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Three examples of covariant integral quantization

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In this paper we describe and apply integral quantization, a procedure based on operator-valued measures and the resolution of the identity. We explore its covariance properties in the context of group representation theory. Three applications based on group representations are carried out. The first one concerns the covariant integral quantization based on the spin one-half irreducible representation of $\mathrm{SU}(2)$. In this case we show that the quantization of both the quadratic Hopf variables and the Euler angles reproduces the $\mathrm{SU}(2)$ Lie algebra. The second example revisits integral quantization based on the Weyl-Heisenberg group. By completing previous results, we show the universality of the canonical commutation relation in such quantizations. In the last example we revisit and enrich the integral quantization based on the affine group of the real line and we give a short account of a relevant application.

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1. Introduction

In digital signal quantization processes, one maps a large set of input values to a smaller set (such as rounding values to some unit of precision). In physics or mathematics, the term has a seemingly different meaning. For instance, quantization is commonly viewed as a procedure that associates to a certain algebra $\mathscr{A}_{\mathrm{cl}}$ of classical observables an algebra $\mathscr{A}_{\mathrm{qt}}$ of quantum observables. The algebra $\mathscr{A}_{\mathrm{cl}}$ is usually realized as a commutative Poisson algebra of functions on a symplectic space X (or a phase space). On the other hand, the algebra $\mathscr{A}_{\mathrm{qt}}$, realized as an algebra of operators acting in a Hilbert space $\mathscr{H}$ without regard to their domains, is in general noncommutative and the quantization process should provide the correspondence $\mathscr{A}_{\mathrm{cl}} \to \mathscr{A}_{\mathrm{qt}} : f \to A_f$ of observables $f(x)$ in the space X and operators A_f acting in $\mathscr{H}$. In the most common example the space X represents the phase space of canonically conjugated variables of position and momentum (q,p) and the correspondence $\mathscr{A}_{\mathrm{cl}} \to \mathscr{A}_{\mathrm{qt}}$ allows one to build self-adjoint operators $(\hat{Q} \equiv A_q, \hat{P} \equiv A_p)$ obeying the so called *canonical commutation relation* $[\hat{Q},\hat{P}] = i\hbar I$, where I represents the identity operator in $\mathscr{A}_{\mathrm{qt}}$.

In the approach followed in this paper and developed at length in the recent works [1, 2] and in chapter 11 of [3], on a minimal level, we understand quantization of a classical set X and functions on it as a procedure fulfilling three requirements: linearity, existence of identity and self-adjointness. More precisely, quantization is:

1. A **linear** map
$$\mathfrak{Q} : \mathscr{C}(X) \to \mathscr{A}(\mathscr{H}),$$
where $\mathscr{C}(X)$ is a vector space of complex-valued functions $f(x)$ on a set X and $\mathscr{A}(\mathscr{H})$ is a vector space of linear operators
$$\mathfrak{Q}(f) \equiv A_f$$
in some complex Hilbert space $\mathscr{H}$ such that;

2. $f = 1$ is mapped to the identity operator I on $\mathscr{H}$;

3. A real function f is mapped to an (essentially) self-adjoint operator A_f in $\mathscr{H}$.

In order to incorporate the physical nature of the system at hand, one needs to add structure to X such as measure, topology, manifold structure, closure under algebraic operations, etc. Besides, one also has the freedom to physically interpret the spectra of classical $f \in \mathscr{C}(X)$ or quantum $A_f \in \mathscr{A}(\mathscr{H})$, so that they can be chosen as observables. And finally, one adds the requirement of an unambiguous classical limit of the quantum physical quantities, the limit operation being associated with a change of scale.

The fact that the position and momentum operators do not commute ($[\hat{Q},\hat{P}] = iI$) leads in general to an ordering problem in the quantization procedure. In the canonical quantization procedure one explicitly chooses some ordering rule, as well as in Wigner-Weyl quantization [4] and in the integral version of the canonical quantization based on the Weyl-Heisenberg algebra or group. The latter can be viewed as a particular case of a more general technique for analysis of set of functions $f(x)$ on X whose field of application goes far beyond Mechanics, or Signal Analysis, where one can recover some of its main aspects. This procedure of quantization, based on operator-valued measures, is called *integral quantization* [1, 2, 3, 5].

The aim of the present paper is to continue the exploration of the method through three examples based on group representations of SU(2), the Weyl-Heisenberg group and the Affine group, respectively. We highlight compatibilities, similarities, differences and issues between our approach and more traditional or canonical ones.

In section 2, we give a rapid survey of integral quantization and explain covariant integral quantization in the case where X is a group or group coset. In section 3 we examine the particular case when X is the group $\mathrm{SU}(2)$ with its spin one-half representation. In section 4 we examine general features of the Weyl-Heisenberg covariant integral quantization in some detail. In section 5 we consider an application of the integral quantization based on the affine group and give a short account of a recent application in quantum cosmology. Finally, we give in section 6 some insights about the continuation of our explorations in future works.

2. General integral quantization

Integral quantization [1], which should be distinguished from Path Integral Quantization (essentially based on canonical quantization), offers many ways to give a classical object a quantum version. Given a measure space (X,ν), where the measure ν is understood in a “wide” sense, let

$$X \ni x \mapsto \mathsf{M}(x) \in \mathcal{L}(\mathcal{H}),$$

be an X-labeled family of bounded operators on a Hilbert space $\mathcal{H}$ resolving the identity I

$$\int_X \mathsf{M}(x)\, \mathrm{d}\nu(x) = I, \quad \text{in a weak sense.}$$

Using the traditional notation of quantum mechanics, we will define

$$\mathsf{M}(x) \equiv \rho(x)$$

for the case when $\mathsf{M}(x)$ is positive and unit trace. In this case, if X is equipped with a suitable topology, then we can define a normalized positive operator-valued measure (POVM) $\mathfrak{m}$ on the σ-algebra $\mathscr{B}(X)$ of Borel through the following map Δ [5]

$$\mathscr{B}(X) \ni \Delta \mapsto \mathfrak{m}(\Delta) = \int_\Delta \rho(x)\, \mathrm{d}\nu(x).$$

Then, the integral quantization of complex-valued functions $f(x) \in \mathscr{C}(X)$ is formally defined as the linear map, if it can be made mathematically rigorous, as

$$f \mapsto A_f = \int_X f(x)\, \mathsf{M}(x)\, \mathrm{d}\nu(x).$$

If this map can be defined, the operator $A_f \in \mathscr{A}(\mathcal{H})$ has to be understood in terms of the sesquilinear form

$$B_f(\psi_1,\psi_2) = \int_X f(x)\, \langle \psi_1 | \mathsf{M}(x) | \psi_2 \rangle\, \mathrm{d}\nu(x),$$

defined on a dense subspace of $\mathcal{H}$. If f is real and at least semi-bounded and $M(x)$ is positive, the Friedrich's extension [6] of B_f univocally defines a self-adjoint operator. If f is not semi-bounded, there is no natural choice of a self-adjoint operator associated with B_f (see, e.g., [7, 8]). In this last case, in order to construct B_f as an observable, we need more structure on $\mathcal{H}$.

The classical limit is achieved by means of the construction of the so-called lower (Lieb) or covariant (Berezin) symbol

$$A_f \mapsto \check{f}(x) := \int_X f(x')\,\mathrm{tr}(\widetilde{\rho}(x)\rho(x'))\,\mathrm{d}\nu(x')\,,$$

where $X \ni x \mapsto \widetilde{\rho}(x) \in \mathcal{L}^+(\mathcal{H})$ is another (or the same) family of positive unit trace operators. This construction gives the probabilistic interpretation of the theory for the case when $\mathsf{M}(x) = \rho(x)$. It is a generalization of the so-called Bargmann-Segal transform (see for instance [9, 10]). Besides, from functional properties of the lower symbol $\check{f}$ one may investigate certain quantum features, such as, e.g., spectral properties of A_f.

2.1 Covariant integral quantizations

In the integral quantization's explicit construction of a quantum theory, Lie group representations give a wide range of possibilities. Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space $\mathcal{H}$. Consider a bounded operator M on $\mathcal{H}$ and suppose that the operator

$$R := \int_G \mathsf{M}(g)\,\mathrm{d}\mu(g)\,,\ \mathsf{M}(g) := U(g)\,\mathsf{M}U^\dagger(g)\,,$$

is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$ we have

$$U(g_0)\,RU^\dagger(g_0) = \int_G \mathsf{M}(g_0 g)\,\mathrm{d}\mu(g) = R\,,$$

and so R commutes with all operators $U(g)$, $g \in G$. Thus, from Schur's Lemma, $R = c_M I$ with

$$c_M = \int_G tr(\rho_0 \mathsf{M}(g))\,\mathrm{d}\mu(g)\,,$$

where the unit trace positive operator ρ_0 is chosen in order to make the integral converge. This family of operators provides the following resolution of the identity

$$\int_G \mathsf{M}(g)\,\mathrm{d}\nu(g) = I,\ \mathrm{d}\nu(g) := \frac{\mathrm{d}\mu(g)}{c_M}\,. \tag{2.1}$$

Let us look in more detail the above procedure in the case of square integrable UIRs (e.g. affine group). For a square-integrable UIR U for which $|\eta\rangle$ is an admissible unit vector, i.e.,

$$c(\eta) := \int_G \mathrm{d}\mu(g)\,|\langle\eta|U(g)|\eta\rangle|^2 < \infty,$$

the resolution of the identity is obeyed by

$$|\eta_g\rangle\langle\eta_g| = \rho(g)\ ,\ \rho := |\eta\rangle\langle\eta|\,,\ |\eta_g\rangle = U(g)|\eta\rangle\,.$$

This allows an integral quantization of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g)\, f(g) \mathrm{d}\nu(g)\,,$$

which is covariant in the sense that

$$U(g) A_f U^\dagger(g) = A_{U_r(g) f}\,.$$

In the case when $f \in L^2(G, \mathrm{d}\mu(g))$, the quantity $(U_r(g) f)(g') := f(g^{-1} g')$ is the regular representation. From the lower symbol we obtain a generalization of the Berezin or heat kernel transform on G

$$\check{f}(g) := \int_G tr(\rho(g)\, \rho(g'))\, f(g') \mathrm{d}\nu(g')\,.$$

In the absence of square-integrability over G, there exists a definition of square-integrable covariant coherent states with respect to a left coset manifold $X = G/H$, with H a closed subgroup of G, equipped with a quasi-invariant measure ν [3].

3. SU(2) as unit quaternions acting in $\mathbb{R}^3$

Here, as a first example, we consider the case $G =$SU(2) and we pick for U the spin one-half UIR.

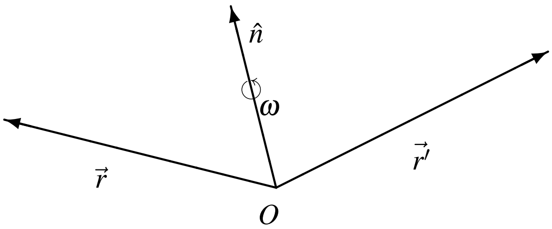
3.1 Rotations and quaternions

A convenient representation is possible thanks to quaternion calculus. We recall that the quaternion field as a multiplicative group is $\mathbb{H} \simeq \mathbb{R}_+ \times$SU(2). The correspondence between the canonical basis of $\mathbb{H} \simeq \mathbb{R}^4$, $(1 \equiv e_0, e_1, e_2, e_3)$, and the Pauli matrices is $e_a \leftrightarrow (-1)^{a+1} i\sigma_a$, with $a = 1, 2, 3$. Hence, the 2×2 matrix representation of these basis elements is the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow e_0\,, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \leftrightarrow e_1 \equiv \hat{\imath}\,, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \leftrightarrow e_2 \equiv \hat{\jmath}\,, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \leftrightarrow e_3 \equiv \hat{k}\,.$$

Any quaternion decomposes as $q = (q_0, \vec{q})$ (resp. $q^a e_a, a = 0, 1, 2, 3$) in scalar-vector notation (resp. in Euclidean metric notation). We also recall that the multiplication law explicitly reads in scalar-vector notation: $qq' = (q_0 q_0' - \vec{q} \cdot \vec{q}', q_0' \vec{q} + q_0 \vec{q}' + \vec{q} \times \vec{q}')$. The (quaternionic) conjugate of $q = (q_0, \vec{q})$ is $\bar{q} = (q_0, -\vec{q})$, the squared norm is $\|q\|^2 = q\bar{q}$, and the inverse of a nonzero quaternion is $q^{-1} = \bar{q}/\|q\|^2$. Unit quaternions, i.e., quaternions with norm 1, the multiplicative subgroup isomorphic to SU(2), constitute the three-sphere S^3.

On the other hand, any proper rotation in space is determined by a unit vector $\hat{n}$ defining the rotation axis and a rotation angle $0 \leq \omega < 2\pi$ about the axis.



The action of such a rotation, $\mathscr{R}(\omega,\hat{n})$, on a vector $\vec{r}$ is given by:

$$\vec{r}' \overset{def}{=} \mathscr{R}(\omega,\hat{n})\cdot\vec{r} = \vec{r}\cdot\hat{n}\,\hat{n} + \cos\omega\,\hat{n}\times(\vec{r}\times\hat{n}) + \sin\omega\,(\hat{n}\times\vec{r})\,. \tag{3.1}$$

The latter is expressed in scalar-vector quaternionic form as

$$(0,\vec{r}') = \xi\,(0,\vec{r})\bar{\xi}\,,$$

where

$$\xi := \left(\cos\frac{\omega}{2}, \sin\frac{\omega}{2}\,\hat{n}\right) \in \mathrm{SU}(2)\,,$$

or, in matrix form,

$$\begin{aligned}\xi &= \begin{pmatrix} \xi_0 + i\xi_3 & -\xi_2 + i\xi_1 \\ \xi_2 + i\xi_1 & \xi_0 - i\xi_3 \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{\omega}{2} + in^3\sin\frac{\omega}{2} & \left(-n^2 + in^1\right)\sin\frac{\omega}{2} \\ \left(n^2 + in^1\right)\sin\frac{\omega}{2} & \cos\frac{\omega}{2} - in^3\sin\frac{\omega}{2} \end{pmatrix}\,,\end{aligned} \tag{3.2}$$

in which case quaternionic conjugation corresponds to the transposed conjugate of the corresponding matrix.

In particular, for a given unit vector

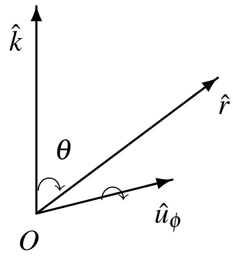
$$\begin{aligned}&\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \overset{def}{=} (\theta,\phi)\,,\\ &0 \leq \theta \leq \pi\,, \quad 0 \leq \phi < 2\pi\,,\end{aligned}$$

one considers the specific rotation $\mathscr{R}_{\hat{n}}$ that maps the unit vector pointing to the north pole, $\hat{k} = (0,0,1)$, to $\hat{n}$,

$$(0,\hat{n}) = \left(0, \mathscr{R}(\theta_{\hat{n}}, \hat{u}_{\phi_{\hat{n}}})\hat{k}\right) \equiv \xi_{\hat{n}}\left(0,\hat{k}\right)\bar{\xi}_{\hat{n}}\,, \quad \hat{u}_{\phi_{\hat{n}}} \overset{def}{=} (-\sin\phi_{\hat{n}}, \cos\phi_{\hat{n}}, 0)\,, \tag{3.3}$$

with

$$\xi_{\hat{n}} = \left(\cos\frac{\theta_{\hat{n}}}{2}, \sin\frac{\theta_{\hat{n}}}{2}\,\hat{u}_{\phi_{\hat{n}}}\right)\,. \tag{3.4}$$



In the above four-euclidean realization we naturally choose hypersherical coordinates

$$\begin{aligned} q &= \|q\|\,\xi = \|q\|\,(\cos\alpha, \sin\alpha\,\hat{n}) \\ &= \|q\|\,(\cos\alpha, \sin\alpha\sin\theta\cos\phi, \sin\alpha\sin\theta\sin\phi, \sin\alpha\cos\theta) \\ &\equiv (\|q\|\,, \alpha\,, \theta,\ \phi)\,, \quad 0\le\alpha,\theta\le\pi\,,\ 0\le\phi<2\pi\,. \end{aligned}$$

Therefore (3.2) becomes

$$\xi = \begin{pmatrix} \cos\alpha + i\sin\alpha\cos\theta & \sin\alpha\sin\theta\,(-\sin\phi + i\cos\phi) \\ \sin\alpha\sin\theta\,(\sin\phi + i\cos\phi) & \cos\alpha - i\sin\alpha\cos\theta \end{pmatrix}.$$

Note the relation between α and the rotation angle ω introduced in (3.1): $\alpha = \omega/2$. When expressed in terms of these hyperspherical coordinates (α,θ,ϕ) of S^3, the Haar measure on SU(2) is given by

$$\dot{\xi} = \sin^2\alpha\,\sin\theta\,\mathrm{d}\alpha\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{3.5}$$

and gives the volume $\int_{\mathrm{SU(2)}}\dot{\xi} = 2\pi^2$. Other parametrizations of quaternions are possible. For instance, we can also use the angular coordinates, which correspond to the bicomplex decomposition $\mathbb{H} = \mathbb{C} + \mathbb{C}e_1$, where $\mathbb{C} = \mathbb{R} + \mathbb{R}e_3$ (use the algebra $e_1e_2 = e_3 = -e_2e_1$ + even permutations, and view e_3 as the imaginary unit $i = \sqrt{-1}$). Explicitly, $q = z_1 + z_2e_1 = z_1 + e_1\bar{z}_2$, $z_1\,, z_2 \in \mathbb{C}$, and put $z_1 = \|q\|\cos\omega e^{i\psi_1}$, $z_2 = \|q\|\sin\omega e^{i\psi_2}$, with $0\le\omega\le\pi/2$, $0\le\psi_1\,,\ \psi_2\le 2\pi$. In term of a 2×2 matrix, this notation corresponds to the usual parametrization of SU(2):

$$q = \|q\|\xi \equiv \|q\| \begin{pmatrix} \cos\omega e^{i\psi_1} & i\,\sin\omega e^{i\psi_2} \\ i\,\sin\omega e^{-i\psi_2} & \cos\omega e^{-i\psi_1} \end{pmatrix}, \tag{3.6}$$

and the Haar measure on SU(2) is given by

$$\dot{\xi} = \frac{1}{2}\,\sin 2\omega\,\mathrm{d}\omega\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2\,, \tag{3.7}$$

which gives the same volume $2\pi^2$.

3.2 POVM on SU(2) from 2×2 density matrices

The unit ball $\mathbb{B}$ in $\mathbb{R}^3$ parametrizes the set of 2×2 complex density matrices ρ. Indeed, given a 3-vector $\vec{a}\in\mathbb{R}^3$ such that $\|\vec{a}\|\le 1$, a general density matrix ρ can be written as

$$\rho = \frac{1}{2}(1 + \vec{a}\cdot\vec{\sigma})\,,\ \vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)\,.$$

If $\|\vec{a}\| = 1$, i.e. $\vec{a} \in S^2$ ("Bloch sphere" in this context), with spherical coordinates (θ, ϕ), then ρ is the pure state

$$\rho = |\theta, \phi\rangle \langle \theta, \phi| \,.$$

Note that the above column vector has to be viewed as the spin $j = 1/2$ coherent state in the Hermitian space $\mathbb{C}^2$ with orthonormal basis $|j = 1/2, m = \pm 1/2\rangle$:

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \sin\frac{\theta}{2} e^{i\phi}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle .$$

Let us now "transport" the density matrix ρ by using the two-dimensional complex representation of rotations in space, namely the matrix SU(2) representation. For that it is convenient to use the (complex) quaternionic representation of ρ as follows:

$$\rho = \frac{1}{2}(1 - i(0, \vec{r})) \equiv \frac{1}{2}(1 - i\underset{\smile}{\mathbf{r}}) \equiv \rho_{\underset{\smile}{\mathbf{r}}} \,, \; r_i = (-1)^{i+1} a_i \,,$$

where $\underset{\smile}{\mathbf{r}}$ stands for the pure vector quaternion $(0, \vec{r})$. For $\xi \in SU(2)$, one defines the family of density matrices labelled by ξ:

$$\rho(\xi) := \xi \rho \bar{\xi} = \frac{1}{2}(1 - i\xi \underset{\smile}{\mathbf{r}} \bar{\xi}) \,.$$

It is then straightforward to prove the resolution of the identity (2.1)

$$\int_{\mathrm{SU}(2)} \rho(\xi) \, \frac{\dot{\xi}}{\pi^2} = I \,. \tag{3.8}$$

3.3 Integral quantization of functions (or distributions) on SU(2)

As it is now well understood, the resolution of the identity (3.8) opens the path to quantizations of objects, functions or distributions, living on SU(2) or S^3 along the linear map

$$f(\xi) \mapsto A_f = \int_{\mathrm{SU}(2)} f(\xi) \, \rho(\xi) \, \frac{\dot{\xi}}{\pi^2} \in \mathrm{M}(2, \mathbb{C}) \,. \tag{3.9}$$

A first point to be noticed is that rotational invariance of the measure combined with (3.3) and (3.4) allows to write (3.9), after the change $\xi \mapsto \xi \bar{\xi}_{\hat{r}}$, as

$$\begin{aligned} A_f &= \int_{\mathrm{SU}(2)} f(\xi) \, \xi \xi_{\hat{r}} \rho_{r\hat{k}} \bar{\xi}_{\hat{r}} \bar{\xi} \, \frac{\dot{\xi}}{\pi^2} \\ &= \int_{\mathrm{SU}(2)} f(\xi \bar{\xi}_{\hat{r}}) \, \rho_{r\hat{k}}(\xi) \, \frac{\dot{\xi}}{\pi^2} \,, \end{aligned}$$

with $r = \|\vec{r}\|$ and

$$\begin{aligned} \rho_{r\hat{k}}(\xi) &= \frac{1}{2}\left(I - ir\xi \left(0, \hat{k}\right) \bar{\xi}\right) = \frac{1}{2} I + \\ &+ \frac{r}{2} \begin{pmatrix} 1 - 2\sin^2\alpha \sin^2\theta & 2\sin\theta \sin\alpha \, e^{i\phi}(\sin\alpha\cos\theta - i\cos\alpha) \\ 2\sin\theta\sin\alpha \, e^{-i\phi}(\sin\alpha\cos\theta + i\cos\alpha) & -1 + 2\sin^2\alpha\sin^2\theta \end{pmatrix} . \end{aligned} \tag{3.10}$$

A second point is that the quantization of cartesian coordinates ξ_a, $a = 0,1,2,3$ gives null operators. The proof derives easily from the null average of cartesian components on S^d, in any dimension d,

$$\frac{1}{|\mathrm{S}^d|}\int_{\mathrm{S}^d} \xi_a\, \dot{\xi} = 0\,. \tag{3.11}$$

This result is applied to our case through quaternion calculus:

$$\begin{aligned}\int_{\mathrm{SU}(2)} \bar{\xi}\,\xi\,\rho\,\bar{\xi}\,\frac{\dot{\xi}}{\pi^2} &= \int_{\mathrm{SU}(2)} \rho\,\bar{\xi}\,\frac{\dot{\xi}}{\pi^2} = \rho\int_{\mathrm{SU}(2)} \bar{\xi}\,\frac{\dot{\xi}}{\pi^2} \\ &= \sum_{a=0}^{3} \rho\,\bar{e}_a \int_{\mathrm{SU}(2)} \xi_a\,\frac{\dot{\xi}}{\pi^2} = 0\,.\end{aligned}$$

The upper symbols corresponding to the Pauli matrices can be obtained via the Hopf map $S^3 \to S^2$, where the Hopf fibration is understood in terms of the transitive action of rotations on S^2. Note that these upper symbols are by far not unique, since the kernel of the quantization map is not reduced to 0, as we already noticed with (3.11). For definiteness, let us fix the north pole $\left(0,\hat{k}\right) \in S^2$, then $\chi = \xi\left(0,\hat{k}\right)\bar{\xi}$ is the image of the rotation by the unit quartenion ξ. The components of χ correspond to the operators

$$A_{\chi_1} = \frac{a_3}{3}\sigma_1\,,\; A_{\chi_2} = -\frac{a_3}{3}\sigma_2\,,\; A_{\chi_3} = \frac{a_3}{3}\sigma_3\,. \tag{3.12}$$

The quantization of the angles give

$$A_\alpha = A_\theta = \frac{\pi}{2}\sigma_0 \equiv \frac{\pi}{2}\left(1,\mathbf{0}\right)\,,\; A_\phi = \pi\sigma_0 + \frac{1}{4}a_2\sigma_1 + \frac{1}{4}a_1\sigma_2 \equiv \left(\pi, \frac{i}{4}a_2, -\frac{i}{4}a_1, 0\right)\,.$$

Hence, the quantization of polar angles α and θ just gives the identity up to the factor $\pi/2$ representing their classical mean value. The quantization of the azimuthal angle ϕ yields an hermitian matrix A_ϕ whose eigenvalue spectrum is equally distanced from the classical mean value π:

$$\lambda_1 = \pi - \frac{\sqrt{a_1^2+a_2^2}}{4}\,,\; \lambda_2 = \pi + \frac{\sqrt{a_1^2+a_2^2}}{4}\,.$$

In angular coordinates (ω,ψ_1,ψ_2), one arrives at the same upper symbols for the Pauli matrices as (3.12), another manifestation of their non uniqueness. The quantization of the angles is

$$A_\omega = \frac{\pi}{16}a_3\sigma_0 - \frac{\pi}{4}\sigma_3\,,\; A_{\psi_1} = \pi\sigma_0 - \frac{a_2}{4}\sigma_1 + \frac{a_1}{4}\sigma_2\,, A_{\psi_2} = \pi\sigma_0 + \frac{a_2}{4}\sigma_1 + \frac{a_1}{4}\sigma_2\,.$$

Redefining the angles as $\beta = 2\omega - \frac{\pi}{8}a_3$, $\alpha = \psi_1 + \psi_2 - 2\pi$ and $\gamma = \psi_2 - \psi_1$, we get

$$A_{\omega - a_3\pi/16} = -\frac{\pi}{4}\sigma_3\,,\; A_{\psi_1+\psi_2} = \frac{a_1}{2}\sigma_2\,,\; A_{\psi_2-\psi_1} = \frac{a_1}{2}\sigma_1\,. \tag{3.13}$$

Comparing the spin one-half representation matrix in Euler angles

$$D^{1/2}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}e^{-i\frac{\gamma}{2}} & -e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}e^{i\frac{\gamma}{2}} \\ e^{i\frac{\alpha}{2}}\cos\frac{\beta}{2}e^{-i\frac{\gamma}{2}} & e^{i\frac{\alpha}{2}}\cos\frac{\beta}{2}e^{i\frac{\gamma}{2}} \end{pmatrix}$$

with the bicomplex representation (3.6) with angles (ω, ψ_1, ψ_2), one sees that the new angles are essentially Euler angles:

$$2\omega = \beta\,, \;\psi_2 - \psi_1 = \gamma + \frac{\pi}{2}\,, \;\psi_1 + \psi_2 - 2\pi = -\alpha + \frac{\pi}{2}\,.$$

For $a_1, a_2 \neq 0$ and the rescaling $A_\beta \mapsto \frac{1}{\pi} A_\beta$, $A_\alpha \mapsto a_1 A_\alpha$ and $A_\gamma \mapsto a_2 A_\gamma$, the algebra of Euler angle operators becomes

$$\left[A_\beta, A_\alpha\right] = iA_\gamma\,, \;\left[A_\gamma, A_\beta\right] = iA_\alpha\,, \;\left[A_\alpha, A_\gamma\right] = iA_\beta\,,$$

i.e., the Lie algebra of SU(2), as we also had with the quantization of the Hopf map. The Euler parametrization of SU(2) seems to be privileged from an algebraic point of view. On the other hand, this phenomenon should not be viewed as exceptional: it is a well-known feature of the two-dimensional representation of SU(2) and the Fourier expansion of the angles, which in general allows one to write group elements as simple combinations of sines and cosines, and in this case, follows from simplifications when integrating against (3.10).

4. Weyl-Heisenberg covariant integral quantization(s)

Let $\mathscr{H}$ be a separable (complex) Hilbert space with orthonormal basis $\{|e_n\rangle\}$. Lowering and raising operators a and $a^\dagger$ are defined by their action on this basis

$$a\,|e_n\rangle = \sqrt{n}|e_{n-1}\rangle\,, \quad a|e_0\rangle = 0\,, \quad a^\dagger\,|e_n\rangle = \sqrt{n+1}|e_{n+1}\rangle\,, \tag{4.1}$$

and the triplet $\{a, a^\dagger, I\}$ generate the Weyl-Heisenberg algebra characterized by the canonical commutation relation

$$\left[a, a^\dagger\right] = I\,. \tag{4.2}$$

It follows from (4.1) that the number operator $N := a^\dagger a$ is diagonal with spectrum $\mathbb{N}$, $N|e_n\rangle = n|e_n\rangle$. At the root of quantum mechanics, the Stone-von Neumann theorem asserts that there exists an essentially unique UIR of the W-H algebra or group. It is square integrable with respect to the center $C \sim \mathbb{R}$ in the sense given in [3], and the measure space which has to be considered here is the Euclidean or complex plane $X = G_{\mathrm{WH}}/C \sim \mathbb{C}$ with measure $\mathrm{d}^2 z/\pi$. To each $z \in \mathbb{C}$ corresponds the (unitary) displacement operator $D(z)$,

$$\mathbb{C} \ni z \mapsto D(z) = e^{za^\dagger - \bar{z}a}\,, \quad D(-z) = (D(z))^{-1} = D(z)^\dagger\,.$$

The canonical commutation relation (ccr) (4.2) or QM noncommutativity is encoded by the addition formula

$$D(z)D(z') = e^{\frac{1}{2}(z\bar{z}' - \bar{z}z')} D(z+z')\,.$$

The family of displaced operators $\mathsf{M}(z) := D(z)\mathsf{M}D(z)^\dagger$ resolves the identity

$$\int_{\mathbb{C}} \mathsf{M}(z)\,\frac{\mathrm{d}^2 z}{\pi} = I\,,$$

where

$$M := \int_{\mathbb{C}} D(z)\,\varpi(z)\frac{\mathrm{d}^2 z}{\pi}\,, \tag{4.3}$$

and $\varpi(z)$ is a function on the complex plane obeying $\varpi(0) = 1$ and chosen in such a way that the above operator-valued integral defines (in a weak sense) a bounded operator M on $\mathscr{H}$.

The resulting quantization map is given by

$$f \mapsto A_f = \int_{\mathbb{C}} f(z)\,\mathsf{M}(z)\,\frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \mathscr{F}(-z)\,D(z)\,\varpi(z)\frac{\mathrm{d}^2 z}{\pi},$$
$$\mathscr{F}(z) = \int_{\mathbb{C}} f(\xi)\,e^{z\bar{\xi}-\bar{z}\xi}\,\frac{\mathrm{d}^2 \xi}{\pi}. \tag{4.4}$$

Covariance with respect to translations reads $A_{f(z-z_0)} = D(z_0)A_{f(z)}D(z_0)^{\dagger}$.

We now show that the ccr is a permanent outcome of the above quantization, whatever the chosen complex function $\varpi(z)$, provided integrability and derivability at the origin is ensured. Since $q = \frac{\sqrt{2}}{2}(z+\bar{z})$ and $p = \frac{\sqrt{2}}{2i}(z-\bar{z})$, we first calculate A_z and $A_{\bar{z}}$. Taking into account that $\mathscr{F}(-z) = \partial_{\bar{z}}\pi\delta^2(-z)$, where $\pi\delta^2(z) = \int_{\mathbb{C}} e^{z\bar{\xi}-\bar{z}\xi}\frac{d^2\xi}{\pi}$, one has from (4.4) $A_z = -\left[\partial_{\bar{z}}D(z)\,\varpi(z) + D(z)\,\partial_{\bar{z}}\varpi(z)\right]_{z=0}$. Then, using $\partial_{\bar{z}}D(z) = -\left(a - \frac{z}{2}\right)D(z)$ we obtain finally

$$A_z = a\varpi(0) - \partial_{\bar{z}}\varpi|_{z=0}.$$

Similarly, we obtain for $A_{\bar{z}}$ the following expression

$$A_{\bar{z}} = a^{\dagger}\varpi(0) + \partial_z\varpi|_{z=0},$$

after using the relation $\partial_z D(z) = \left(a^{\dagger} - \frac{\bar{z}}{2}\right)D(z)$. As a result, we have

$$A_q = \frac{\sqrt{2}}{2}\left[\left(a+a^{\dagger}\right)\varpi(0) - \partial_{\bar{z}}\varpi|_{z=0} + \partial_z\varpi|_{z=0}\right],$$
$$A_p = \frac{\sqrt{2}}{2i}\left[\left(a-a^{\dagger}\right)\varpi(0) - \partial_{\bar{z}}\varpi|_{z=0} - \partial_z\varpi|_{z=0}\right],$$

and therefore the commutation relation becomes the ccr,

$$A_qA_p - A_pA_q = i\left[a, a^{\dagger}\right],$$

where we used $\varpi(0) = 1$.

Now, beyond this equivalence w.r.t. the ccr, it is clear that for different choices of the weight function ϖ, one will arrive at different quantizations. Let us consider a particular aspect of these differences in relation with the ordering problem. In general, one has $A_{q\star p} = A_qA_p$, where $\star$ is the induced Moyal product in the space of symbols. Therefore, for a general choice of ϖ, it follows that $A_{qp} \neq A_qA_p$. Next, we relate A_{qp} and A_qA_p using the freedom in choosing ϖ with appropriate differentiability properties at the origin. Let us calculate the operator corresponding to the classical function $qp = \frac{1}{2i}\left(z^2 - \bar{z}^2\right)$. We first obtain A_{z^2} and $A_{\bar{z}^2}$,

$$A_{z^2} = a^2\varpi(0) - 2a\,\partial_{\bar{z}}\varpi|_{z=0} + \partial_{\bar{z}}^2\varpi\big|_{z=0},$$
$$A_{\bar{z}^2} = \left(a^{\dagger}\right)^2\varpi(0) + 2a^{\dagger}\,\partial_z\varpi|_{z=0} + \partial_z^2\varpi\big|_{z=0}.$$

Then,

$$A_{qp} = \frac{1}{2i}\left[\left(a^2 - \left(a^{\dagger}\right)^2\right)\varpi(0) - 2a\,\partial_{\bar{z}}\varpi|_{z=0} - 2a^{\dagger}\,\partial_z\varpi|_{z=0} + \partial_{\bar{z}}^2\varpi\big|_{z=0} - \partial_z^2\varpi\big|_{z=0}\right].$$

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Using the previous results for A_q and A_p, we obtain

$$A_{qp} = \frac{1}{2\varpi(0)}\left(A_qA_p + A_pA_q\right) + \partial_{\bar{z}}^2\varpi\big|_{z=0} - \partial_z^2\varpi\big|_{z=0} - \frac{1}{\varpi(0)}\left(\partial_{\bar{z}}\varpi|_{z=0}\right)^2 + \frac{1}{\varpi(0)}\left(\partial_z\varpi|_{z=0}\right)^2 .$$

Therefore, since $A_qA_p = A_pA_q + i$ and $\varpi(0) = 1$, we have

$$\begin{aligned} A_{qp} &= A_qA_p - \frac{i}{2} + \partial_{\bar{z}}^2\varpi\big|_{z=0} - \partial_z^2\varpi\big|_{z=0} - \left(\partial_{\bar{z}}\varpi|_{z=0}\right)^2 + \left(\partial_z\varpi|_{z=0}\right)^2 \\ &\equiv A_qA_p + (\text{constant} \in \mathbb{C}) , \end{aligned}$$

and the constant can take any value we wish. For instance, with $\varpi(z) = (az+1)e^{s|z|^2}$, $a, s \in \mathbb{C}$, $\mathrm{Re}\, s < 1$, we have

$$A_{qp} = A_qA_p - \frac{i}{2} + a^2 ,$$

and putting $a = e^{i\pi/4}/\sqrt{2}$ leads to the so-called xp-quantization [11]:

$$A_{qp} = A_qA_p .$$

For $\varpi(z) = e^{s|z|^2+az+b\bar{z}}$, with $a, b, s \in \mathbb{C}$, $\mathrm{Re}\, s < 1$, it follows that

$$A_{qp} = A_qA_p - \frac{i}{2}$$

is independent of a or b. Thus, the Cahill-Glauber function ϖ ($a = 0 = b$) [12, 13],

$$\varpi_s(z) = e^{s|z|^2/2}, \quad \mathrm{Re}\, s < 1 ,$$

never gives the xp-quantization. We note that the Cahill-Glauber function ϖ for $s = 0$ corresponds to the Wigner-Weyl quantization, while the cases $s = -1, 1$ correspond, respectively, to the CS (anti-normal) and normal quantization (in an asymptotic sense for (4.4)).

For ϖ chosen *real* and *even*, one has $A_z = a$, $A_{\overline{f(z)}} = A^\dagger_{f(z)}$, or equivalently, from $z = (q + ip)/\sqrt{2}$,

$$A_q = \frac{a+a^\dagger}{\sqrt{2}} := Q , A_p = \frac{a-a^\dagger}{i\sqrt{2}} := P , \ [Q,P] = iI .$$

Moreover, iff $|\varpi(z)| = 1$,

$$\mathrm{tr}(A_f^\dagger A_f) = \int_{\mathbb{C}} |f(z)|^2 \frac{\mathrm{d}^2 z}{\pi} ,$$

which means that the map $L^2(\mathbb{C}, d^2z/\pi) \ni f \mapsto A_f \in \mathscr{H}_{\text{Hilbert}-\text{Schmidt}}$ is invertible through a trace formula. Moreover, any real even ϖ defining a bounded operator through (4.3) yields the correct energy spectrum for the harmonic oscillator. For a general function ϖ (with $\varpi(0) = 1$), we have, from the classical expressions $q^2 = \frac{1}{2}\left(z^2 + 2z\bar{z} + \bar{z}^2\right)$ and $p^2 = -\frac{1}{2}\left(z^2 - 2z\bar{z} + \bar{z}^2\right)$,

$$\begin{aligned} A_{q^2} &= (A_q)^2 + \frac{1}{2}\left[(\partial_z - \partial_{\bar{z}})^2\varpi\right]_{z=0} - \frac{1}{2}\left(\partial_z\varpi|_{z=0} - \partial_{\bar{z}}\varpi|_{z=0}\right)^2 , \\ A_{p^2} &= (A_p)^2 - \frac{1}{2}\left[(\partial_z + \partial_{\bar{z}})^2\varpi\right]_{z=0} + \frac{1}{2}\left(\partial_z\varpi|_{z=0} + \partial_{\bar{z}}\varpi|_{z=0}\right)^2 , \end{aligned}$$

where we used

$$A_{|z|^2} \equiv A_J = a^\dagger a + \frac{1}{2} - \partial_z \partial_{\bar{z}} \varpi|_{z=0} + a\, \partial_z \varpi|_{z=0} - a^\dagger\, \partial_{\bar{z}} \varpi|_{z=0} \,.$$

Here $|z|^2$ is the energy for the harmonic oscillator and $A_{|z|^2}$ is the quantum energy operator. Performing a general Bogoliubov transform as

$$b = ua + va^\dagger + \gamma, \quad b = \bar{u}a^\dagger + \bar{v}a + \bar{\gamma} \tag{4.5}$$

one can show that in the case where $A_{|z|^2}$ is hermitian, i.e., when $\partial_z \varpi|_{z=0}$ is imaginary, it is not possible to bring $A_{|z|^2}$ to the harmonic oscillator's canonical form [14]. We therefore only obtain the quantum harmonic oscillator's energy spectrum for even ϖ. In that case, the difference between the ground state energy

$$E_0 = 1/2 - \partial_z \partial_{\bar{z}} \varpi|_{z=0} \,,$$

and the minimum of the quantum potential energy

$$E_m = [\min(A_{q^2}) + \min(A_{p^2})]/2 = -\, \partial_z \partial_{\bar{z}} \varpi|_{z=0} \,,$$

is

$$E_0 - E_m = 1/2 \,.$$

This difference (experimentally verified, see for instance [15]) is independent of the particular quantization which has been chosen. It has been proven [16] that these constant shifts in energy are inaccessible to measurement.

5. Affine quantization and application to quantum cosmology

Since the complex plane is viewed as the phase space for the motion of a particle on the line, the half-plane can be viewed as the phase space for the motion of a particle on the half-line [17]. Let our measure space (X, ν) be the upper half-plane $X \equiv \Pi_+ := \{(q,p) \,|\, p \in \mathbb{R}, q > 0\}$ equipped with the left invariant measure $\mathrm{d}q\mathrm{d}p$. Together with the multiplication $(q,p)(q_0,p_0) = (qq_0, p_0/q + p)$, $q \in \mathbb{R}_+^*$, $p \in \mathbb{R}$, Π_+ is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line. $\mathrm{Aff}_+(\mathbb{R})$ has two non-equivalent UIR [18, 19]. Both are square integrable and this is the rationale behind *continuous wavelet analysis* (see references in [3]). The UIR $U_+ \equiv U$ is realized in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}_+^*, \mathrm{d}x)$:

$$U(q,p)\psi(x) = (e^{ipx}/\sqrt{q})\psi(x/q) \,. \tag{5.1}$$

As in the case of the Weyl-Heisenberg group, we choose a suitably localized weight function $\varpi(q,p)$ on the half-plane such that the integral

$$\int_{\Pi_+} U(q,p)\, \varpi(q,p)\, \mathrm{d}q\, \mathrm{d}p := \mathsf{M} \tag{5.2}$$

defines a bounded operator, at least in a weak sense. In addition, we provide a condition on the weight function $\varpi\,(q,p)$ so that M is a self-adjoint positive operator. From the condition $M = M^\dagger$, where

$$M^\dagger = \int_{\Pi_+} U\left(\frac{1}{q}, -qp\right) \overline{\varpi(q,p)}\, \mathrm{d}q\, \mathrm{d}p$$
$$= \int_{\Pi_+} U(q,p)\, \frac{1}{q}\overline{\varpi\left(\frac{1}{q}, -pq\right)}\, \mathrm{d}q\, \mathrm{d}p = M$$

we find that the weight function must satisfy

$$\varpi(q,p) = \frac{1}{q}\overline{\varpi\left(\frac{1}{q}, -qp\right)}. \tag{5.3}$$

If for instance, we consider the following ansatz for ϖ:

$$\varpi\,(q,p) = F\,(q)\, G\left(qp^2\right)\,,\ \varpi\,(q,p) \in \mathbb{R}\,. \tag{5.4}$$

Then, from condition (5.3), the function F should obey:

$$F\,(q) = \frac{1}{q}F\left(\frac{1}{q}\right).$$

Elementary solutions of this functional equation are of the form $q^{-\frac{1+\alpha\beta}{2}}\,|1 \pm q^{\alpha}|^{\beta}\, l(|\ln q|)$, where the function l is arbitrary. Next, we investigate boundedness and positiveness of M as a quadratic form,

$$B\,(\psi) := \langle\psi|M\,|\psi\rangle = \sqrt{2\pi}\int_0^{\infty}\int_0^{\infty}\frac{1}{\sqrt{q}}\widehat{\varpi}^{p}\,(q,x)\,\overline{\psi\,(x)}\,\psi\left(\frac{x}{q}\right)\mathrm{d}x\mathrm{d}q\,,\ \psi \in \mathscr{H}\,,$$

where $\widehat{\varpi}^{p}$ is the partial Fourier transform

$$\widehat{\varpi}^{p}\,(q,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\varpi\,(q,p)\,\mathrm{d}p\,.$$

For concreteness, we choose a Gaussian×polynomial form for the function $p \mapsto \varpi\,(q,p)$,

$$\varpi\,(q,p) = F\,(q)\,P_n(\sqrt{q}p)e^{-qp^2}\,, \quad n = \text{degree of } P_n\,.$$

Then its Fourier transform w.r.t. p has the same form,

$$\widehat{\varpi}^{p}(q,x) = \frac{F\,(q)}{\sqrt{2q}}Q_n\left(\frac{x}{\sqrt{q}}\right)e^{-\frac{x^2}{4q}}\,, \quad n = \text{degree of } Q_n\,.$$

Hence,

$$B\,(\psi) = \sqrt{\pi}\int_0^{\infty}\int_0^{\infty}\frac{1}{q}F\,(q)\,Q_n\left(\frac{x}{\sqrt{q}}\right)e^{-\frac{x^2}{4q}}\overline{\psi\,(x)}\,\psi\left(\frac{x}{q}\right)\mathrm{d}x\mathrm{d}q\,.$$

The above expression becomes after changing $x \mapsto x$, $y = x/q$,

$$B\,(\psi) = \sqrt{\pi}\int_{\Pi_{++}}\mathscr{K}\,(x,y)\,\overline{\psi\,(x)}\,\psi\,(y)\,\mathrm{d}x\mathrm{d}y\,,$$

where Π_{++} is the positive quarter of the plane and $\mathscr{K}$ is the symmetrical integral kernel

$$\mathscr{K}\,(x,y) = \frac{1}{y}F\left(\frac{x}{y}\right)Q_n(\sqrt{xy})e^{-\frac{1}{4}xy} = \frac{1}{x}F\left(\frac{y}{x}\right)Q_n(\sqrt{xy})e^{-\frac{1}{4}xy}\,.$$

For instance, with the choice $F(q) = q^{-\frac{1+\alpha\beta}{2}} |1 \pm q^\alpha|^\beta \, l(|\ln q|)$ and $Q_n(\sqrt{xy}) = \sqrt{xy}$, this kernel reads

$$\mathscr{K}(x,y) = (xy)^{-\frac{\alpha\beta}{2}} |x^\alpha + y^\alpha|^\beta \, l(|\ln x - \ln y|) e^{-\frac{1}{4}xy}.$$

An interesting question is to establish the values of α and β, and functions l, for which this kernel defines a bounded operator. For instance, for $\alpha = 2$, $\beta = -1/2$ and $l = 1$, this kernel is positive and square integrable (use polar coordinates)

$$\int_{\Pi_{++}} (\mathscr{K}(x,y))^2 \mathrm{d}x\mathrm{d}y = \frac{\pi}{2},$$

and so defines a Hilbert-Schmidt integral operator, which is of course bounded self-adjoint. We leave the continuation of this investigation for a future work.

Now let us turn to the specific case for which $M = |\psi\rangle\langle\psi|$ where ψ is a unit-norm state in $L^2(\mathbb{R}_+^\dagger, \mathrm{d}x) \cap L^2(\mathbb{R}_+^\dagger, \mathrm{d}x/x)$ (also called "fiducial vector" or "wavelet"). The action of UIR U produces all affine coherent states, i.e. wavelets, defined as $|q,p\rangle = U(q,p)|\psi\rangle$.

Due to the square-integrability of the UIR U, the corresponding quantization reads as

$$f \;\mapsto\; A_f = \int_{\Pi_+} f(q,p) |q,p\rangle\langle q,p| \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}},$$

which arises from the resolution of the identity

$$\int_{\Pi_+} |q,p\rangle\langle q,p| \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}} = I, \; c_\gamma := \int_0^\infty |\psi(x)|^2 \frac{\mathrm{d}x}{x^{2+\gamma}}.$$

This quantization is canonical (up to a multiplicative constant) for q and p

$$A_p = P = -i\partial/\partial x, \; A_{q^\beta} = (c_{\beta-1}/c_{-1}) Q^\beta, \; Qf(x) = xf(x).$$

The quantization of kinetic energy gives

$$A_{p^2} = P^2 + KQ^{-2}, \quad K = K(\psi) = \int_0^\infty (\psi'(u))^2 u \frac{\mathrm{d}u}{c_{-1}}.$$

Therefore, wavelet quantization prevents a quantum free particle moving on the positive line from reaching the origin. It is well known that the operator $P^2 = -d^2/dx^2$ in $L^2(R_+^*, \mathrm{d}x)$ is not essentially self-adjoint, whereas the above regularized operator, defined on the domain of smooth function of compact support, is essentially self-adjoint for $K \geq 3/4$ [20]. Then quantum dynamics of the free motion is unique.

As usual, the semi-classical aspects are included in the phase space. The quantum states and their dynamics have phase space representations through wavelet symbols. For the state $|\phi\rangle$ one has

$$\Phi(q,p) = \langle q,p|\phi\rangle/\sqrt{2\pi},$$

with the associated probability distribution on phase space given by

$$\rho_\phi(q,p) = \frac{1}{2\pi c_{-1}} |\langle q,p|\phi\rangle|^2.$$

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Having the (energy) eigenstates of some quantum Hamiltonian H at our disposal, we can compute the time evolution

$$\rho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}} |\langle q,p| e^{-iH}|\phi\rangle|^2$$

for any state ϕ.

The integral quantization based on the affine group has applications in quantum cosmology [17]. For a Friedmann–Lemaître–Robertson–Walker (FLRW) model filled with barotropic fluid with equation of state $p = w\rho$, the resolution of the Hamiltonian constraint leads to a model of a singular universe, or equivalently, to a particle moving on the half-line $(0,\infty)$ with Hamiltonian

$$h(q,p) = \alpha(w)p^2 + 6\tilde{k}q^{\beta(w)},\ q>0\,,\{q,p\} = 1,$$

where

$$\tilde{k} = \left(\int d\omega\right)^{2/3} k, \alpha(w) = \frac{3(1-w)^2}{32},\ \beta(w) = \frac{2(3w+1)}{(3(1-w)},$$

and $k = 0,-1$ or 1 (in units of inverse area) depending on whether the universe is flat, open or closed. Let us assume a closed universe with radiation content $w = 1/3$. The affine quantization with a fiducial vector like $\psi(x) \propto \exp(-(\alpha(\nu)x + \beta(\nu)/x)$, and parameter $\nu > 0$, on R^*_+ yields the quantum Hamiltonian

$$A_h = H = \frac{1}{24}P^2 + \frac{a_P^2 K(\nu)}{24}\frac{1}{Q^2} + 6\frac{a_P^2}{\sigma^2}\frac{c_1}{c_{-1}}Q^2\,,$$

where a_P is the Planck area. For $K(\nu) \geq 3/4$, this *wavelet quantization* removes the quantum singularity and thus a well-defined quantum evolution exists, as opposed to canonical quantization. Let us now pay some attention to the "semiclassical" behavior of the Friedmann equation. As we have already mentioned, in general the lower symbol $\check{f}(q,p)$ differs from its classical counterpart $f(q,p)$: it is a quantum-corrected effective observable. Thus, computing the lower symbol of Hamiltonian leads to the semiclassical Friedmann equation for scale factor $a(t)$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} + c^2 a_P^2 (1-w)^2 \frac{\nu}{128}\frac{1}{V^2} = \frac{8\pi G}{3c^2}\rho\,.$$

Note that the repulsive potential depends explicitly on volume. This excludes non-compact universes from quantum modeling. As a result, the singularity resolution is confirmed, i.e., as the singular geometry is approached ($a \to 0$), the repulsive potential grows faster ($\sim a^{-6}$) than the fluid density ($\sim a^{-3(1+w)}$) and therefore at some point the two terms become equal and the contraction is brought to a halt.

We remark that the form of the repulsive potential does not depend on the state of the fluid filling the universe and the origin of singularity avoidance is a pure quantum geometrical result.

6. Conclusion

The main idea behind quantization is to obtain a map between classical observables, functions on a set X, and quantum observables, operators on a Hilbert space $\mathscr{H}$. In general, because of the appearance of unbounded operators, one does not obtain an algebra of operators, due to domain issues. However, it is posssible to bypass such subtleties by working with the lower symbols of operators, which live in the classical setting, but carry some information from the noncommutative and probabilistic quantum setting.

We have applied the method of covariant integral quantization to three particular cases: the groups SU(2) and its spin one-half representation, the Weyl-Heisenberg group and the affine group $\mathrm{Aff}_+(\mathbb{R})$.

In the first case we obtain the SU(2) Lie algebra in terms of operators corresponding to the classical Euler angles and to the quadratic Hopf map. As a natural follow-up to this work, we will consider the covariant integral quantization of the group SU(2) for a general $spin - j$ UIR.

For the Weyl-Heisenberg group we show that the canonical commutation relation is obtained for general complex weight function ϖ, generalizing previous results. Moreover, we obtain the expected energy spectrum for the quantum harmonic oscillator for even ϖ differentiable at the origin.

In the third case, for the quantization of the Affine group, we examined the possibility of quantizing with a general weight function, similarly to the above Weyl-Heisenberg case, and we have given sufficient conditions for the existence of a density matrix, generalizing the affine coherent state projectors. We recalled that an important issue of the affine quantization based on coherent states projectors is the regularized quantum geometry where the singularity at the origin of the configuration space is avoided by the appearance of a centrifugal quantum potential. It would be interesting to see if we still have these features if a general density matrix is used.

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