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Dispersive approach to isospin breaking in $\pi\pi$ scattering

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We report on recent work [1, 2] concerning isospin breaking in the $K_{\ell 4}$ form factors induced by the difference between charged and neutral pion masses. Starting from suitably subtracted dispersion representations, the form factors can be constructed in an iterative way up to two loops in the low-energy expansion by implementing analyticity, crossing, and unitarity due to two-meson intermediate states. This provides a connection between the phases of the two-loop form factors of the $K^\pm \to \pi^+\pi^- e^\pm \nu_e$ channel measured experimentally (out of the isospin limit) and the difference of S- and P-wave $\pi\pi$ phase shifts studied theoretically (in the isospin limit). The isospin-breaking correction consists of the sum of a universal part, involving only $\pi\pi$ rescattering, and a process-dependent contribution, involving the form factors in the coupled channels. The dependence on the two S-wave scattering lengths a_0^0 and a_0^2 in the isospin limit is worked out in a general way, in contrast to previous analyses based on one-loop chiral perturbation theory. The two-loop universal and process-dependent contributions are estimated and cancel partially to yield an isospin-breaking correction close to the one-loop case. The recent results on the phases of $K^\pm \to \pi^+\pi^- e^\pm \nu_e$ form factors obtained by the NA48/2 collaboration at the CERN SPS are reanalysed including this isospin-breaking correction to extract values for the scattering lengths a_0^0 and a_0^2, as well as for low-energy constants and order parameters of two-flavour χPT.

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1. Introduction

One of the best tests of our understanding of low-energy QCD comes from $\pi\pi$ scattering, as it probes the spontaneous breaking of chiral symmetry, responsible for the existence of light pions as Goldstone Bosons. It provides a very stringent test of $N_f = 2$ Chiral Perturbation Theory (χPT), the effective theory for low-energy pion dynamics built on the chiral limit $m_u = m_d = 0$, of its structure and of its range of validity [3, 4]. A particularly clean probe of $\pi\pi$ (re)scattering consists in the angular analysis of $K \to \pi\pi\ell\nu$ ($K_{\ell 4}$) decays, yielding information on the interference between the S and P waves [5]. Dispersive methods, i.e. Roy equations, can then be used to reconstruct the low-energy $\pi\pi$ amplitude using unitarity, analyticity, and data at higher energies, with two subtraction parameters chosen as the scattering lengths a_0^0 and a_0^2 [6]. The reconstructed amplitude can be checked against the prediction from $N_f = 2$ χPT. In order to match higher-energy data on $\pi\pi$ phase shifts, Roy equations require the values of (a_0^0, a_0^2) to lie within a large so-called Universal Band, out of which the domain favoured by χPT represents only a small region.

Until 2001, the only available data on $K_{\ell 4}$ decay into two charged pions came from the old Geneva-Saclay and more recent BNL-E865 experiments [7]. A first analysis using the Roy equations together with a theoretical estimate of the scalar radius of the pion led to a determination of the scattering lengths in close agreement with the predictions from two-loop χPT [8]. Another analysis of the data available at that time (including $I = 2$ low-energy phase shifts) favoured a slightly larger value for a_0^2, 1 σ away from the two-loop χPT prediction [9]. Recently, the NA48/2 collaboration has collected high-statistics $K_{e4}^{\pm}$ data at the CERN SPS [10]. After the announcement of the preliminary results of NA48/2 [11], it was pointed out that the high level of accuracy reached by the experiments in extracting the $\pi\pi$ phase shifts required taking into account isospin-breaking effects [12]. These effects stem from different sources. First, the contributions from real and virtual photons can be removed, estimating the Coulomb exchanges and incorporating radiative processes through a Monte-Carlo treatment [13]. Second, the effect of the mass difference between charged and neutral pions on the one hand, which is also dominantly of electromagnetic origin, and between *up* and *down* quarks on the other hand, must be determined from a theoretical analysis.

These remaining corrections will be called "isospin-breaking" for simplicity, being understood that the other photon effects mentioned above have been taken care of beforehand by appropriate means, or can otherwise be considered to be negligible. A computation of these corrections was performed using next-to-leading-order χPT [14], leading to a significant energy-dependent correction in the phase shifts, restoring the agreement between the NA48/2 results and two-loop χPT. However, this correction was evaluated in the framework of χPT, with a given set of counterterms with values corresponding to a rather narrow range of scattering lengths a_0^0 and a_0^2. The underlying assumption is that the correction remains the same even for values of (a_0^0, a_0^2) that are reasonable from the dispersive point of view, i.e. consistent with Roy equations and higher-energy data, but cannot be accommodated from the chiral point of view, because they differ too much from the current-algebra results. If the correction had a strong dependence on a_0^0 and a_0^2, the latter would not be exhibited by the one-loop computation performed in the framework of χPT, but it could affect the outcome of the analysis of the data provided by the NA48/2 experiment.

We report here on a dispersive computational framework of isospin-breaking corrections in the phases of the form factors where the values of the scattering lengths are not unnecessarily restricted from the outset, recently developed in Refs. [1, 2] and overcoming this potential issue. In presence of isospin breaking, several $\pi\pi$ channels can rescatter into a given final state, contributing to isospin-breaking effects in direct link with the structure of the $\pi\pi$ amplitude itself. As shown in Refs. [15, 16] in the isospin limit [1], the use of analyticity, unitarity and crossing is sufficient to reconstruct the $\pi\pi$ amplitude up to two loops in terms of a limited number of subtraction constants (subthreshold or threshold parameters). Refs. [1, 2] has used the same approach to derive a general expression for the isospin-breaking correction in the phases of the two-loop form factors, where the values of a_0^0 and a_0^2 remain as free parameters and are not fixed from the outset as in Ref. [14].

2. Properties of the $K_{\ell 4}$ form factors

Crossing. In the Standard Model, the amplitudes corresponding to $K_{\ell 4}$ decays are defined from the matrix elements of the type $\langle \pi^a(p_a)\pi^b(p_b)|iA_\mu^{4-i5}(0)|K(k)\rangle$ involving the $\Delta S = \Delta Q = +1$ axial current. Through crossing, this amplitudes can be related to $\langle \pi^a(p_a)\bar{K}(k)|iA_\mu^{4-i5}(0)|\bar{\pi}^b(p_b)\rangle$ and $\langle \bar{K}(k)\pi^b(p_b)|iA_\mu^{4-i5}(0)|\bar{\pi}^a(p_a)\rangle$, which can be all treated with common notation:

$$\mathscr{A}_\mu^{ab}(p_a,p_b;p_c) = \langle a(p_a)\, b(p_b)|iA_\mu(0)|\bar{c}(p_c)\rangle. \tag{2.1}$$

In practice the sets of interest are $\{a,b,c\} = \{\pi^+,\pi^-,K^-\}$, $\{\pi^0,\pi^0,K^-\}$ or $\{\pi^0,\pi^-,\bar{K}^0\}$. This matrix element possesses the general decomposition into invariant form factors

$$\mathscr{A}_\mu^{ab}(p_a,p_b;p_c) = (p_a+p_b)_\mu F^{ab}(s,t,u) + (p_a-p_b)_\mu G^{ab}(s,t,u) + (p_c-p_a-p_b)_\mu R^{ab}(s,t,u). \tag{2.2}$$

They depend on the variables $s=(p_a+p_b)^2,\ t=(p_c-p_a)^2,\ u=(p_c-p_b)^2$, obeying the "mass-shell" condition $s+t+u = M_a^2+M_b^2+M_c^2+s_\ell \equiv \Sigma_\ell$, with $s_\ell \equiv (p_c-p_a-p_b)^2$ being the square of the dilepton invariant mass. The decomposition (2.2) leads to form factors which are free from kinematical singularities, but which do not have simple decompositions into partial waves. For the latter, it is more convenient to introduce another set of form factors $\mathscr{F}^{ab},\mathscr{G}^{ab},\mathscr{R}^{ab}$, which are linear combinations of the former, with projections on $\pi\pi$ partial waves denoted $f_l^{ab}(s,s_l)$, $g_l^{ab}(s,s_l)$, $r_l^{ab}(s,s_l)$. It turns out that crossing provides relations among $\mathscr{F}$ and $\mathscr{G}$-type form factors on one side, and among $\mathscr{R}$-type form factors on the other hand ($\mathscr{R}$ form factors are linked to the divergence of the matrix elements of the current $A_\mu(x)$, so that they cannot mix under crossing with the other form factors, related to transverse components of the same current).

Chiral counting. The low-energy behaviour of the partial waves [17] is based on the chiral counting $M_P \sim \mathscr{O}(E)$, $s,t,u,s_\ell \sim \mathscr{O}(E^2)$, where M_P stands for the mass of any of the light pseudoscalar states. S and P waves are dominant at low energies:

$$\begin{aligned}
&\mathrm{Re}\, f_0^{ab},\ \mathrm{Re}\, f_1^{ab},\ \mathrm{Re}\, g_1^{ab} \sim \mathscr{O}(E^0)), \qquad \mathrm{Im}\, f_0^{ab},\ \mathrm{Im}\, f_1^{ab},\ \mathrm{Im}\, g_1^{ab} \sim \mathscr{O}(E^2)\,,\\
&\mathrm{Re}\, f_{l\geq 2}^{ab}(s,s_\ell),\ \mathrm{Re}\, g_{l\geq 2}^{ab}(s,s_\ell) \sim \mathscr{O}(E^2), \qquad \mathrm{Im}\, f_{l\geq 2}^{ab}(s,s_\ell),\ \mathrm{Im}\, g_{l\geq 2}^{ab}(s,s_\ell) \sim \mathscr{O}(E^6)\,.
\end{aligned} \tag{2.3}$$

The chiral counting of the partial waves translates into the decompositions

$$\begin{aligned}
F^{ab}(s,t,u) &= F_S^{ab}(s,s_\ell) + F_P^{ab}(s,s_\ell)\cos\theta_{ab} + F_>^{ab}(s,\cos\theta_{ab},s_\ell),\\
G^{ab}(s,t,u) &= G_P^{ab}(s,s_\ell) + G_>^{ab}(s,\cos\theta_{ab},s_\ell).
\end{aligned} \tag{2.4}$$

[1] The isospin limit is defined as the limit in which the values of the neutral pion and kaon masses tend towards the charged ones, $M_{\pi^0} \to M_{\pi^\pm}$, $M_{K^0} \to M_{K^\pm}$, while keeping the latter fixed.

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where θ_{ab} denotes the angle made by the line of flight of particle a in the (a,b) rest frame with the direction of $\vec{p}_a+\vec{p}_b$ in the rest frame of particle $\bar{c}$, The contributions of $\ell \geq 2$ partial waves are collected in $F_>^{ab}$ and in $G_>^{ab}$, with the counting $\mathrm{Re}F_>^{ab}$, $\mathrm{Re}G_>^{ab} \sim \mathscr{O}(E^2)$ and $\mathrm{Im}F_>^{ab}$, $\mathrm{Im}G_>^{ab} \sim \mathscr{O}(E^6)$, while S and P waves are collected in

$$\begin{aligned} F_S^{ab}(s,s_\ell) &= f_0^{ab}(s,s_\ell) - \frac{M_a^2-M_b^2}{s} g_1^{ab}(s,s_\ell), \\ F_P^{ab}(s,s_\ell) &= f_1^{ab}(s,s_\ell) - \frac{M_c^2-s-s_\ell}{s} \frac{\lambda_{ab}^{\frac{1}{2}}(s)}{\lambda_{\ell c}^{\frac{1}{2}}(s)} g_1^{ab}(s,s_\ell), \qquad G_P^{ab}(s,s_\ell) = g_1^{ab}(s,s_\ell). \end{aligned} \tag{2.5}$$

involving the Källen's function $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$ with $\lambda_{ab}(s) = \lambda(s,M_a^2,M_b^2)$ and $\lambda_{\ell c}(s) = \lambda(s,s_\ell,M_c^2)$,

Analyticity. The form factors $F^{ab}(s,t,u)$ and $G^{ab}(s,t,u)$ are assumed to have the usual analyticity properties with respect to the variable s, for fixed values of t and of u (and of $s_\ell \geq 0$), with a cut on the positive s-axis (fixed by unitarity) and a cut on the negative s-axis (unitarity in the crossed channel). Up to and including two loops, the discontinuities along the positive s-axis at low energies (at a fixed s_ℓ) originate from mesonic two-particle intermediate states

$$\begin{aligned} \mathrm{Im}\, f_l^{ab}(s,s_\ell) &= \sum_{\{a',b'\}} \frac{1}{\mathscr{S}_{a'b'}} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{s} \mathrm{Re}\left\{ t_l^{a'b';ab}(s) \left[f_l^{a'b'}(s,s_\ell) \right]^\star \right\} \theta(s-s_{a'b'}) + \mathscr{O}(E^8), \\ \mathrm{Im}\, g_l^{ab}(s,s_\ell) &= \sum_{\{a',b'\}} \frac{1}{\mathscr{S}_{a'b'}} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{s} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{\lambda_{ab}^{\frac{1}{2}}(s)} \mathrm{Re}\left\{ t_l^{a'b';ab}(s) \left[g_l^{a'b'}(s,s_\ell) \right]^\star \right\} \theta(s-s_{a'b'}) + \mathscr{O}(E^8), \end{aligned} \tag{2.6}$$

where $l=0,1$ and $s_{a'b'} = (M_{a'}+M_{b'})^2$ stands for the lowest invariant mass squared of the corresponding intermediate state. The symmetry factor reads $\mathscr{S}_{a'b'}=1$ except for $\{a',b'\} = \{\pi^0,\pi^0\}$ or $\{\eta,\eta\}$, where $\mathscr{S}_{a'b'}=2$. The partial waves $t_l^{a'b';ab}(s)$ of the mesonic scattering amplitudes $A^{a'b';ab}(s,\hat{t})$, $\hat{t}=(p_a-p_{a'})^2$, are defined as usual with the chiral counting [15]

$$\begin{aligned} &\mathrm{Re}\, t_l^{a'b';ab}(s) \sim \mathscr{O}(E^2),\ l=0,1, \qquad \mathrm{Re}\, t_l^{a'b';ab}(s) \sim \mathscr{O}(E^4),\ l\geq 2, \\ &\mathrm{Im}\, t_l^{a'b';ab}(s) \sim \mathscr{O}(E^4),\ l=0,1, \qquad \mathrm{Im}\, t_l^{a'b';ab}(s) \sim \mathscr{O}(E^8),\ l\geq 2. \end{aligned} \tag{2.7}$$

An important observation is that the scattering amplitudes start at least at $\sim \mathscr{O}(E^2)$, so that the unitarity condition requires the imaginary part of the form factors to arise one higher order (in the chiral counting) compared to their real part.

3. Phases of the form factors

We are eventually interested in the phases of the F_S, F_P, and G_P components of the F and G form factors corresponding to the decay channel $K^+ \to \pi^+\pi^-\ell^+\nu_\ell$ as defined in Eq. (2.5) and more precisely, in the differences of these phases that are observable in the interferences occurring in the differential decay distribution. These form factors have the generic low-energy structure

$$\begin{aligned} F^{+-}(s,t,u) &= \widehat{F}_S^{+-}(s,s_\ell)e^{i\delta_S(s,s_\ell)} + \widehat{F}_P^{+-}(s,s_\ell)e^{i\delta_P(s,s_\ell)}\cos\theta + \mathrm{Re}F_>^{+-}(s,\cos\theta,s_\ell) + \mathscr{O}(E^6), \\ G(s,t,u) &= \widehat{G}_P^{+-}(s,s_\ell)e^{i\delta_P(s,s_\ell)} + \mathrm{Re}G_>^{+-}(s,\cos\theta,s_\ell) + \mathscr{O}(E^6), \end{aligned} \tag{3.1}$$

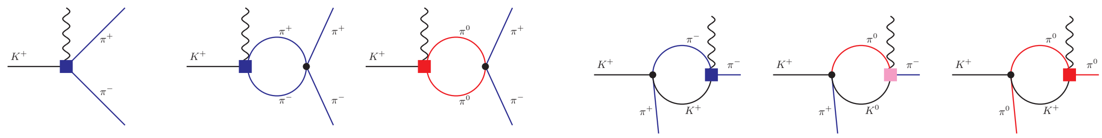

Figure 1: K^+_{e4} form factors: tree-level representation (left) and typical rescattering diagrams involved in the reconstruction of the K^+_{e4} form factors in the s-channel (center) and in the t- and u-channels (right).

where the real functions $\widehat{F}_S^{+-}(s,s_\ell)$, $\widehat{F}_P^{+-}(s,s_\ell)$, and $\widehat{G}_P^{+-}(s,s_\ell)$ correspond to the quantities appearing in Eq. (2.4), but with their phases removed, $\widehat{F}_S^{+-}(s,s_\ell) = e^{-i\delta_S(s,s_\ell)} F_S^{+-}(s+i0,s_\ell)$, etc. Order by order, the phases are related to the chiral expansion of the real parts of the partial-wave projections

$$\mathrm{Re}\, t_l^{a'b';+-}(s) = \varphi_l^{a'b';+-}(s) + \psi_l^{a'b';+-}(s) + \mathcal{O}(E^6), \tag{3.2}$$

for $l=0,1$, with the shorthand notation $+-$ denoting $\pi^+\pi^-$. We have $\varphi_{0,1}^{a'b';+-}(s) \sim \mathcal{O}(E^2)$ and $\psi_{0,1}^{a'b';+-}(s) \sim \mathcal{O}(E^4)$. We write a similar expansion for the form factors themselves, e.g.

$$\mathrm{Re}\, F_S^{+-}(s,s_\ell) = F_{S[0]}^{+-} + F_{S[2]}^{+-}(s,s_\ell) + \mathcal{O}(E^4), \qquad \mathrm{Re}\, G_P^{+-}(s,s_\ell) = G_{P[0]} + G_{P[2]}^{+-}(s,s_\ell) + \mathcal{O}(E^4), \tag{3.3}$$

Using the unitarity condition Eq. (2.7), we obtain the expressions valid up to $\mathcal{O}(E^6)$ corrections

$$\delta_S = \sum_{\{a',b'\}} \frac{1}{\mathscr{S}_{a'b'}} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{s} \left[\varphi_0^{a'b';+-}(s)\, \frac{F_{S[0]}^{a'b'} + F_{S[2]}^{a'b'}(s,s_\ell)}{F_{S[0]}^{+-} + F_{S[2]}^{+-}(s,s_\ell)} + \psi_0^{a'b';+-}(s)\, \frac{F_{S[0]}^{a'b'}}{F_{S[0]}^{+-}} \right] \theta(s - s_{a'b'}) + \ldots, \tag{3.4}$$

$$\delta_P = \sum_{\{a',b'\}} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{s} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{\lambda_{ab}^{\frac{1}{2}}(s)} \left[\varphi_1^{a'b';+-}(s)\, \frac{G_{P[0]}^{a'b'} + G_{P[2]}^{a'b'}(s,s_\ell)}{G_{P[0]}^{+-} + G_{P[2]}^{+-}(s,s_\ell)} + \psi_1^{a'b';+-}(s)\, \frac{G_{P[0]}^{a'b'}}{G_{P[0]}^{+-}} \right] \theta(s - s_{a'b'}) + \ldots \tag{3.5}$$

The phases $\delta_S(s,s_\ell)$ and $\delta_P(s,s_\ell)$ depend on s_ℓ through the order $\mathcal{O}(E^2)$ corrections to the form factors, as soon as a second intermediate state $a'b' \neq +-$ is involved. In the case of the P-wave phase shift, there can be no contribution from states with two identical particles due to Bose symmetry. Hence, for δ_P in the specific case $ab = +-$ and for $s \le M_K^2$, the sum boils down to the single $\pi^+\pi^-$ intermediate state, the contribution from form factors drops out altogether and there is no s_ℓ dependence. While Watson's theorem does not apply to the $\delta_S(s,s_\ell)$ phase shift due to the occurrence of two distinct possible intermediate states [$\pi^0\pi^0$ and $\pi^+\pi^-$ for $s \le M_K^2$], it is still operative in the $l=1$ channel. This explains both why the phases of $F_P(s,s_\ell)$ and of $G_P(s,s_\ell)$ are identical, and why this common phase $\delta_P(s)$ actually does not depend on s_ℓ, as indicated in Eq. (3.1).

4. Two-loop representation of $K_{\ell 4}$ form factors

One can derive a representation of the $K_{\ell 4}$ form factors $F^{ab}(s,s_\ell)$ and $G^{ab}(s,s_\ell)$ that holds up to and including two loops in the low-energy expansion, proceeding as in the case of the $\pi\pi$ amplitude in Ref. [15], or as discussed for the scalar form factor of the pion in Ref. [18] (in the isospin limit) and in Ref. [1] (with isospin breaking included). As compared to the latter case, one has to deal with some additional kinematic complexities when addressing the $K_{\ell 4}$ form factors.

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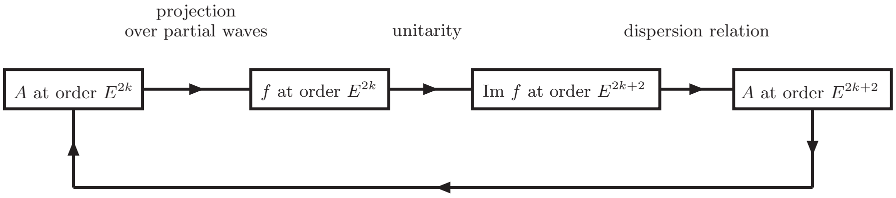

Figure 2: Recursive construction for two-loop representations of the K^+_{e4} form factors and $\pi\pi$ scattering amplitudes in the low-energy regime. A denotes the amplitude of interest and f its partial waves.

As a starting point, we consider fixed-t dispersion relations with two subtractions for all form factors, in all three channels. Assuming the usual analyticity properties for the form factors, with a first cut extending to infinity along part of the real positive s-axis, and a similar second cut along the real negative s-axis, due to the u-channel singularities, we obtain the following dispersion relations

$$\mathbf{A}^{ab}(s,t) = \begin{pmatrix} F^{ab}(s,t) \\ G^{ab}(s,t) \end{pmatrix} = \mathbf{P}^{ab}(t|s,u) + \frac{s^2}{\pi}\int \frac{dx}{x^2}\frac{1}{x-s-i0}\,\mathrm{Im}\mathbf{A}^{ab}(x,t) + \frac{u^2}{\pi}\int \frac{dx}{x^2}\frac{1}{x-u-i0}\,\lambda_a\lambda_c\mathscr{C}_{us}\mathrm{Im}\mathbf{A}^{cb}(x,t). \qquad (4.1)$$

where $\lambda_{a,c}$ and $\mathscr{C}_{us}$ are phases and matrix implementing the expected structure of the $K_{\ell 4}$ form factors under u-crossing, and $\mathbf{P}^{ab}(t|s,u)$ denotes a pair of subtraction functions that are polynomials of the first degree in s and u, with coefficients given by arbitrary functions of t. We may express $\mathrm{Im}\mathbf{A}^{ab}$ in terms of the imaginary parts of the form factors $F_S^{ab}(s)$, $F_P^{ab}(s)$ and g_1^{ab}, and exploit the chiral counting to absorb parts of the dispersive integrals in the (yet unspecified) functions $\mathbf{P}^{ab}$

$$\mathbf{A}^{ab}(s,t,u) = \mathbf{P}^{ab}(s,t,u) + \left[\Phi_+^{ab}(s) - (t-u)\Phi_-^{ab}(s)\right] + \lambda_a\lambda_c\mathscr{C}_{us}\left[\Phi_+^{cb}(u) - (t-s)\Phi_-^{cb}(u)\right] + \lambda_b\lambda_c\mathscr{C}_{st}\left[\Phi_+^{ac}(t) - (s-u)\Phi_-^{ac}(t)\right] + \mathscr{O}(E^6). \qquad (4.2)$$

where the functions Φ_+ and Φ_- are defined through their analyticity properties in the complex s-plane: their singularities are restricted to a cut along the positive real axis, and their discontinuities along this cut are linear combinations of $\mathrm{Im}\, f_0^{ab}(s)$, $\mathrm{Im}\, f_1^{ab}(s)$ and $\mathrm{Im} g_1^{ab}(s)$. Crossing relations can be used to show that $\mathbf{P}^{ab}(s,t,u)$ is a pair of polynomials of at most second order in all three variables s, t, and u, with arbitrary constant coefficients (which may depend on the masses and on s_ℓ).

The low-energy discontinuities are limited to two-meson intermediate states [cf. examples of typical diagrams at one loop shown in Fig. 1] up to and including two-loop order. This provides an iterative set-up to construct the K_{e4} form factors at two loops through a two-step process, as illustrated schematically in Fig. 2. The starting point is provided by the form factors and amplitudes at lowest order. Since these are given by at most first order polynomials in the corresponding Lorentz invariant kinematical variables, the computation of the lowest partial waves required for the one-loop discontinuities is a simple exercise. Likewise, finding the appropriate explicit representation of the one-loop functions with the prescribed discontinuities presents no particular difficulties. Things become less tractable at the second iteration, which requires the partial-wave projections of the one-loop form factors and scattering amplitudes. Ref. [1] used this approach both for the vector and scalar form factors of the pion, as well as for the $\pi\pi$ scattering amplitudes φ and ψ, whereas the case of $K_{\ell 4}$ form factors has been discussed in ref. [2].

5. Isospin breaking in the phases of the two-loop form factors

Since the low-energy $\pi\pi$ scattering amplitudes play a central role in this discussion, we simplify the notation, so that quantities related to the process $\pi^+\pi^- \to \pi^+\pi^-$ ($\pi^0\pi^0 \to \pi^0\pi^0$) will be distinguished by a $+-$ (00) superscript or subscript, e.g. $\varphi_0^{+-}(s) \equiv \varphi_0^{+-;+-}(s)$. For the inelastic channel $\pi^+\pi^- \to \pi^0\pi^0$, we use the superscript/subscript x, so that $\varphi_0^x(s) \equiv \varphi_0^{+-;00}(s)$, for instance. The general formulas (3.4) and (3.5) read, for $4M_\pi^2 \le s \le (M_K - m_\ell)^2$,

$$\begin{aligned}
\delta_S(s,s_\ell) - \delta_0(s) \;=\; & \sigma(s)\left\{\left[\varphi_0^{+-}(s) - \overset{o}{\varphi}{}_0^{+-}(s)\right] + \left[\psi_0^{+-}(s) - \overset{o}{\psi}{}_0^{+-}(s)\right]\right.\\
& \left. -\frac{1}{2}\left[\varphi_0^x(s) - \overset{o}{\varphi}{}_0^x(s)\right] - \frac{1}{2}\left[\psi_0^x(s) - \overset{o}{\psi}{}_0^x(s)\right]\right\}\\
& +\frac{1}{2}\left[\sigma(s) - (1+2\sqrt{3}\varepsilon_2)\sigma_0(s)\right]\left[\varphi_0^x(s) + \psi_0^x(s)\right]\\
& +\frac{1}{2}\sigma_0(s)\varphi_0^x(s)\left[(1+2\sqrt{3}\varepsilon_2)\mathrm{f}_0^{+-}(s,s_\ell) - \mathrm{f}_0^{00}(s,s_\ell)\right] + \mathscr{O}(E^6), \qquad (5.1)\\
\delta_P(s) - \delta_1(s) \;=\; & \sigma(s)\left\{\left[\varphi_1^{+-}(s) - \overset{o}{\varphi}{}_1^{+-}(s)\right] + \left[\psi_1^{+-}(s) - \overset{o}{\psi}{}_1^{+-}(s)\right]\right\} + \mathscr{O}(E^6). \qquad (5.2)
\end{aligned}$$

The phase-space factors for two charged or two neutral pions are $\lambda_{+-}^{\frac{1}{2}}(s) = s\sigma(s)$, and $\lambda_{00}^{\frac{1}{2}}(s) = s\sigma_0(s)$, and for any quantity A, $\overset{o}{A}$ denotes its counterpart in the isospin limit. We have used $F_{S[0]}^{00}/F_{S[0]}^{+-} = -(1+2\sqrt{3}\varepsilon_2)$ (ε_2 being related to $\pi\eta$ mixing), and denoted $F_{S[2]}^{+-}(s,s_\ell) = F_{S[0]}^{+-}\cdot \mathrm{f}_0^{+-}(s,s_\ell)$ and $F_{S[2]}^{00}(s,s_\ell) = -F_{S[0]}^{+-}\cdot \mathrm{f}_0^{00}(s,s_\ell)$, so that f_0^{00} and f_0^{+-} combine F- and G-form factor partial waves.

In agreement with Ref. [1], isospin-breaking effects take place in the S-wave phase shift through two types of contributions: the first two lines in Eq. (5.1) are universal as they depend only on $\pi\pi$ (re)scattering, whereas the last two are process-dependent as they involve isospin-breaking in the $K_{\ell 4}$ form factors. For the third term, this dependence is not as explicit as for the last one, but one should recall that the factor $-(1+2\sqrt{3}\varepsilon_2)$ originates from the ratio $F_{S[0]}^{00}/F_{S[0]}^{+-}$. On the other hand, isospin breaking in the P-wave phase shift Eq. (5.2) is indeed universal. In order to relate the data from $K_{e4}^{\pm}$ decays to the $\pi\pi$ phases shifts $\delta_0(s) - \delta_1(s)$ in the isospin limit, we evaluate the isospin-breaking correction $\Delta_{\mathrm{IB}}(s,s_\ell) = [\delta_S(s,s_\ell) - \delta_0(s)] - [\delta_P(s) - \delta_1(s)]$, at next-to-leading order. This requires the determination of the partial-wave projections $\mathrm{f}_0^{+-}(s,s_\ell)$ and $\mathrm{f}_0^{00}(s,s_\ell)$ of the $K_{\ell 4}$ form factors and the $\pi\pi$ partial waves $\varphi_{0,1}^{+-}(s)$, $\varphi_0^x(s)$, $\psi_{0,1}^{+-}(s)$, and $\psi_0^x(s)$.

The iterative procedure described above allows one to describe these quantities in terms of a large set of subtraction constants not fixed by the general properties (unitarity, analyticity, chiral counting) on which we have built our approach. Additional information must be provided on these quantities, which is obtained by matching the expression of the subtraction constants onto χPT: *a)* The quantities related to the $\pi\pi$ partial-wave projections, $\varphi_{0,1}^{+-}(s), \varphi_0^x(s)$ or $\psi_{0,1}^{+-}(s), \psi_0^x(s)$ can be expressed in terms of the corresponding threshold parameters, which can be related to the two S-wave scattering a_0^0 and a_0^2 in the isospin limit using the results of Ref. [1]. *b)* For the other lowest-order two-meson scattering amplitudes contributing to the real parts of the form factors at one-loop, we have used leading-order expressions from three-flavour χPT [2] c) Finally, for the subtraction constants for the $K_{\ell 4}$ form factors, we have matched the dispersive representation with $N_f = 3$ χPT expressions.

For our numerical analysis, we use the inputs for strong and electromagnetic low-energy constants described in Refs. [1, 2]. We assume (as already done in Ref. [14]) that the low-energy constants involved in the theory without virtual photons are identical to those in the full theory. This identification induces a systematic theoretical error whose size is difficult to assess, but which will be assumed to be small compared to the other sources of uncertainties. We have varied a_0^0 in the range [0.18,0.30] and a_0^2 in the Universal Band obtained from the analysis of Roy equations and the compatibility with high-energy inputs. The main contribution to $\Delta_{\rm IB}$ can be seen as coming, on the one hand, from pure phase-space effects which dominates in the low-energy region, and on the other hand, from the significant (especially at higher energies) universal contribution and the form-factor dependent one, with opposite signs. As in the case of the scalar and vector pion form factors [1], the form-factor dependent part tends to decrease the size of the correction, and a significant cancellation takes place between the universal and non-universal contributions to isospin breaking in the two-loop phase shifts. The contributions to $\Delta_{\rm IB}(s,s_\ell)$ from the P-wave term are completely universal and very small, in agreement with Ref. [1]. At large s, the correction is reduced compared to the leading-order results, and varies significantly in the (a_0^0,a_0^2) plane, as illustrated in Fig. 3, which can be compared to Ref. [14]. Going away from $s_\ell = 0$ does not change the above picture. The dependence on the dilepton invariant mass s_ℓ comes from the partial-wave projection of the form factors f_0^{00} and f_0^{+-}, but this dependence is very mild: varying over the allowed phase space $0 \le s_\ell \le (M_{K^+} - \sqrt{s})^2$ changes $\Delta_{\rm IB}(s,s_\ell)$ by less than 1%.

6. Re-analysis of NA48/2 results

We can use our computation of the isospin-breaking correction $\Delta_{\rm IB}(s,s_\ell)$ as a function of the two scattering lengths a_0^0 and a_0^2 to perform an analysis of the available phase shifts from the NA48/2 experiment [10]. We proceed along the lines of Ref. [9], using the same solutions of the Roy equations in the isospin limit, and correcting the measured phase shifts based on the interference between S and P waves ($S-P$ fit). Actually, the S-P interference from the $K_{e4}^{\pm}$ angular analysis provides a strong correlation between a_0^0 and a_0^2, but a weaker constraint on each of them separately. We can circumvent this problem by performing the extended fit described in Ref. [9], where we supplement the NA48/2 data set with information from the $I = 2$ S wave [3] in order to constrain each of the two scattering lengths more tightly (extended fit). The results of these analyses are shown in Fig. 3 and summarised in Tab. 1. We perform the analysis either with or without isospin-breaking corrections. In the first case, our results agree with the NA48/2 collaboration for the S-P fit (so-called Model B in Ref. [10]: $a_0^0 = 0.222 \pm 0.013$ and $a_0^2 = -0.043 \pm 0.009$) but with slightly larger errors once isospin-breaking corrections are included. This is not surprising

[2] This might not look quite at the same level of generality as in the case of the $\pi\pi$ amplitudes. In some cases, like for instance πK scattering, we could have used existing phenomenological information [19]. However, the numerical weight of all these contributions is quite small, well below the level of the uncertainties generated by the other terms.

[3] The isospin-breaking corrections attached to the $I = 2$ channel cannot be estimated in our framework but are certainly subleading compared to the large uncertainties for this set of data.

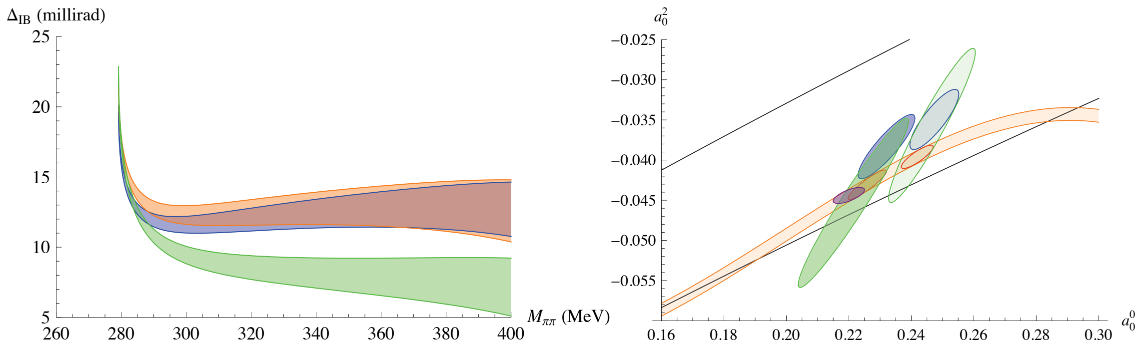

Figure 3: On the left: Isospin breaking in the phase of the two-loop form factors, $\Delta_{\rm IB}(s,s_\ell)$, as a function of the dipion invariant mass $M_{\pi\pi}=\sqrt{s}$, for $s_\ell=0$. The middle (light-blue) band corresponds to the $(a_0^0,a_0^2)=(0.182,-0.052)$, whereas the other two cases shown correspond to $(a_0^0,a_0^2)=(0.205,-0.055)$ (upper orange band) and to $(a_0^0,a_0^2)=(0.24,-0.035)$ (lower green band). On the right: Results of the fits to the NA48/2 data in the (a_0^0,a_0^2) plane. The two black solid lines indicate the universal band where the two S-wave scattering lengths comply with dispersive constraints (Roy equations) and high-energy data on $\pi\pi$ scattering. The orange band is the theoretical constraint coming from the scalar radius of the pion [8]. The small dark (purple) ellipse represents the prediction based on $N_f=2$ χPT [8]. The three other ellipses on the left represent, in order of increasing sizes, the 1-σ ellipses corresponding to the scalar (orange ellipse [8]), S-P (blue ellipse) and extended (green ellipse) fits, respectively, including isospin-breaking corrections. The light-shaded ellipses on the right represent the same outputs, without isospin-breaking corrections.

since our isospin-breaking correction varies with a_0^0 and a_0^2. Once isospin-breaking corrections are included, the mild discrepancy previously observed between the two fits [9] is recovered, whereas the larger uncertainty of the S-P fit covers both solutions. By comparing the dispersive and chiral descriptions of the low-energy $\pi\pi$ amplitude in the isospin limit following Refs. [15, 16, 9], we can extract the $N_f=2$ chiral low-energy constants $\bar\ell_3,\bar\ell_4$, or equivalently the two-flavour quark condensate $\Sigma(2)=-\lim_{m_u,m_d\to 0}\langle 0|\bar uu|0\rangle$ and pion decay constant $F(2)=\lim_{m_u,m_d\to 0}F_\pi$ measured in physical units (called $X(2)$ and $Z(2)$). As shown in Tab. 1, the minor difference in a_0^2 between the two fits yields significant differences in the estimate of the $N_f=2$ order parameters and low-energy constants. For comparison, we also show the results obtained without including the isospin corrections.

A natural extension of our work would consist in working out not only the phases, but also the real parts of the $K_{\ell 4}$ form factors, in order to compute isospin breaking in these quantities which are experimentally available. A full analytical treatment seems out of reach, but the outcome would involve a limited number of one-dimensional dispersive integrals amenable to a numerical treatment. For instance, it would provide a theoretical framework suitable to analyse the cusp recently observed by the NA48/2 experiment in $K^\pm\to\pi^0\pi^0 e^\pm\nu_e$ [20].

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	With isospin-breaking corrections		Without isospin-breaking corrections	
	S-P	Extended	*S-P*	Extended
a_0^0	0.221 ± 0.018	0.232 ± 0.009	0.247 ± 0.014	0.247 ± 0.008
a_0^2	-0.0453 ± 0.0106	-0.0383 ± 0.0040	-0.0357 ± 0.0096	-0.0349 ± 0.0038
$\rho_{a_0^0,a_0^2}$	0.964	0.881	0.945	0.842
χ^2/N	7.6/6	16.6/16	7.2/6	15.7/16
$\bar{\ell}_3$	3.15 ± 9.9	-10.2 ± 5.7	-39.9 ± 20.3	-43.5 ± 19.1
$\bar{\ell}_4$	5.3 ± 0.8	4.4 ± 0.6	5.2 ± 0.8	5.2 ± 0.7
$X(2) = \frac{2m\Sigma(2)}{F_\pi^2 M_\pi^2}$	0.88 ± 0.05	0.80 ± 0.06	0.72 ± 0.05	0.71 ± 0.05
$Z(2) = \frac{F^2(2)}{F_\pi^2}$	0.87 ± 0.03	0.89 ± 0.02	0.87 ± 0.02	0.87 ± 0.02

Table 1: Scattering lengths and chiral low-energy constants for the fits to NA48/2 data only ($S-P$) or together with data on $I = 2$ channel (Extended), with and without the isospin-breaking correction Δ_{IB}.

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