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Status of standard model calculations of lepton $g-2$

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The contributions from the standard model interactions to the anomalous magnetic moments of the two lightest charged leptons, the electron and the muon, are reviewed. Comparison with the very accurate experimental values is made, using the most recent high-precision determination of the fine structure constant.

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1. Introduction

The anomalous magnetic moments of the electron and of the muon have been experimentally measured with very high accuracies. The latest experiments of the Penning-trap type conducted at Harvard [1, 2] have improved by more than an order of magnitude the relative precision on the value of the magnetic moment g_e of the electron as compared to the value obtained previously [3] by the group of H.G. Dehmelt in Seattle. The present best determination [2] of the electron's magnetic moment at the level of 0.28ppt gives the anomalous magnetic moment [$a_e \equiv (g_e - 2)/2$]

$$a_e^{\text{exp}} = 1\,159\,652\,180.73(0.28) \cdot 10^{-12}, \tag{1.1}$$

at an impressive relative precision of 0.24ppb.

The finite lifetime of the muon, $\tau_\mu \sim 2 \cdot 10^{-6}$ s, precludes the use of similar experimental techniques in order to measure its magnetic moment. Instead, muons, produced from the decay of pions obtained by sending a proton beam on a target, are collected in a storage ring where they are accelerated before decaying into electrons and neutrinos. Successive experiments of this type, conducted over several decades, first at CERN, from the beginning of the sixties to the middle of the seventies, and more recently at BNL, have brought the relative precision on the anomalous magnetic moment of the muon below the ppm level. The combination of the results obtained by the E821 experiment at BNL [4] leads to the value

$$a_\mu^{\text{exp}} = 11659208.9(6.3) \cdot 10^{-10} \tag{1.2}$$

which shows a relative precision of 0.54ppm, the total error being dominated by statistics [$\pm 5.4 \cdot 10^{-10}$, vs. $\pm 3.3 \cdot 10^{-10}$ for systematics].

As far as the τ lepton is concerned, the experimental situation is unfortunately far from being close to the one of the electron, or even of the muon. The very short lifetime, $\tau_\tau \sim 2.9 \cdot 10^{-13}$ s, of the τ, which is heavy enough to decay into hadrons, precludes the implementation of the experimental approach used for the muon. Only rather crude bounds [at 95% confidence level] have been obtained by the LEP and LEP2 experiments:

$$-0.052 < a_\tau^{\text{exp}} < +0.058 \text{ [5]}, \; -0.068 < a_\tau^{\text{exp}} < +0.065 \text{ [6]}, \; -0.052 < a_\tau^{\text{exp}} < +0.013 \text{ [7]}. \tag{1.3}$$

I will therefore only discuss the cases of the electron and of the muon in the sequel, and refer to Refs. [8] and [9] for the theory of a_τ.

The experimental measurements of the anomalous magnetic moments of the electron and of the muon constitute quite impressive achievements, and the natural question that arises is whether theoretical predictions are able to match this precision. At stake is the possibility for an indirect evidence of physics beyond the standard model (for a survey, see Ref. [10]), should the comparison between the experimental results and the theoretical calculations reveal a sufficiently significant and robust discrepancy. The present contribution reviews the status of the theoretical determinations of a_e and of a_μ within the standard model.

2. General aspects

The response of a charged lepton ℓ^-, $\ell = e, \mu, \tau$, to an external electromagnetic field is given by the matrix element

$$\langle \ell^-; p' | \mathscr{J}_\rho(0) | \ell^-; p \rangle \equiv \bar{\mathrm{u}}(p') \Gamma_\rho^{(\ell)}(p', p) \mathrm{u}(p) \tag{2.1}$$

of the conserved electric current $\mathscr{J}_\rho(x)$. In the most general situation, where neither invariance under parity nor under time reversal is assumed, the matrix element $\Gamma_\rho^{(\ell)}(p', p)$ decomposes into four independent Lorentz invariant form factors $F_i(k^2)$, $i = 1, 2, 3, 4$, with $k_\mu \equiv (p' - p)_\mu$ [most of the time, the superscript (ℓ) will be omitted]:

$$\Gamma_\rho(p', p) = F_1(k^2)\gamma_\rho + \frac{i}{2m_\ell} F_2(k^2)\sigma_{\rho\nu}k^\nu - F_3(k^2)\gamma_5\sigma_{\rho\nu}k^\nu + F_4(k^2)(k^2\gamma_\rho - 2m_\ell k_\rho)\gamma_5. \tag{2.2}$$

Restricting one's attention to static quantities, $k \to 0$, the Dirac form factor is normalized by the electric charge, $F_1(0) = 1$ in the normalization chosen here for the electric current, while the Pauli form factor describes the anomalous magnetic moment, $F_2(0) = a_\ell \equiv \frac{1}{2}(g_\ell - 2)$, i.e. half the deviation of the gyromagnetic factor g_ℓ from its tree-level value $g_\ell^{\mathrm{tree}} = 2$ in the standard model. The two remaining form factors describe, in the same static limit, an electric dipole moment $d_\ell = e_\ell F_3(0)$, and an axial radius, the so-called anapole moment $F_4(0)$, sensitive to the gradient of the external field. A more complete discussion and corresponding references can be found in Ref. [11].

Bofore entering the detailed account of the various contributions of the standard model degrees of freedom to the anomalous magnetic moment a_ℓ, it is useful to keep in mind a few general and simple, but nevertheless useful, remarks and observations:

- Within the framework of a renormalizable quantum field theory, $F_2(k^2)$, $F_3(k^2)$, and $F_4(k^2)$ can only arise through loop corrections. These loop contributions have to be *finite and calculable*.

- The anomalous magnetic moments a_ℓ are dimensionless. Therefore, contributions to a_ℓ arising from loops containing only photons and leptons of type ℓ are universal.

- Massive degrees of freedom with $M \gg m_\ell$ contribute in general to a_ℓ through powers of m_ℓ^2/M^2 times logarithms (decoupling).

- Light degrees of freedom with $m \ll m_\ell$ give rise to logarithmic contributions to a_ℓ, e.g. $\ln(m_\ell^2/m^2)$. Note that $\pi^2 \ln(m_\mu/m_e) \sim 50$.

It has become common practice to decompose the various standard model contributions to a_ℓ into parts arising from quantum electrodynamics, from the strong interactions, and finally from the weak interactions,

$$a_\ell^{\mathrm{SM}} = a_\ell^{\mathrm{QED}} + a_\ell^{\mathrm{had}} + a_\ell^{\mathrm{weak}}. \tag{2.3}$$

I will adopt this decomposition, and discuss the three contributions in turn.

3. Contributions from quantum electrodynamics

The quantum electrodynamics (QED) contribution, $a_\ell^{\rm QED}$, to the anomalous magnetic moment of a charged lepton is defined as the sum of the contributions that arise from loops made only of virtual photons and leptons. Due to the smallness of the fine structure constant, a perturbative approach is appropriate:

$$a_\ell^{\rm QED} = \sum_{n\geq 1} A_1^{(2n)}\left(\frac{\alpha}{\pi}\right)^n + \sum_{n\geq 2} A_2^{(2n)}(m_\ell/m_{\ell'})\left(\frac{\alpha}{\pi}\right)^n + \sum_{n\geq 3} A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})\left(\frac{\alpha}{\pi}\right)^n. \tag{3.1}$$

The first sum accounts for the universal, mass-independent, contributions, that start at the one-loop level, and that correspond to the case where the lepton in the loop is identical to the external lepton (i.e. one-flavour QED). The remaining contributions describe multi-flavour QED. Mass-dependent contributions involving a single ratio of lepton masses start only at the two-loop level, and are described by the coefficients $A_2^{(2n)}(m_\ell/m_{\ell'})$. Finally, from the three-loop level onwards, one can also have contributions involving two ratios of lepton masses, which are accounted for by the coefficients $A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$. In the standard model with only three generic families, this exhausts all the possibilities.

The lowest-order contribution $A_1^{(2)}$ arising from the vertex correction at one-loop is known since long time [12], and reads

$$A_1^{(2)} = \frac{1}{2}. \tag{3.2}$$

At next-to-leading order, there are seven graphs contributing to $A_1^{(4)}$ (see Fig. 2 in Ref. [11]), and the corresponding expression, also known analytically, was first given correctly in Refs. [13] and [14]:

$$A_1^{(4)} = \frac{197}{144} + \left(\frac{1}{2} - 3\ln 2\right)\zeta(2) + \frac{3}{4}\zeta(3) = -0.328\,478\,965\,579\,193... \tag{3.3}$$

with $\zeta(p) = \sum_{n=1}^{\infty} 1/n^p$, $\zeta(2) = \pi^2/6$. At this same order, there also appears a mass-dependent contribution. It arises from the insertion of a vacuum polarization loop coming from the lepton ℓ' into the photon propagator of the one-loop vertex correction of the lepton ℓ. Its expression reads

$$\begin{aligned} A_2^{(4)}(m_\ell/m_{\ell'}) &= \frac{1}{3}\int_{4m_{\ell'}^2}^{\infty} ds \sqrt{1-\frac{4m_{\ell'}^2}{s}}\,\frac{s+2m_{\ell'}^2}{s^2}\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)\frac{s}{m_\ell^2}} \\ &= \frac{1}{3}\int_{4m_{\ell'}^2}^{\infty}\frac{ds}{s}K(s)R^{\ell'^+\ell'^-}(s), \end{aligned} \tag{3.4}$$

and is equivalent to the formulas given in Refs. [15] and [16] after changes of the integration variables. The function $K(s)$ in the second expression is defined by the second integral in the first expression, and $R^{\ell'^+\ell'^-}(s)$ stands for the ratio $\sigma(\ell^+\ell^- \to \ell'^+\ell'^-)/\sigma^{\rm pt}(s)$, with $\sigma^{\rm pt}(s) = 4\pi\alpha^2/(3s)$, at tree level. The integrations in Eq. (3.4) can be done explicitly in order to obtain analytical expressions, which were given in Ref. [17], for $m_\ell/m_{\ell'} > 1$ and, in a more compact way, in Refs. [18] and [19] for the general case. In practice, one may also use series expansions [20]-[23], the first terms of which read

$$A_2^{(4)}(m_\ell/m_{\ell'}) = \frac{1}{3}\ln\left(\frac{m_\ell}{m_{\ell'}}\right) - \frac{25}{36} + \frac{\pi^2}{4}\frac{m_{\ell'}}{m_\ell} - 4\left(\frac{m_{\ell'}}{m_\ell}\right)^2 \ln\left(\frac{m_\ell}{m_{\ell'}}\right) + 3\left(\frac{m_{\ell'}}{m_\ell}\right)^2 + \mathscr{O}\left[\left(\frac{m_{\ell'}}{m_\ell}\right)^3\right], \tag{3.5}$$

for $m_\ell \gg m_{\ell'}$, and

$$A_2^{(4)}(m_\ell/m_{\ell'}) = \frac{1}{45}\left(\frac{m_\ell}{m_{\ell'}}\right)^2 + \frac{1}{70}\left(\frac{m_\ell}{m_{\ell'}}\right)^4 \ln\left(\frac{m_\ell}{m_{\ell'}}\right) + \frac{9}{19600}\left(\frac{m_\ell}{m_{\ell'}}\right)^4 + \mathscr{O}\left[\left(\frac{m_\ell}{m_{\ell'}}\right)^3 \ln\left(\frac{m_\ell}{m_{\ell'}}\right)\right], \tag{3.6}$$

for $m_{\ell'} \gg m_\ell$. These expansions display the general properties discussed at the end of Sec. 2. In principle, they can be carried out to any desired order, the only limitation in precision coming from the accuracy of our knowledge of the mass ratios they involve. Using the values given by the CODATA compilation [24]

$$m_\mu/m_e = 206.7682843(52),\, m_\mu/m_\tau = 5.94649(54)\cdot 10^{-2},\, m_e/m_\tau = 2.87592(26)\cdot 10^{-4}, \tag{3.7}$$

one finds [25][26]

$$\begin{aligned} A_2^{(4)}(m_e/m_\mu) &= 5.197\,386\,67(26)\cdot 10^{-7},\ \ A_2^{(4)}(m_e/m_\tau) = 1.837\,98(34)\cdot 10^{-9},\\ A_2^{(4)}(m_\mu/m_e) &= 1.094\,258\,312\,0(83),\quad A_2^{(4)}(m_\mu/m_\tau) = 7.807\,9(15)\cdot 10^{-5}. \end{aligned} \tag{3.8}$$

At order $\mathscr{O}(\alpha^3)$, there are 72 diagrams to consider. A quite impressive series of computational achievements [27]-[37] led to the following analytical expression for the sixth-order mass-independent contribution

$$\begin{aligned} A_1^{(6)} &= \frac{87}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right] - \frac{239}{2160}\pi^4 \\ &\quad + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} \qquad \left[a_p = \sum_1^\infty 1/(2^n n^p)\right] \\ &= 1.181\,241\,456\ldots \end{aligned} \tag{3.9}$$

The mass-dependent contributions at sixth order are also known analytically and were obtained in Refs. [38] and [39]. The corresponding numerical values read [25][26]:

$$\begin{aligned} &A_2^{(6)}(m_e/m_\mu) = -7.373\,941\,55(27)\cdot 10^{-6},\, A_2^{(6)}(m_e/m_\tau) = -6.5830(11)\cdot 10^{-8},\\ &A_2^{(6)}(m_\mu/m_e) = 22.868\,380\,04(23),\, A_2^{(6)}(m_\mu/m_\tau) = 36.070(13)\cdot 10^{-5}, \end{aligned} \tag{3.10}$$

and [23, 25, 26]

$$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) = 0.1909(1)\cdot 10^{-12},\, A_3^{(6)}(m_\mu/m_e, m_\mu/m_\tau) = 5.2776(11)\cdot 10^{-4}. \tag{3.11}$$

The eighth-order contribution consists of 891 Feynman diagrams. Its value is known only from a numerical integration of the corresponding Feynman-parametrized loop integrals. The latest result of the mass-independent part reads [25]

$$A_1^{(8)} = -1.9106(20). \tag{3.12}$$

It is interesting to follow the evolution of this numerical determination during time, as displayed in Eq. (9) of Ref. [40]. The discussion following it shows that these numerical evaluations present tough difficulties that in some cases have led to erroneous values. Having such delicate calculations done by a single group can be potentially problematic. An independent evaluation of these contribution through different techniques and by a different group [41] can thus only be welcome.

The contribution at order $\mathscr{O}(\alpha^5)$ to a_ℓ arises from 12672 diagrams. Its numerical evaluation was completed only recently [40][42]-[50][25] The mass-independent part amounts to

$$A_1^{(10)} = 9.168(571), \tag{3.13}$$

whereas the values of the relevant mass-dependent contributions are

$$A_2^{(10)}(m_e/m_\mu) = -0.00382(39), \tag{3.14}$$

and

$$A_2^{(10)}(m_\mu/m_e) = 742.18(87), A_2^{(10)}(m_\mu/m_\tau) = -0.068(5), A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau) = 2.011(10). \tag{3.15}$$

Summing up all the contributions at a given order, one arrives at the perturbative expansion

$$a_\ell^{\text{QED}} = C_\ell^{(2)}\left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \dots \tag{3.16}$$

with the values[25][26] of the coefficients $C_\ell^{(n)}$ gathered in the following table:

	$\ell = e$	$\ell = \mu$
$C_\ell^{(2)}$	0.5	0.5
$C_\ell^{(4)}$	$-0.328\,478\,444\,00$	$0.765\,857\,425(17)$
$C_\ell^{(6)}$	$1.181\,234\,017$	$24.050\,509\,96(32)$
$C_\ell^{(8)}$	$-1.9144(35)$	$130.879\,6(63)$
$C_\ell^{(10)}$	$9.16(58)$	$753.29(1.04)$

Since $(\alpha/\pi)^2 \sim 6.76 \cdot 10^{-14}$, we see that the tenth-order QED contributions are relevant at the level of the experimental precision on a_e and on a_μ, in the latter case because of the large value of the coefficients $C_\mu^{(10)}$ (cf. the end of Sec. 2).

4. Contributions from quantum chromodynamics

Formally speaking, the contributions from the strong interactions arise from loops involving, in addition to the leptons and photon, also the quarks and gluons as described by quantum chromodynamics (QCD). Unfortunately, the kinematical regime relevant for static quantities, such as the anomalous magnetic moments of the leptons, corresponds to large distances, where a perturbative approach to the QCD effects is not useful. The first QCD contribution arises at order $\mathscr{O}(\alpha^2)$, through a hadronic vacuum polarization insertion into the one-loop vertex diagram, see Fig. 1. This contribution, $a_\ell^{\text{HVP-LO}}$, is evaluated using the second expression in Eq. (3.4), with the same function $K(s)$, but with $R^{\ell'^+\ell'^-}(s)$ replaced by the hadronic ratio $R^{\text{had}}(s)$ [51][52][53] [and

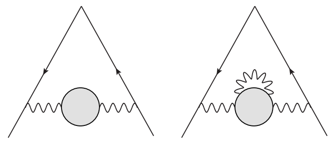

Figure 1: Lowest-order hadronic corrections to a_ℓ (left). The shaded blob represents the hadronic vacuum polarization function. The contribution on the right, although of higher order, is also included in $a_\ell^{\text{HVP-LO}}$.

the range of integration starts at $4M_{\pi^\pm}^2$], that can be extracted from available data for the cross-section of $e^+e^- \to$ hadrons. Note that $K(s)$ is positive, so that $a_\ell^{\text{HVP-LO}}$ is positive. Furthermore, $K(s) \sim m_\ell^2/(3s)$ as $s \to \infty$, so that the low-energy region dominates. Two recent determinations in the case of the muon give very close results

$$a_\mu^{\text{HVP-LO}} = 692.3(4.2) \cdot 10^{-12} \text{ [54]}, \; a_\mu^{\text{HVP-LO}} = 694.9(4.3) \cdot 10^{-12} \text{ [55]}. \tag{4.1}$$

For more details, I refer to the references already quoted, ot to [56]-[59], and to [60][61] for different approaches.

At order $\mathscr{O}(\alpha^3)$, there are further hadronic corrections[62][63], shown in Fig. 2, which involve the same R ratio, but convoluted with a different function $K^{(2)}(s)$, which is no longer positive definite. One finds for this NLO hadronic vacuum polarization contribution [55]

$$a_\mu^{\text{HVP-NLO}} = -9.84(7) \cdot 10^{-12}, \tag{4.2}$$

At order $\mathscr{O}(\alpha^3)$, there is also another hadronic contribution arising from the diagram in Fig. 3. This hadronic (virtual) light-by-light scattering contribution cannot be related to available data, and has thus to be evaluated by other means. For a critical overview, see [64]. Ref. [65] provides a recent update. Related issues are discussed in [66]-[68] For reference, I quote the “best estimate” of Ref. [69]

$$a_\mu^{\text{HLxL}} = 10.5(2.6) \cdot 10^{-10}, \tag{4.3}$$

and the somewhat more conservative value [70]

$$a_\mu^{\text{HLxL}} = 11.5(4.0) \cdot 10^{-10}. \tag{4.4}$$

In the case of the electron, these hadronic contributions are suppressed by approximatively a factor $(m_e/m_\mu)^2$ as compared to the muon, and the most recent values are [72]

$$a_e^{\text{HVP-LO}} = 1.866(11) \cdot 10^{-12}, \; a_e^{\text{HVP-NLO}} = -0.2234(14) \cdot 10^{-12}, \tag{4.5}$$

and

$$a_e^{\text{HLxL}} = 0.035(10) \cdot 10^{-12} \text{ [71]}, \; a_e^{\text{HLxL}} = 0.039(13) \cdot 10^{-12} \text{ [70]}. \tag{4.6}$$

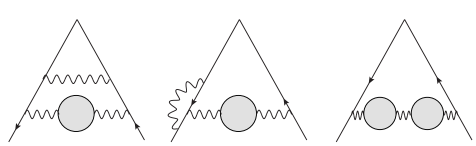

Figure 2: Representative Feynman graphs for the higher-order hadronic corrections $a_\ell^{\text{HVP-NLO}}$.

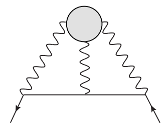

Figure 3: Hadronic Light-by-Light contribution to a_ℓ.

5. Contributions from weak interactions

At the one-loop level, contributions from the weak interactions arise through the vertex corrections due to the exchange of a virtual Z^0 gauge boson or a virtual scalar boson, and from the $\gamma - W^+ - W^-$ vertex, with the $W^\pm$ lines connected to the external lepton lines through two tree-level charged-current vertices. These one-loop contributions have been worked out more than forty years ago by several groups [73]-[77], and the result reads

$$\begin{aligned} a_\ell^{\mathrm{weak}(1)} &= \frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{m_\ell^2}{8\pi^3} \left[\frac{5}{3} + \frac{1}{3}\left(1 - 4s_w^2\right)^2 + \mathscr{O}\left(\frac{m_\ell^2}{M_Z^2} \ln \frac{M_Z^2}{m_\ell^2} \right) + \mathscr{O}\left(\frac{m_\ell^2}{M_H^2} \ln \frac{M_H^2}{m_\ell^2} \right) \right] \\ &= 19.48 \cdot 10^{-10}. \end{aligned} \tag{5.1}$$

At the two-loop level, one conveniently distinguishes between contributions without internal fermion loops (called bosonic), and those containing internal fermion loops (called fermionic). The former were evaluated in Refs. [78]-[80], and the latter were treated in Refs. [81]-[84] A recent reanalysis [85] gives, for the sum of one-loop and two-loop corrections, the total value

$$a_\mu^{\mathrm{weak}} = 15.36(10) \cdot 10^{-10}. \tag{5.2}$$

The corresponding value for the electron case is expected to be $\sim 40\,000$ times smaller,

$$a_e^{\mathrm{weak}} = 0.0297(5) \cdot 10^{-12}, \tag{5.3}$$

which lies one order of magnitude below the current experimental precision.

6. Determination of the fine-structure constant

In order to answer the question raised at the end of Sec. 1, it is necessary to know the value of the fine-structure constant at the level of precision that matches the experimental accuracy. In the case of the electron, the precision required is thus $\Delta\alpha/\alpha \sim 0.24$ppb. Up to a decade ago, the most precise knowledge on the value of α was provided by the quantum Hall effect, with however a relative precision of only ~ 20 ppb, i.e. two orders of magnitude above what is required. The situation has changed quite dramatically during the years 2000, with a series of measurements of the recoil velocity through photon absorption of Cesium [86] and Rubidium [87]-[89]atoms. The latest value obtained in Ref. [89] is (see also [90])

$$\alpha^{-1}[Rb11] = 137.035\,999\,037(91), \tag{6.1}$$

with a relative error of only 0.66ppb, i.e. quite close to what is actually required. The comparison between the experimental value of $a_e[HV08]$ obtained in 2008 [2] and the prediction using the value $\alpha^{-1}[Rb11]$ quoted above gives the quite impressive agreement

$$a_e[HV08] - a_e[\text{theory}] = -1.05(0.82)\cdot 10^{-12}. \tag{6.2}$$

Conversely, using the theoretical calculations in order to extract α from the value of a_e measured by the latest Harvard experiment yields

$$\alpha^{-1}[HV08] = 137.035\,999\,172\,7(68)_{\alpha^4}(46)_{\alpha^5}(19)_{\text{had+weak}}(331)_{a_e[HV08]}, \tag{6.3}$$

which corresponds to a relative uncertainty of 0.25ppb, more than twice smaller than the relative error on $\alpha^{-1}[Rb11]$, and with the largest contribution to the error coming from the experimental uncertainty on $a_e[HV08]$.

7. Conclusion and perspectives for the future

The anomalous magnetic moments of the electron and of the muon have been measured very precisely (to 0.24ppb and to 0.54ppm, respectively), improving previous measurements by several factors. Recent high-precision measurements of the fine structure constant in atomic physics allow to reach the level of accuracy required in order to test QED with a_e, and even to become sensitive to the contributions from the strong interactions $a_e^{\text{HVP-LO}}$ [compare Eqs. (4.5) and (6.2)]. Although a_e to date still provides the most precise determination of the fine structure constant, it ist certainly of interest and worthwhile to pursue the efforts to measure α with the accuracy required to test the prediction for a_e at the level where this quantity has been measured.

The anomalous magnetic moments of the muon probes all the interactions of the standard model, and perhaps even beyond. Indeed, there is a persistent discrepancy between the measured value and the SM prediction at the level of 3 to 3.6 σ,

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (28.7 \pm 8.0)\cdot 10^{-10}\ [3.6\sigma], \tag{7.1}$$

with the values from Refs. [54] and [69]. The discrepancy shrinks somewhat below the 3σ level if one takes Refs. [55] and [70]

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (25.0 \pm 8.6)\cdot 10^{-10}\ [2.9\sigma]. \tag{7.2}$$

Whether this discrepancy is real or not will be probed soon by two forthcoming experiments, at FNAL (E989) and at J-PARC (E34), see [91], which aim at measuring a_μ with a relative precision of 0.14ppm.

The interpretation of these future experiments requires further theoretical improvement on the evaluation of the hadronic contributions. As far as hadronic vacuum polarization is concerned, improvements can be expected from forthcoming data [59]. A more refined theoretical understanding of the hadronic light-by-light scattering contribution, which remains a difficult challenge, would also be welcome. Finally, simulations of QCD on the lattice could provide valuable alternative determinations of these hadronic contributions, provided they can meet the required precision. For prospects in this direction, see Ref. [92].

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