PROCEEDINGS OF SCIENCE

Magnetized Branes and the Six-Torus

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In the framework of Type IIB String Theory compactified on a general six-torus T^6 with arbitrary complex structure, Yukawa couplings are determined for the chiral matter described by open strings ending on D9-branes having different oblique magnetization.

Proceedings of the Corfu Summer Institute 2012 "School and Workshops on Elementary Particle Physics and Gravity"
September 8-27, 2012
Corfu, Greece

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[†]XXIII Ciclo di Dottorato in Fisica Fondamentale e Applicata

PoS(Corfu2012)108

1. Introduction

Many new ideas and tools have been introduced in the last years in order to connect the ten-dimensional Superstring Theory to the four-dimensional Standard Model. This is the goal of the so-called *String Phenomenology* which in fact studies how the (beyond) Standard Model physics could be obtained as a low-energy limit of String Theory [1, 2, 3].

In orientifolds of Type II String Theory, the main ingredients that String Phenomenology uses for achieving its aim are Dp-branes together with the compactification of the extra dimensions and the derivation of chiral spinors. In particular, the gauge groups of the Standard Model are localized in the world-volume of suitable configurations of Dp-branes and the chiral matter can be introduced by "dressing" the compact directions with magnetic fields [4, 5, 6, 7, 8, 9]. The open strings ending on different piles of branes with different magnetizations are named *dy-charged* or *twisted* strings and they exactly describe the chiral matter of the low-energy theory.

In order to promote magnetized branes in a compact space as vacua interesting for the String Phenomenology, one needs to have information about the low-energy effective actions living in the world-volume of such configurations of branes [10, 11, 12]. These actions are model dependent and a particular interest, especially after the Higgs discovery, is focused on the determination of the Yukawa couplings and their dependence on the details of the compact manifold, i.e. moduli and geometry of the extra dimensions. Yukawa couplings in Type II brane-world scenario arise from an overlap integral of wave-functions of the three participant fields in the extra dimensions. The wave-functions, depending on the Bose-Fermi statistics of the fields, are solutions of the internal Laplace-Beltrami or Dirac equation with suitable boundary conditions dictated by the compact geometry and by the presence of the magnetic fields. In the *bottom-up* approach one usually neglects the global aspects of the compactification and solves these equations locally. However, for the simplest compact manifold, the factorized torus, the boundary conditions imposed by the magnetized torus geometry have exactly determined the holomorphic part of the Yukawa couplings that turns out to be proportional to the Jacobi θ-function [13, 14, 15, 16, 17]. The global properties of the compact manifold are then important to fix the complete structure of the effective actions and it results to be interesting to compute such couplings in the case of non factorized geometries as the one of the torus T^6 with arbitrary complex structure.

In this talk - which is based on the paper of ref. [18] - these couplings are studied in a configuration of M D9-branes in the background $\mathbb{R}^{1,3} \times T^6$. In the same spirit as the one of ref. [13] (see also [16, 19]), constant magnetic fields are turned on, along the compact directions, in the abelian sector of the $U(M)$ gauge group defined on the world-volume of the M branes. Depending on the choice of such constant fields, the single stack of branes is now separated in different piles of magnetized branes. The ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory living in the world-volume of a stack of D9-branes is dimensionally reduced to four dimensions by expanding the ten-dimensional bosonic or fermionic fields in a basis of eigenfunctions of the internal Laplace or Dirac operator. The eigenfunctions of these operators have to be invariant, up to gauge transformations, when translated along the one-cycles of the torus. They are easily determined in the complex frame where both the metric and the difference $F^{ab} = F^a - F^b$ of the magnetic fields on the two piles a and b of branes between which the strings are stretched, are diagonal matrices in their off-diagonal boxes. In this frame, supersymmetry has been also partially imposed by requiring the field F^{ab} to be a $(1,1)$-form in the coordinate system defining the complex torus. The wave-functions of twisted open strings turn out to be proportional to the Riemann Theta function only when the background gauge field, in the original system of coordinates defining the torus, is a matrix with null diagonal blocks. They depend on the first Chern class I_{ab} associated with the difference of the gauge fields on the a and b branes and on a generalized complex structure that is a matrix whose entries are related to the original complex structure of the torus or to its complex conjugate, depending on the signs of the eigenvalues of the non-vanishing blocks of the gauge field F^{ab}.

The Yukawa couplings are obtained by evaluating an overlap integral over three of such functions. The integral is computed after using an identity between the product of two Riemann θ-functions. The identity has been derived in refs. [18, 20] by extending the analysis given in ref. [21] and here revised. The resulting expression is compatible with the known results obtained under different assumptions [22, 20]; the non trivial and holomorphic part of these couplings, the Riemann θ-function, is again determined by the boundary conditions due to the geometry of the magnetized torus. Here holomorphicity means, as in the factorized case, that the θ-function can never depend on a variable and its complex conjugate, although the argument of such a function can be either holomorphic or antiholomorphic along different directions. These properties are related to the signs of the first Chern classes evaluated along the corresponding compactified directions of the torus.

The paper is organized as follows.

In section 2, generalities about dimensional reduction and magnetic fluxes are given. In section 3, the bosonic and fermionic wave-functions for the lowest states are derived together with the mass spectrum of the Kaluza-Klein states. In section 4, the Yukawa couplings for a general magnetized six-torus T^6 are computed. Finally, in the appendix details about the proof of an identity involving the product of two wave-functions are given.

2. Open Fluxes and Torus Geometry

A configuration made of a stack of M D9-branes in the compact background $\mathbb{R}^{1,3}\times T^6$ is going to be studied in this paper. Branes backreaction on the space-time geometry is neglected and the analysis is focused on the open string degrees of freedom. Their interaction with the closed string degrees of freedom is described by the supersymmetric DBI and by the Chern-Simons actions. In the following, attention will be drawn to the low-energy limit of the DBI action which, for this particular brane configuration, is the ten-dimensional $\mathscr{N}=1$ super Yang-Mills with gauge group $U(M)$:

$$S=\frac{1}{g^2}\int d^{10}X^{\hat{N}}\,\mathrm{Tr}\Big(-\frac{1}{4}\mathscr{F}_{\hat{M}\hat{N}}\mathscr{F}^{\hat{M}\hat{N}}+\frac{i}{2}\bar{\lambda}\Gamma^{\hat{M}}D_{\hat{M}}\lambda\Big) \tag{2.1}$$

where $\hat{M},\hat{N}=0,\dots,9,\;\; g^2=4\pi e^{\phi_{10}}(2\pi\sqrt{\alpha'})^6$ and

$$\mathscr{F}_{\hat{M}\hat{N}}=\nabla_{\hat{M}}A_{\hat{N}}-\nabla_{\hat{N}}A_{\hat{M}}-i[A_{\hat{M}},A_{\hat{N}}]\;\;;\;\;D_{\hat{M}}\lambda=\nabla_{\hat{M}}\lambda-i[A_{\hat{M}},\lambda]$$

with λ being a ten-dimensional Weyl-Majorana spinor. Chiral matter is introduced by turning on magnetic fields with constant field strength along the compact directions of the world-volume of

N_a branes, with $\sum_{a=1}^{n} N_a = M$. The integer n labels the branes having different magnetization. The original gauge group is then broken into the product $U(M) \simeq \prod_{a=1}^{n} U(N_a)$ and this breaking can be used to engineer the gauge groups of the Standard Model. The chiral matter is given by the twisted open strings charged with respect two of these groups and transforms in the bifundamental representation of the gauge group $U(N_a) \times U(N_b)$. In the following, the complete breaking $U(M) \simeq U(1)^M$ is considered, but the extension to other gauge configurations is straightforward. The breaking is realized by first separating the generators U_a of the Cartan subalgebra from the ones out of it, e_{ab}, in the definitions of the gauge field and of the gaugino:

$$A_{\hat{M}} = B_{\hat{M}} + W_{\hat{M}} = B^a_{\hat{M}} U_a + W^{ab}_{\hat{M}} e_{ab} \;\; ; \;\; \lambda = \chi + \Psi = \chi^a U_a + \Psi^{ab} e_{ab}$$

and then expanding the Lagrangian around the background fields which are present only along the compact directions in T^6 of the branes:

$$\begin{aligned} B^a_M(x^\mu, X^N) &= \langle B^a_M \rangle (X^N) + \delta B^a_M(x^\mu, X^N) \\ W^{ab}_M(x^\mu, X^N) &= 0 + \Phi^{ab}_M(x^\mu, X^N) \,. \end{aligned} \tag{2.2}$$

Here $\mu = 0, \ldots, 3$ and $M, N = 1, \ldots, 6$. The fields B^a_M and Φ^{ab}_M are, respectively, adjoint and chiral scalars, from the point of view of the four-dimensional Lorentz group. The background fields $\langle B^a_M \rangle$ are taken with a constant field strength corresponding to the background constant magnetic fields along the compact dimensions. In particular, the gauge

$$\langle B^a_M \rangle (X^N) = -\frac{1}{2} F^a_{MN} X^N$$

is chosen.

The four-dimensional effective action is obtained by compactifying the extra dimensions on the torus T^6, defined by imposing the identification

$$x^m \equiv x^m + 2\pi R_1^{(m)} m_1^m \;\; ; \;\; y^m \equiv y^m + 2\pi R_2^{(m)} m_2^m \quad m_1^m, m_2^m \in \mathbb{Z}$$

on the space-time coordinates $(x^m, y^m) \equiv (X^{2m-1}, X^{2m})$ $(m = 1, 2, 3)$, being $R_1^{(m)}$ and $R_2^{(m)}$ the radii of the torus along the direction m. In the following, in order to compare the string with the field theory results, it is convenient to use the following rescaling:

$$(x^m, y^m) \to \left(x^m \frac{R_1^{(m)}}{R}, \, y^m \frac{R_2^{(m)}}{R} \right),$$

with R being an arbitrary dimensionful parameter, and then to define the torus geometry through the identification:

$$x^m \equiv x^m + 2\pi R m_1^m \;\; ; \;\; y^m \equiv y^m + 2\pi R m_2^m \,.$$

The description of the torus as a complex manifold is based on the introduction of the coordinates:

$$w^m = \frac{x^m + U^m_{\ n} y^n}{2\pi R}$$

together with their complex conjugate variables. Here, U is a complex matrix parametrizing the complex structure of the manifold. The lattice identification is given by

$$w^m \equiv w^m + m_1^m + U^m{}_n m_2^n \,.$$

The twisted sectors of the theory, as previously discussed, are present because a background magnetic field $F^{ab} = F^a - F^b$ acts on the world-volume of two piles of branes a and b. In the system of complex coordinates it takes the following form [18]:

$$F^{ab} = -\frac{(2\pi R)^2}{8} F^{(\mathscr{W}\mathscr{W})ab}_{MN} d\mathscr{W}^M \wedge d\mathscr{W}^N \tag{2.3}$$

with $(\mathscr{W}^1, \ldots, \mathscr{W}^6) \equiv (w^1, \ldots, w^3, \bar{w}^1, \ldots \bar{w}^3)$ and

$$F^{(ww)ab} = (\mathrm{Im}U^{-1})^t \left[\bar{U}^t F^{(xx)ab}\bar{U} - \bar{U}^t F^{(xy)ab} + F^{(xy)ab\,t}\bar{U} + F^{(yy)ab}\right] \mathrm{Im}U^{-1}$$
$$F^{(w\bar{w})ab} = (\mathrm{Im}U^{-1})^t \left[-\bar{U}^t F^{(xx)ab}U + \bar{U}^t F^{(xy)ab} - F^{(xy)ab\,t}U - F^{(yy)ab}\right] \mathrm{Im}U^{-1}$$

while $F^{(\bar{w}w)ab} = [F^{(w\bar{w})ab}]^*$ and $F^{(ww)ab} = [F^{(\bar{w}\bar{w})ab}]^*$; furthermore the index t denotes the transposed of the matrices it refers to. Supersymmetric configurations require the gauge field to be a $(1,1)$-form. Imposing such constraint necessarily makes $iF^{(w\bar{w})}$ an Hermitian matrix [20] which is diagonalized by an unitary matrix $\bar{C}^{-1}_{ab}$:

$$(C^{-1}_{ab})^m{}_r \, F^{(w,\bar{w})ab}_{m\bar{n}} \; (\bar{C}^{-1}_{ab})^{\bar{n}}{}_{\bar{s}} = \frac{2}{i}\frac{\lambda_r^{ab}}{(2\pi R)^2}\delta_{r\bar{s}} \,.$$

Here, $r,s = 1,\ldots 3$ and, since $\bar{C}^{-1}_{ab}$ is an unitary matrix, one has $(C^{-1}_{ab})^m{}_r h_{m\bar{n}} (\bar{C}^{-1}_{ab})^{\bar{n}}{}_{\bar{s}} = \delta_{rs}$ where $h_{m\bar{n}}$ refers to the metric of the complex torus which can also be written in terms of the holomorphic and anti-holomorphic vielbeins: $h_{m\bar{n}} = e^r{}_{\bar{m}} \delta_{r\bar{s}} \bar{e}^{\bar{s}}{}_{\bar{n}}$. Complex coordinates having a trivial metric can now be introduced by defining: $w^r = e^r{}_m w^m$ together with their complex conjugate.

The diagonalization naturally introduces a new system of complex coordinates $(\mathscr{Z}^1_{ab}, \ldots, \mathscr{Z}^6_{ab}) = (z^1_{ab}, \ldots, z^3_{ab}, \bar{z}^1_{ab}, \ldots, \bar{z}^3_{ab})$, defined by

$$w^m = \left(C^{-1}_{ab}\right)^m{}_r z^r_{ab} \;\; ; \;\; \bar{w}^m = \left(\bar{C}^{-1}_{ab}\right)^{\bar{s}}{}_{\bar{r}} \bar{z}^r_{ab} \,.$$

In this frame both the metric and the field strengths of the gauge field are diagonal matrices in the non-vanishing blocks, being the metric equal to:

$$\frac{ds^2}{(2\pi R)^2} = \frac{1}{2} d\mathscr{Z}^I_{ab} \mathscr{G}_{IJ} d\mathscr{Z}^J_{ab} \;\; ; \;\; \mathscr{G} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

while the magnetic field strength is

$$F^{ab} = \frac{1}{2} F^{(\mathscr{Z}\mathscr{Z})ab}_{IJ} d\mathscr{Z}^I_{ab} \wedge d\mathscr{Z}^J_{ab} \;\; ; \;\; F^{(\mathscr{Z}\mathscr{Z})ab}_{IJ} = \frac{i}{2} \begin{pmatrix} 0 & \mathscr{I}^{ab}_\lambda \\ -\mathscr{I}^{ab}_\lambda & 0 \end{pmatrix} \tag{2.4}$$

with

$$\mathscr{I}^{ab}_\lambda = \mathrm{diag}\left(\lambda_1^{ab} \ldots \lambda_d^{ab}\right)$$

and $I,J = 1,\ldots,6$ denote the flat indices.

3. The Wave-Functions.

The quadratic terms in the scalar fields of the four-dimensional action, derived in detail in ref. [13], see also [15, 16], are obtained by starting from eq. (2.1) and expanding the fields, defined in the second line of eq. (2.2), in a basis of eigenfunctions of the internal Laplace-Beltrami operator:

$$-\tilde{D}_N\tilde{D}^N\phi^{ab}_{\mathscr{M}}(X^N) = m^2_{\mathscr{M}}\phi^{ab}_{\mathscr{M}}(X^N) \ ; \ \Phi^{ab}_M = \sum_{\mathscr{M}} \varphi^{ab}_{M,\mathscr{M}}(x^\mu)\otimes\phi^{ab}_{\mathscr{M}}(X^N)$$

with suitable boundary conditions determined by the torus geometry. Here, the covariant derivative depends only on the constant background gauge fields

$$\tilde{D}_N\phi^{ab}_{\mathscr{M}} = \partial_N\phi^{ab}_{\mathscr{M}} - i(\langle B^a_N\rangle - \langle B^b_N\rangle)\phi^{ab}_{\mathscr{M}} \ .$$

The mass spectrum of the Kaluza-Klein states is easily determined in the system of the $\mathscr{Z}$-coordinates. In such a frame the mass operator is [18]:

$$\left[M^2_{\mathscr{M}}\right]^{ab} \equiv \ \mathrm{diag}\ (\tilde{m}^{2ab}_{\mathscr{M}}\,\mathbb{I} - 2\,\mathscr{I}^{ab}_{\lambda}, \tilde{m}^{2ab}_{\mathscr{M}}\mathbb{I} + 2\,\mathscr{I}^{ab}_{\lambda})$$

with $\tilde{m}^{ab}_{\mathscr{M}} = 2\pi R\, m^{ab}_{\mathscr{M}}$ while, due to the block diagonal expression of the background gauge field, the commutation relations $[\tilde{D}^{(\mathscr{Z})}_I, \tilde{D}^{(\mathscr{Z})}_J] = -i F_{IJ}$ of the covariant derivatives reduce to the algebra of decoupled creation and annihilation bosonic operators. This identification depends on the signs of the eigenvalues λ_r, being for positive λ_r [1]:

$$a^\dagger_r = \sqrt{\frac{2}{|\lambda_r|}}\, i\tilde{D}^{(\mathscr{Z})}_r \ ; \ a_r = \sqrt{\frac{2}{|\lambda_r|}}\, i\tilde{D}^{(\mathscr{Z})}_{r+d} \tag{3.1}$$

with the role of the creation and annihilation operators exchanged for negative λ_r. In both cases one has $[a_r, a^\dagger_r] = 1$ and the Laplace equation becomes

$$\sum_{r=1}^{3}|\lambda_r|(2N_r+1)\phi_{\mathscr{M}} = \tilde{m}^2_{\mathscr{M}}\phi_{\mathscr{M}} \ ; \ N_r = a^\dagger_r a_r \ .$$

The eigenvalues of the mass operator result to be:

$$M^2_{\pm;s} = \sum_{r=1}^{3}|\lambda_r|(2N_r+1) \mp 2\lambda_s \ . \tag{3.2}$$

The lightest state is massless if the $\mathscr{N}=1$ susy condition $|\lambda_r|+|\lambda_s| = |\lambda_t|$ $(r \neq s \neq t)$ is imposed. Then, by applying creation operators on the massless state, two towers of Kaluza-Klein states are generated. Their spectrum, when the $\mathscr{N}=1$ susy condition is imposed, is contained in the expression:

$$M^2_k = 2\sum_{r=1}^{3}|\lambda_r|(N_r+k) \ ; \ k = 0, 1 \ . \tag{3.3}$$

[1] In this analyis the a, b labels are omitted when possible.

The eigenfunctions relative to the ground state are obtained by solving the first order differential equation

$$a_r \phi_0 = 0 \,\forall r \quad \Leftrightarrow \quad \left(\frac{\partial}{\partial \bar{\mathrm{z}}^r} + \frac{1}{4}|\lambda_r|\,\mathrm{z}^r \right) \phi_0 = 0$$

where the complex coordinates $(\mathrm{z}^1, \ldots, \mathrm{z}^3, \bar{\mathrm{z}}^1 \ldots, \bar{\mathrm{z}}^3)$, with

$$\mathrm{z}^r = z^r \left(\frac{1+\mathrm{sign}(\lambda_r)}{2}\right) + \bar{z}^r \left(\frac{1-\mathrm{sign}(\lambda_r)}{2}\right) = (C^{(\lambda)})^r{}_m \left(\frac{x^m + \Omega^m{}_n y^n}{2\pi R} \right) , \tag{3.4}$$

have been introduced in order to take into account that the identification between the covariant derivatives and the creations or annihilations operators depends on the signs of the eigenvalues λ_r. In eq. (3.4)

$$\begin{aligned} C^{(\lambda)^r} &= \left(\frac{1+\mathrm{sign}(\lambda_r)}{2} \right) C^r + \left(\frac{1-\mathrm{sign}(\lambda_r)}{2} \right) \bar{C}^r = \bar{C}^{(-\lambda)\, r} \\ \tilde{C}^{(\lambda)\, r} &= \left(\frac{1+\mathrm{sign}(\lambda_r)}{2} \right) C^r U + \left(\frac{1-\mathrm{sign}(\lambda_r)}{2} \right) \bar{C}^r \bar{U} = \bar{\tilde{C}}^{(-\lambda)\, r} \end{aligned} \tag{3.5}$$

and $\Omega = (C^{(\lambda)})^{-1} \tilde{C}^{(\lambda)}$ is named the *generalized complex structure* because, when all the λ_rs have the same sign, it coincides with the complex structure of the torus or its complex conjugate.

The wave-function of the ground state is:

$$\phi_0 = e^{-\frac{1}{4} \vec{\bar{\mathrm{Z}}}^t |\mathscr{I}_\lambda| \vec{\mathrm{Z}} + \frac{1}{4} \vec{\mathrm{Z}}^t C^{(\lambda)^{-t}} \bar{C}^{(\lambda)t} |\mathscr{I}_\lambda| \vec{\mathrm{Z}}} \theta(\vec{\mathrm{z}}) \tag{3.6}$$

where $(Z^1, \ldots, Z^6) \equiv (\mathrm{z}^1, \ldots, \mathrm{z}^3, \bar{\mathrm{z}}^1 \ldots, \bar{\mathrm{z}}^3)$ and with $\theta(\vec{\mathrm{z}})$ being an holomorphic function of the coordinates which is determined by the boundary conditions. It is interesting to notice that the wave-function (3.6), when rewritten in the original system of coordinates $\mathscr{Z}^I = (z^r, \bar{z}^r)$, may depend on both the holomorphic and anti-holomorphic variables. However, it never simultaneously depends on a variable and its complex conjugate, i.e. on z^r and $\bar{z}^r$ (same r) and, therefore θ is an holomorphic function of the complex coordinates.

Boundary conditions are dictated by the transformation properties of the scalar fields under the torus translations [23, 24]. The behavior of the vector potential $A_r^{(\mathrm{Z})}$ under the lattice translations

$$A_r^{(\mathrm{Z})}(\bar{\mathrm{z}} + \bar{C}^{(\lambda)} \eta_{(s)}) \equiv A_r^{(\mathrm{Z})}(\bar{\mathrm{z}}) + \partial_r \chi^{(1)}_{(s)} \;\; ; \;\; A_r^{(\mathrm{Z})}(\bar{\mathrm{z}} + \bar{C}^{(\lambda)}\, \Omega\, \eta_{(s)}) \equiv A_r^{(\mathrm{Z})}(\bar{\mathrm{z}}) + \partial_r \chi^{(2)}_{(s)}$$

defines the corresponding gauge transformations

$$\begin{aligned} \chi^{(1)}_{(s)} &= -\frac{i}{4} \mathrm{z}^r |\lambda_r| (\bar{C}^{(\lambda)})^r{}_n \eta^n_{(s)} + \frac{i}{4} \bar{\mathrm{z}}^r |\lambda_r| (C^{(\lambda)})^r{}_n \eta^n_{(s)} \\ \chi^{(2)}_{(s)} &= -\frac{i}{4} \mathrm{z}^r |\lambda_r| (\bar{C}^{(\lambda)})^r{}_m \bar{\Omega}^n{}_m\, \eta^n_{(s)} + \frac{i}{4} \bar{\mathrm{z}}^r |\lambda_r| (C^{(\lambda)})^r{}_m \Omega^m{}_n \eta^n_{(s)} \end{aligned}$$

with $\eta^t_{(s)} = (\overbrace{0, \ldots, 0, 1}^{s\ times}, 0, \ldots)$. The holomorphic function appearing in the definition of the ground state is determined by imposing the identifications

$$\begin{aligned} \phi_0(\vec{\mathrm{z}} + C^{(\lambda)} \eta_{(s)}, \vec{\bar{\mathrm{z}}} + \bar{C}^{(\lambda)} \eta_{(s)}) &= e^{i\chi^{(1)}_{(s)}} \phi_0(\vec{\mathrm{z}}, \vec{\bar{\mathrm{z}}}) \\ \phi_0(\vec{\mathrm{z}} + C^{(\lambda)}\, \Omega\, \eta_{(s)}, \vec{\bar{\mathrm{z}}} + \bar{C}^{(\lambda)}\, \bar{\Omega}\, \eta_{(s)}) &= e^{i\chi^{(2)}_{(s)}} \phi_0(\vec{\mathrm{z}}, \vec{\bar{\mathrm{z}}}). \end{aligned} \tag{3.7}$$

The full wave-function of the ground state, in the real coordinates system and in the case $F^{(xx)} = F^{(yy)} = 0$, is

$$\phi_0 \equiv \phi^{ab}_{\Omega_{ab};\vec{j}}(x^m, y^m) = \mathscr{N}_{ab}\, e^{i\vec{y}^t \frac{I_{ab}}{(2\pi R)^2}\vec{x} + i\vec{y}^t \Omega^t_{ab} \frac{I^t_{ab}}{(2\pi R)^2}\vec{y}} \sum_{\vec{n}\in\mathbb{Z}^d} e^{i\pi(\vec{n}+\vec{j})^t I_{ab}\Omega_{ab}(\vec{n}+\vec{j}) + 2i\pi(\vec{n}+\vec{j})^t I_{ab}\left(\frac{\vec{x}+\Omega_{ab}\vec{y}}{2\pi R}\right)} \tag{3.8}$$

where the overall constant $\mathscr{N}_{ab} = \sqrt{2g}V^{-1/2}_{T^{2d}}\left[\det(I_{ab}\mathrm{Im}\Omega_{ab})\right]^{1/4}$ is fixed by requiring canonical normalization for the kinetic terms of the scalars. Here, $V_{T^{2d}}$ is the torus volume and $I = 2\pi R^2 F^{(xy)t}$.

The wave-function (3.8) can be easily compared with the corresponding expression given in ref. [20] for the torus T^4. The two expressions coincide if the generalized complex structure here introduced is identified with the modular matrix $i\hat{\Omega}$ defined in that reference. It can also be compared with the one given for the chiral scalars in the case of the factorized torus $(T^2)^d$[14] and the two coincide when eq. (3.8) is specified for this peculiar factorized geometry.

The wave function for the four-dimensional fermions is the solution of the internal Dirac equation:

$$\gamma^M_{(6)}\tilde{D}_M \eta^{ab}_n = m_n \eta^{ab}_n\,, \tag{3.9}$$

being m_n the mass of nth-level of the Kaluza-Klein tower and $\gamma^M_{(6)}$ are the six-dimensional Dirac matrices. In analogy with the dimensional reduction of the bosonic kinetic terms, the eigenfunction problem of the Dirac equation is solved in the complex frame $\mathscr{Z}$ where both the metric and the magnetic background are diagonal matrices in the non-vanishing off-diagonal blocks. In this complex frame, the Clifford algebra becomes:

$$\left\{\gamma^{\mathscr{Z}^r}, \gamma^{\bar{\mathscr{Z}}^s}\right\} = 4\delta^{rs}$$

with all the other anti-commutators vanishing. This algebra is the usual one of fermion creation and annihilation operators and the gamma-matrices can be identified with such operators. According to the identifications (3.1), the massless state living in the kernel of the Dirac equation is obtained by defining a factorized vacuum $\eta_0(\vec{\mathscr{Z}}, \vec{\bar{\mathscr{Z}}}) = u_0 \otimes \phi_0(\vec{\mathscr{Z}}, \vec{\bar{\mathscr{Z}}})$. Here, u_0 is a constant six-dimensional spinor and ϕ_0 is a function of the internal coordinates, both vanishing under the action respectively of all the fermionic and bosonic annihilation operators

$$\begin{aligned} D^{(\bar{\mathscr{Z}})}_r \phi_0(\vec{\mathscr{Z}}, \vec{\bar{\mathscr{Z}}}) = 0 \;\; ; \;\; \gamma^{\mathscr{Z}^r}_{(6)} u_0 = 0 \quad \text{for } \lambda_r > 0 \\ D^{(\mathscr{Z})}_r \phi_0(\vec{\mathscr{Z}}, \vec{\bar{\mathscr{Z}}}) = 0 \;\; ; \;\; \gamma^{\bar{\mathscr{Z}}^r}_{(6)} u_0 = 0 \quad \text{for } \lambda_r < 0 \end{aligned} \tag{3.10}$$

together with the boundary conditions given in eq. (3.7). The solution of eq. (3.10) is then obtained by assuming ϕ_0 to be the wave-function in eq. (3.8) and by defining $u_0 = \gamma^{\mathscr{Z}^r}\chi_0$ for positive eigenvalues λ_r and $u_0 = \gamma^{\bar{\mathscr{Z}}^r}\chi_0$ for negative eigenvalues, being χ_0 an arbitrary eight-component constant spinor.

The whole spectrum of the Kaluza-Klein fermions is obtained, according to the standard procedure, by squaring eq. (3.9):

$$\begin{aligned} -\left(\gamma^{\mathscr{Z}^I}_{(6)} D^{\mathscr{Z}}_I \gamma^{\mathscr{Z}^J}_{(6)} D^{\mathscr{Z}}_J\right)\eta_n &= \sum_{r=1}^{3}\left(|\lambda_r|(2N_r+1) - \frac{1}{4}\left[\gamma^{Z^r}_{(6)}, \gamma^{\bar{Z}^r}_{(6)}\right]|\lambda_r|\right)\eta_n \\ &= (2\pi R)^2 m_n^2 \eta_n\,, \end{aligned} \tag{3.11}$$

where the bosonic number operator, defined in the previous sections, has been introduced and the expression of the background gauge field given in eq. (2.4) is used.

The vacuum state shown in eq. (3.10) satisfies the previous equation with $m = 0$ and, applying on it an arbitrary number of bosonic oscillators

$$(a_1^\dagger)^{N_1}\,(a_2^\dagger)^{N_2}\,(a_3^\dagger)^{N_3}\prod_{r=1}^{3}\gamma^{Z^r}\chi_0\otimes\phi_{\vec{j}}(\vec{Z},\bar{\vec{Z}})\;,$$

a set of Kaluza-Klein states are generated with masses

$$m^2 = \frac{2}{(2\pi R)^2}\sum_{r=1}^{3}|\lambda_r|N_r.$$

The next levels in the fermion Fock space, satisfying eq. (3.11), are obtained by applying one fermion creation operator and an arbitrary number of bosonic creation operators:

$$(a_1^\dagger)^{N_1}\,(a_2^\dagger)^{N_2}\,(a_3^\dagger)^{N_3}\gamma^{\bar{Z}^k}\prod_{r=1}^{3}\gamma^{Z^r}\chi_0\otimes\phi_{\vec{j}}(\vec{Z},\bar{\vec{Z}})\quad k=1,2,3.$$

A tower of KK states is generated with masses given by:

$$m_k^2 = \frac{1}{(2\pi R)^2}\sum_{r=1}^{3}|\lambda_r|(2N_r)+2\frac{|\lambda_k|}{(2\pi R)^2}\qquad k=1,2,3\;.$$

Other KK towers are obtained by acting on the vacuum with two or three fermion creation oscillators and an arbitrary number of bosonic oscillators

$$(a_1^\dagger)^{N_1}\,(a_2^\dagger)^{N_2}\,(a_3^\dagger)^{N_3}\gamma^{\bar{Z}^k}\gamma^{\bar{Z}^l}\eta_0\;\;;\;\;(a_1^\dagger)^{N_1}\,(a_2^\dagger)^{N_2}\,(a_3^\dagger)^{N_3}\prod_{k=1}^{3}\gamma^{\bar{Z}^k}\eta_0$$

with $k,l = 1,2,3$. These are three and one tower of massive states having respectively the same and opposite chirality of the vacuum [18]. Their mass spectrum is given by:

$$m_{k,l}^2 = \frac{2}{(2\pi R)^2}\sum_{r=1}^{3}|\lambda_r|N_r+2\frac{|\lambda_k|+|\lambda_l|}{(2\pi R)^2}\;\;;\;\;m^2=\frac{2}{(2\pi R)^2}\sum_{r=1}^{3}|\lambda_r|(N_r+1)\;.$$

All the mass formulas can be collected in a more concise relation by introducing the fermion number operator $N_r^f = 0,1$ and by writing

$$m_n^2 = \frac{2}{(2\pi R)^2}\sum_{r=1}^{3}|\lambda_r|(N_r+N_r^f)\;.$$

The mass of the Kaluza-Klein fermions coincides with the one given in eq. (3.3) valid when the susy condition $|\lambda_r|+|\lambda_s| = |\lambda_t|$ $(r\neq s\neq t)$ is imposed showing the consistency and accuracy of the dimensional reduction procedure.

The wave-functions of the chiral matter are derived in the background dependent system of complex coordinates $\mathscr{Z}_{ab}$ where the off-diagonal blocks of the background magnetic fields are diagonal. By definition, in each of these frames a wave-function is associated with the corresponding dy-charged sector of the theory. The calculation of the effective actions demands the evaluation of overlap integrals among three or more of these functions. It is therefore necessary to re-express such states in terms of quantities defined in a unique system of coordinates as the w^ms. In this frame one has:

$$\eta_0(\vec{w},\bar{\vec{w}}) = \prod_{r=1}^{3}\left(C_s^r \frac{(1+\text{sign}\lambda_r)}{2}\gamma^{w^s} + \bar{C}_s^r \frac{(1-\text{sign}\lambda_r)}{2}\gamma^{\bar{w}^s}\right)\chi_0 \otimes \phi_{\Omega;\vec{j}}(\vec{w},\bar{\vec{w}})$$

where $C_r^s, \bar{C}_s^r$ are the inverse matrices of the ones defined in eq. (3.5) and $\phi_{\Omega;\vec{j}}$ is a scalar function of the coordinates. It is defined in eq. (3.8) in terms of the real variables (x^m, y^m) and the relation among these coordinates and the complex ones is given in sect. 2. By using these relations it is straightforward to re-write the expression of the wave-function in the complex frame, however the calculus of the Yukawa couplings will be performed in the real system of coordinates and therefore it is not necessary to give a such expression here.

4. Yukawa Couplings

The Yukawa couplings are obtained by considering the trilinear couplings, involving one boson and two fermions, of the ten-dimensional $\mathcal{N}=1$ SYM action reduced to four dimensions according to the Kaluza-Klein compactification procedure outlined in the previous section. In refs. [13, 15, 16] this dimensional reduction is studied in great detail; here we just quote the result:

$$\begin{aligned} S_3^\Phi = \frac{1}{2g^2}\int d^4x\sqrt{G_4}\int d^6X^N\sqrt{G_6}\,\bar{\psi}_0^{ca}(x^\mu)\,\gamma_{(4)}^5\Big[\varphi_{i,0}^{ab}(x^\mu)\,\psi_0^{bc}(x^\mu)\otimes(\eta_0^{ac})^\dagger(x^n,y^n) \\ \times\ \gamma_{(6)}^i\phi^{ab}_{\Omega_{ab};\vec{j}_1}(x^n,y^n)\eta_0^{bc}(x^n,y^n) - \varphi_{i,m}^{bc}(x^\mu)\,\psi_0^{ab}(x^\mu)\otimes(\eta_0^{ac})^\dagger(x^n,y^n)\gamma_{(6)}^i\phi^{bc}_{\Omega_{bc};\vec{j}_2}(x^n,y^n)\eta_0^{ab}(x^n,y^n)\Big] \end{aligned} \tag{4.1}$$

where ψ_0 is the massless fermion ground state. φ_0, instead, is the lightest bosonic excitation which is massless if the supersymmetry condition, given in the text soon after eq. (3.2), is imposed. In the following, in order to fix notations, we choose λ_1^{ab} to be positive. So doing, the massless scalar turns out to be $\phi_{\mathscr{Z}^1}$, while with the opposite choice $\phi_{\bar{\mathscr{Z}}^1}$ would have been the massless state [18]. In the chosen notations, only the first term in eq. (4.1) contributes to the Yukawa coupling for massless particles and one is left with the expression

$$(S_3^\Phi)^{(1)} = \int d^4x\sqrt{G_4}\,\bar{\psi}_0^{ca}\,\gamma_{(4)}^5\,\varphi^{ab}_{\mathscr{Z}^1,0}\,\psi_0^{bc}Y^{\vec{j}_1\vec{j}_2\vec{j}_3}$$

with the Yukawa coupling constants, in the string frame, given by

$$Y^{\vec{j}_1\vec{j}_2\vec{j}_3} = \frac{1}{2g^2}\left[(u_0^{ac})^\dagger\gamma_{(6)}^{\mathscr{Z}^1_{ab}}u_0^{bc}\right]\mathscr{Y}^{\vec{j}_1\vec{j}_2\vec{j}_3}$$

where

$$\mathscr{Y}^{\vec{j}_1\vec{j}_2\vec{j}_3} = \int_{T^6} d^3x\,d^3y\sqrt{G_6}(\phi^{ac}_{\Omega_{ac};\vec{j}_3}(x^n,y^n))^\dagger\phi^{ab}_{\Omega_{ab};\vec{j}_1}(x^n,y^n)\phi^{bc}_{\Omega_{bc};\vec{j}_2}(x^n,y^n)\,. \tag{4.2}$$

The integral in eq. (4.2) has been computed in ref. [18]. The calculation is here summarized in the case in which all the first Chern-classes associated with the three twisted sectors are independent. When this latter condition is not satisfied there are subtleties that are discussed in ref. [18].

The integral can be performed after using the following identity involving the product of two wave-functions[2]:

$$\phi^{ab}_{\Omega_{ab};\vec{j}_1}(x^m,y^m)\,\phi^{bc}_{\Omega_{bc};\vec{j}_2}(x^m,y^m)=\mathcal{N}_{ab}\,\mathcal{N}_{bc}\,e^{i\pi\vec{x}^t\left(\frac{I_{ab}+I_{bc}}{(2\pi R)^2}\right)\vec{y}+i\pi\vec{y}^t\left(\frac{I_{ab}\Omega_{ab}+I_{bc}\Omega_{bc}}{(2\pi R)^2}\right)\vec{y}}$$
$$\times\sum_{\substack{(\vec{l}_3,\vec{l}_4)\in\mathbb{Z}^3\\ \vec{m}\in\tilde{\mathbf{Z}}_{ac}}}\;\sum_{\substack{\vec{p}\in\mathbf{Z}_{bc}\\ \vec{q}\in\mathbf{Z}_{ab}}} e^{i\pi\vec{l}\,'^t Q'\vec{l}\,'+2\pi i\vec{l}\,'^t Q'\begin{pmatrix}\vec{y}\\0\end{pmatrix}+2\pi i\vec{l}\,'^t\mathscr{I}'\begin{pmatrix}\vec{x}\\0\end{pmatrix}} \tag{4.3}$$

with

$$Q'=\begin{pmatrix} I_{ab}\Omega_{ab}+I_{bc}\Omega_{bc} & (\Omega^t_{ab}-\Omega^t_{bc})\alpha^t\\ \alpha(\Omega_{ab}-\Omega_{bc}) & \alpha(\Omega_{ab}I^{-t}_{ab}+\Omega_{bc}I^{-t}_{bc})\alpha^t\end{pmatrix}\;;\;\mathscr{I}'=\begin{pmatrix} I_{ab}+I_{bc} & (I_{ab}I^{-t}_{ab}-I_{bc}I^{-t}_{bc})\alpha^t\\ 0 & \alpha(I^{-t}_{ab}+I^{-t}_{bc})\alpha^t\end{pmatrix},$$

and

$$\vec{l}\,'^t=(\vec{l}_1'^t,\vec{l}_2'^t)=\Big((\vec{j}^t_1 I_{ab}+\vec{j}^t_2 I_{bc}+\vec{m}^t I_{ab})\,(I_{ab}+I_{bc})^{-1}+\vec{l}^t_3;$$
$$(\vec{j}^t_1-\vec{j}^t_2+\vec{m}^t)\left(I^{-1}_{ab}+I^{-1}_{bc}\right)^{-1}\alpha^{-1}+\vec{p}^t\frac{I_{bc}}{\det[I_{bc}]}+\vec{q}^t\frac{I_{ab}}{\det[I_{ab}]}+\vec{l}^t_4\Big). \tag{4.4}$$

The parameter α is chosen in such a way to make the matrix $\alpha(I^{-1}_{ab}+I^{-1}_{bc})$ integer. This request is fundamental in order to obtain the identity written in eq. (4.3). The choice $\alpha=\det[I_{ab}I_{bc}]\mathbb{I}$ satisfies this requirement [20]. The indices of the two summations need some explanation. Let us denote by $\mathbb{Z}^3_{(I^{-1}_{ab}+I^{-1}_{bc})\alpha}$ the set of equivalence classes obtained by identifying the elements of $\mathbb{Z}^3$ under the shift $\vec{n}\to\vec{n}+\vec{t}(I^{-1}_{ab}+I^{-1}_{bc})\alpha$, with $\vec{t},\vec{n}\in\mathbb{Z}^3$. A subset of $\mathbb{Z}^3_{(I^{-1}_{ab}+I^{-1}_{bc})\alpha}$ is obtained by considering the integer vectors lying within a cell generated by $\vec{e}_i\det[I_{ab}]I^{-1}_{ab}$,$(i=1,2,3)$ being $\vec{e}_i$ defined in the appendix [20]. This subset is denoted by $\mathbf{Z}_{ab}$. $\mathbf{Z}_{bc}$ is defined by exchanging I_{ab} with I_{bc}. Finally:

$$\tilde{\mathbf{Z}}_{ac}=\mathbb{Z}^3_{(I^{-1}_{ab}+I^{-1}_{bc})\alpha}\setminus(\mathbf{Z}_{bc}\cup\mathbf{Z}_{ab}).$$

The proof of the identity written in eq. (4.3) is outlined in the appendix. More details are given in refs [18, 20].

The integral over the $\vec{x}$ variable can be easily performed giving:

$$\mathscr{Y}^{\vec{j}_1\vec{j}_2\vec{j}_3}=\sqrt{G_6}\,\mathscr{D}\,\mathcal{N}_{ab}\,\mathcal{N}_{bc}\,\mathcal{N}_{ca}\sum_{\vec{l}_3,\vec{l}_4\in\mathbb{Z}^3}\;\sum_{\vec{p}\in\mathbf{Z}_{bc}}\;\sum_{\vec{q}\in\mathbf{Z}_{ab}}\mathscr{F}_{\Omega,I}(\vec{l}_3,\vec{l}_4)e^{i\pi\vec{l}_2'^t\,\Pi\,\vec{l}_2'}$$

where

$$\Pi=\alpha\left((\Omega_{ab}I^{-t}_{ab}+\Omega_{bc}I^{-t}_{bc})-(\Omega_{ab}-\Omega_{bc})(I_{ca}\Omega_{ca}+I_{ab}\Omega_{ab}+I_{bc}\Omega_{bc})^{-1}(\Omega_{ab}-\Omega_{bc})^t\right)\alpha^{-t} \tag{4.5}$$

and

$$\mathscr{D}\equiv\sum_{\vec{m}\in\tilde{\mathbb{Z}}_{(I^{-1}_{ab}+I^{-1}_{bc})\alpha}}\delta_{(\vec{j}^t_1 I_{ab}+\vec{j}^t_2 I_{bc}+\vec{m}^t I_{ab})(I_{ab}+I_{bc})^{-1};\vec{j}^t_3}.$$

[2] In this paper by respect the ref.[18] a different notation for the indices of the summation is used. The correspondence among the two sets of symbols is: $\mathbf{Z}_{ab}\equiv\mathbb{Z}^3_{\det[I_{ab}]I^{-1}_{ab}}$, $\mathbf{Z}_{bc}\equiv\mathbb{Z}^3_{\det[I_{bc}]I^{-1}_{bc}}$ and $\tilde{\mathbf{Z}}_{ac}=\tilde{\mathbb{Z}}_{(I^{-1}_{ab}+I^{-1}_{bc})\alpha}$

The last integral to be computed is contained in the definition of the following function:

$$\mathscr{F}_{\Omega,I}(\vec{l}_3,\vec{l}_4) \equiv \int_0^1 d^3y e^{-\pi[\vec{y}^t+\vec{l}_1{}'^t+\vec{l}_2{}'^t Q'^{21^t} A^{-1}](-iA)[\vec{y}+\vec{l}_1{}'+A^{-1}Q'^{21^t}\vec{l}_2{}']}$$

with $Q'^{21} = \alpha(\Omega_{ab} - \Omega_{bc})$. The integral is convergent and, after having evaluated it, one has the expression [18]

$$\mathscr{Y}^{\vec{j}_1\vec{j}_2\vec{j}_3} = \int d^3xd^3y\sqrt{G_6}\phi^{ca}_{\Omega_{ca};\vec{j}_1}{}^*\phi^{ab}_{\Omega_{ab};\vec{j}_1}\phi^{bc}_{\Omega_{bc};\vec{j}_2} = \mathscr{N}_{ab}\mathscr{N}_{bc}\mathscr{N}_{ca}\sqrt{G_6}\,\mathscr{D}$$
$$\times\left[\det(-i(I_{ca}\Omega_{ca}+I_{ab}\Omega_{ab}+I_{bc}\Omega_{bc}))\right]^{-1/2}\sum_{\vec{p}\in\mathbf{Z}_{bc}}\sum_{\vec{q}\in\mathbf{Z}_{ab}}\Theta\begin{bmatrix}\alpha^{-t}I^t_{bc}(\vec{j}_3-\vec{j}_2)+\frac{I^t_{bc}}{\det I^t_{bc}}\vec{p}+\frac{I^t_{ab}}{\det I_{bc}}\vec{\tilde{p}}\\ 0\end{bmatrix}(0|\Pi).$$

that simplifies when all the differences of magnetic fields living on the various stacks of magnetized branes are independent but commuting. In this approximation, an analogous string calculus of the Yukawa couplings has been performed in ref. [22]. In this case the quantity $\alpha(I_{ab}^{-1}+I_{bc}^{-1})$ can be made an integer matrix by choosing $\alpha = I_{ab}I_{bc}$ and the product of two wave-functions is still equal to eq. (4.3) specialized with this value of α and without the sums over the vectors $\vec{p}$ and $\vec{q}$. The overlap integral over the three wave-functions is now:

$$\mathscr{Y}^{\vec{j}_1\vec{j}_2\vec{j}_3} = \int d^3xd^3y\sqrt{G_6}\phi^{ca}_{\Omega_{ca};\vec{j}_1}{}^*\phi^{ab}_{\Omega_{ab};\vec{j}_1}\phi^{bc}_{\Omega_{bc};\vec{j}_2} = \mathscr{N}_{ab}\mathscr{N}_{bc}\mathscr{N}_{ca}\sqrt{G_6}\,\mathscr{D}$$
$$\times\ \left[\det(-i(I_{ca}\Omega_{ca}+I_{ab}\Omega_{ab}+I_{bc}\Omega_{bc}))\right]^{-1/2}\Theta\begin{bmatrix}I^{-t}_{ab}(\vec{j}_3-\vec{j}_2)\\ 0\end{bmatrix}(0|\Pi)$$

with Π given by the eq. (4.5) specialized to the value $\alpha = I_{ab}I_{bc}$.

In conclusion, the field theory approach is a very efficient tool in determining the low-energy effective actions supported in the world-volume of magnetized branes. These coefficients and, in particular, the holomorphic part of the Yukawa couplings strongly depend on the global aspects of the internal manifold as one has explicitly shown in the case of compactifications on the torus T^6. It would be interesting to extend this approach to models where few global quantities are explicitly computed. In this respect, models coming from compactification of F-theory are a good arena for this kind of analysis.

A. Appendix

It is useful to give here the proof of eq. (4.3) involving the product of two wave-functions. According to ref. [21] such a product can be written concisely as follows:

$$\phi^{ab}_{\Omega_{ab};\vec{j}_1}(x^m,y^m)\,\phi^{bc}_{\Omega_{bc},\vec{j}_2}(x^m,y^m) = \mathscr{N}_{ab}\,\mathscr{N}_{bc}\,e^{i\pi\vec{y}^t\left(\frac{I^t_{ab}+I^t_{bc}}{(2\pi R)^2}\right)\vec{x}+i\pi\vec{y}^t\left(\frac{I_{ab}\Omega_{ab}+I_{bc}\Omega_{bc}}{(2\pi R)^2}\right)\vec{y}}$$
$$\times\sum_{\vec{l}\in\mathbb{Z}^{2d}} e^{i\pi\vec{l}^tQ\vec{l}+2\pi i\vec{l}^tQ\frac{\vec{Y}}{2\pi R}+2\pi i\vec{l}^t\mathscr{I}\frac{\vec{X}}{2\pi R}}$$

being $Q = \mathrm{diag}(I_{ab}\Omega_{ab}, I_{bc}\Omega_{bc})$, $\mathscr{I} = \mathrm{diag}(I_{ab}, I_{bc})$ and

$$\vec{l} = \begin{pmatrix}\vec{n}_1+\vec{j}_1\\ \vec{n}_2+\vec{j}_2\end{pmatrix}\ ;\ X = \begin{pmatrix}\vec{x}\\ \vec{x}\end{pmatrix}\quad ;\quad Y = \begin{pmatrix}\vec{y}\\ \vec{y}\end{pmatrix}\ .$$

An equivalent representation of the product of two Riemann Theta functions is obtained by introducing the following transformation matrix [20]:

$$T = \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \alpha I_{ab}^{-1} & -\alpha I_{bc}^{-1} \end{pmatrix} \quad ; \quad T^{-1} = \begin{pmatrix} (I_{ab}^{-1}+I_{bc}^{-1})^{-1} I_{bc}^{-1} & (I_{ab}^{-1}+I_{bc}^{-1})^{-1}\alpha^{-1} \\ (I_{ab}^{-1}+I_{bc}^{-1})^{-1} I_{ab}^{-1} & -(I_{ab}^{-1}+I_{bc}^{-1})^{-1}\alpha^{-1} \end{pmatrix}$$

acting as (see also the relation before eq.s (4.4)):

$$Q' = TQT^t \qquad ; \qquad \mathscr{I}' = T\,\mathscr{I}\,T^t \ .$$

We introduce also the vector:

$$\vec{l}'^t \equiv \vec{l}^t T^{-1} = \Big((\vec{n}_1+\vec{j}_1)^t (I_{ab}^{-1}+I_{bc}^{-1})^{-1} I_{bc}^{-1} + (\vec{n}_2+\vec{j}_2)^t (I_{ab}^{-1}+I_{bc}^{-1})^{-1} I_{ab}^{-1} \,;$$
$$(\vec{n}_1+\vec{j}_1)^t (I_{ab}^{-1}+I_{bc}^{-1})^{-1}\alpha^{-1} - (\vec{n}_2+\vec{j}_2)^t (I_{ab}^{-1}+I_{bc}^{-1})^{-1}\alpha^{-1} \Big) \ .$$

By using the following identity:

$$\left(I_{ab}^{-1}+I_{bc}^{-1}\right)^{-1} = I_{bc}\left(I_{ab}+I_{bc}\right)^{-1} I_{ab} = I_{ab}\left(I_{ab}+I_{bc}\right)^{-1} I_{bc} \ ,$$

one can write [18]

$$\begin{aligned} &\left(\vec{n}_1^t\, I_{ab} + \vec{n}_2^t\, I_{bc}\right)\left(I_{ab}+I_{bc}\right)^{-1} = \vec{m}_1^t\left(I_{ab}+I_{bc}\right)^{-1} + \vec{l}_3^t \\ &\left(\vec{n}_1^t - \vec{n}_2^t\right) I_{ab}\left(I_{ab}+I_{bc}\right)^{-1} I_{bc}\alpha^{-1} = \vec{m}_2^t\left(I_{ab}^{-1}+I_{bc}^{-1}\right)^{-1}\alpha^{-1} + \vec{l}_4^t \end{aligned} \tag{A.1}$$

where $\vec{l}_3, \vec{l}_4 \in \mathbb{Z}^3$, $\vec{m}_1$ and $\vec{m}_2$ are suitable integer vectors, while α has to be fixed in such a way that the matrix $\alpha\left(I_{ab}^{-1}+I_{bc}^{-1}\right)$ has integer entries. In the following, we will choose $\alpha = \det\left[I_{ab}I_{bc}\right]\mathbb{I}$ [20] which indeed satisfies the above mentioned constraint. By writing $\vec{m}_1 = m_1^i \vec{e}_i$, with

$$\vec{e}_i^t = (\overbrace{0,\ldots,0,1}^{i\ times},0,\ldots) \ ,$$

the lattice with basis vectors $\vec{e}_i\left(I_{ab}+I_{bc}\right)$ is introduced and, in it, the equivalent points are those which change $\vec{l}_3$ by integer values, because this quantity is summed over all the possible elements of $\mathbb{Z}^3$.

$\mathbb{Z}^3_{(I_{ab}+I_{bc})}$ is the set of equivalent classes obtained by identifying the elements of $\mathbb{Z}^3$ under the shift $\vec{m}_1 + \vec{k}^t\left(I_{ab}+I_{bc}\right)$ ($\forall \vec{k} \in \mathbb{Z}^3$). Inequivalent values of $\vec{m}_1$ lie in the cell determined by the vectors $\vec{e}_i\left(I_{ab}+I_{bc}\right)$ and their number is $|\det[I_{ab}+I_{bc}]|$. Analogously, the number of inequivalent values of $\vec{m}_2 \in \mathbb{Z}^3_{(I_{ab}^{-1}+I_{bc}^{-1})\alpha}$ is $|\det[I_{ab}^{-1}+I_{bc}^{-1}]\alpha|$.

From eqs. (A.1), it is straightforward to obtain the identities:

$$\begin{aligned} \vec{n}_1^t &= (\vec{m}_1^t + \vec{m}_2^t I_{bc})(I_{ab}+I_{bc})^{-1} + \vec{l}_3^t + \vec{l}_4^t \alpha I_{ab}^{-1} \\ \vec{n}_2^t &= (\vec{m}_1^t - \vec{m}_2^t I_{ab})(I_{ab}+I_{bc})^{-1} + \vec{l}_3 - \vec{l}_4 \alpha I_{bc}^{-1} \end{aligned}$$

which are consistent if both αI_{ab}^{-1} and αI_{bc}^{-1} are integer matrices. This latter request is indeed satisfied by the choice $\alpha = \det[I_{ab}\,I_{bc}]\mathbb{I}$. Moreover, one has also to impose

$$\vec{m}_1^t + \vec{m}_2^t I_{bc} = \vec{k}^t\left(I_{ab}+I_{bc}\right) \ ; \ \vec{m}_1^t - \vec{m}_2^t I_{ab} = \vec{k}_1^t\left(I_{ab}+I_{bc}\right)$$

with $\vec{k}$ and $\vec{k}_1$ elements of $\mathbb{Z}^3$. The solution of the last two equations is

$$\vec{m}_1^t = \vec{m}_2^t I_{ab} + \vec{k}_1^t (I_{ab} + I_{bc}) \,. \tag{A.2}$$

The correspondence between $\vec{m}_1$ and $\vec{m}_2$ is not one-to-one since the number of the inequivalent values of $\vec{m}_2$ is bigger than the one of inequivalent $\vec{m}_1$. Following ref. [20], one can replace:

$$\vec{m}_2^t = \vec{\tilde{m}}_2^t + \vec{p}^t \det[I_{ab}](I_{ab} + I_{bc})I_{ab}^{-1} + \vec{q}^t \det[I_{bc}](I_{ab} + I_{bc})I_{bc}^{-1}$$

and the second line of eq. (A.1) becomes:

$$\begin{aligned} \left(\vec{n}_1^t - \vec{n}_2^t\right) I_{ab} \left(I_{ab} + I_{bc}\right)^{-1} I_{bc} \alpha^{-1} &= \vec{\tilde{m}}_2^t \left(I_{ab}^{-1} + I_{bc}^{-1}\right)^{-1} \alpha^{-1} + \vec{p}^t \frac{I_{bc}}{\det I_{bc}} \\ &+ \vec{q}^t \frac{I_{ab}}{\det I_{ab}} + \vec{l}_4^t . \end{aligned} \tag{A.3}$$

From eq. (A.3) one can easily see that shifting $\vec{p} \to \vec{p} + \vec{k}(\det[I_{bc}])[I_{bc}]^{-1}$ for all $\vec{k} \in \mathbb{Z}^3$ corresponds to add $\vec{k}$ to $\vec{l}_4$, providing equivalent values of $\vec{p}$ since $\vec{l}_4$ is summed over all possible integer vectors . The set of inequivalent $\vec{p}$ is denoted by $\mathbf{Z}_{bc}$ and its number is $|\det(\det[I_{bc}]I_{bc}^{-1})|$. A similar definiton holds for $\vec{q} \in \mathbf{Z}_{ab}$ and the dimension of this set results to be $|\det(\det[I_{ab}]I_{ab})^{-1}|$. Consequently, the number of inequivalent $\vec{\tilde{m}}_2$s is $|\det[I_{ab} + I_{bc}]|$ which now matches with the one of inequivalent $\vec{m}_1$.

By starting from eq. (A.3) and repeating the same manipulations which have led to eq. (A.2), one has that this latter equation remains unchanged but with $\vec{m}_2$ replaced by $\vec{\tilde{m}}_2$. The solution of eq. (A.2) is now unique and one gets the expression of $\vec{l}'$ given in eq. (4.4) with $\vec{m} \equiv \vec{\tilde{m}}_2$. After collecting all the results, one derives the identity written in eq. (4.3).

When I_{ab} and I_{bc} commute, the quantities $\alpha(I_{ab}^{-1} + I_{bc}^{-1})$ can be made an integer matrix with the choice $\alpha = I_{ab}I_{bc}$. Eqs. (A.1) become:

$$\begin{aligned} \left(\vec{n}_1^t I_{ab} + \vec{n}_2^t I_{bc}\right) \left(I_{ab} + I_{bc}\right)^{-1} &= \vec{m}_1^t \left(I_{ab} + I_{bc}\right)^{-1} + \vec{l}_3^t \\ \left(\vec{n}_1^t - \vec{n}_2^t\right) \left(I_{ab} + I_{bc}\right)^{-1} &= \vec{m}_2^t \left(I_{ab} + I_{bc}\right)^{-1} + \vec{l}_4^t . \end{aligned}$$

with $\vec{m}_1, \vec{m}_2 \in \mathbb{Z}^3_{(I_{ab}+I_{bc})}$. In this case there is no need to introduce the vectors $\vec{p}$ and $\vec{q}$ and one can trivially impose eq. (A.2).

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