}^n_t$ is given by

$$g(\bar{x}, w) := q + h(\bar{x}) + \bar{r}(\bar{x})\left(-\tilde{\sigma}_0 + w + \frac{\|w\|^2}{2}\sigma_0\right). \bullet$$

□

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