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Reduction at the integrand level beyond NLO

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Reduction techniques at the integrand level play important role for NLO calculations. In this note we review recent developments in using these techniques at NNLO.

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1. Introduction

The discovery of a scalar particle with a mass of 125 GeV [1, 2], $h(125)$, marked the first run of the Large Hadron Collider at CERN at an energy of 7 and 8 TeV. If conclusively identified with the Higgs particle [3], may be one of the most important discoveries in the history of physics, revealing the fundamental concept of the origin of mass. Moreover, the accumulated analysis experience demonstrates that any New Physics effect, if it exists, and in order to be verified, requires a very precise knowledge of the existing Standard Model backgrounds. For this, not only next-to-leading order (NLO) calculations are indispensable, but also next-to-next-to-leading order (NNLO) calculations are highly desirable [4–6]. This implies that (very) large loop integrals have to be computed for very many Feynman diagrams, which has widely been considered the bottleneck of such calculations.

Reduction techniques form a way out. The idea of reducing Feynman integrals with a large number of denominators to a set of simpler integrals (*i.e.* with fewer denominators) at one loop goes surprisingly many years back [7, 8]. A typical integral with n such denominators, in d space-time dimensions, is given by

$$\int d^d q \frac{1}{D_1 D_2 ... D_n} \ ,$$

where $D_i = (q + p_i)^2 - m_i^2$ is the denominator of a generic propagator. In [7] the authors reduce a triangle (integral with 3 denominators) to bubbles (2 denominators) in 2 dimensions while in [8] a pentagon (5 denominators) is reduced to boxes (4 denominators) in 4 dimensions.

Since a few decades now [9], it is well known that a generic one-loop amplitude is decomposable in terms of scalar integrals, with one, two, three and four external legs (in $d = 4$). Passarino and Veltman [10] used Lorentz invariance to express tensor one-loop $n-$point integrals in terms of $m-$point scalar integrals ($m \leq n$). As a consequence, only the evaluation of scalar integrals (integrals with trivial numerators) is needed in order to perform a one-loop calculation.

The next big step comes from unitarity methods [12–15]. Instead of working with specific Feynman diagrams these methods have a big advantage in that they try to decompose the whole one-loop amplitude in terms of the scalar integrals. By cutting propagators[1] the rational coefficients of loop integrals are given in terms of products of tree amplitudes. In generalized unitarity methods [16–21], the notion of *multiple cuts* is introduced. One can cut more than one propagator to find these coefficients. Note that, for $d = 4$, cutting four propagators essentially determines the loop momentum (there is, in general, more than a single solution since the D's are quadratic in the loop momentum).

The Ossola-Papadopoulos-Pittau (OPP) method [22–26] comes as a natural combination of all the above. Since every integral can be decomposed to scalar integrals with up to four denominators (for $d = 4$), every one-loop amplitude is written in terms of coefficients that multiply these scalar integrals. The OPP method works at the integrand level [27, 28], which means that for these decompositions to be possible one must also include spurious terms. Then one has to find a way to calculate the coefficients of the reduction and multiply them with the appropriate scalar integrals, using one of the packages available for the evaluation of them(i.e [29, 30]). Finding the

[1] 'Cutting' a propagator means that loop momenta are chosen for which one or more of the D's vanish so that the integrand becomes singular. One also speaks of 'putting propagators on-shell'.

coefficients is a purely algebraic problem. The method is suitable for a fully numerical implementation [31]. The OPP method has been widely used so far in many one loop calculations (see for example [32–46]).

As we noticed before, in the case of one-loop calculations a basis for any integral is known in advance. Any one-loop integral can be written in terms of scalar boxes, triangles, bubbles and tadpoles. However, in the case of higher-loop integrals the situation is different. A basis is not known *a priori*. It is believed that unitarity methods can also be applied in that case and there are some recent papers in that direction, performing decomposition à la OPP [47–50], or using generalized unitarity [51–53] at two loops.

Two remarks are in order here. The first is that the basis of two-loop integrals does not include only scalar integrals. It includes integrals that also have irreducible scalar products (ISP) as numerators (to some power) that cannot be rewritten as existing denominators of the integral. In the one-loop case these ISP are always spurious and integrate to zero, but for higher loops this does no longer hold. The second remark is that if one is interested in constructing a unitarity-like basis, the set of integrals that ends up with is not necessarily a minimal one: the integrals are not by default Master Integrals (MI). There might be smaller sets of true MI and at two or more loops one can find them by using integration-by-parts (IBP) identities [54–58].

2. Reduction at one loop

A compact form of the OPP reduction master equation at one loop can be given by the following equation:

$$\frac{N(q;\{p_i\})}{D_1 D_2 \dots D_n} = \sum_{m=1}^{\min(n,4)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \dots i_m}(q;\{p_i\})}{D_{i_1} D_{i_2} \dots D_{i_m}} \tag{2.1}$$

where q denotes the loop momentum, $\{p_i\}$ the set of the external momenta, $D_i = (q+P_i)^2 - M_i^2$ is the usual Feynman propagator with momentum flow P_i and mass M_i, $N(q;\{p_i\})$ an arbitrary function of the loop and external momenta, and $S_{m;n}$ stands for all subsets of m indices out of the n ones: for instance if $S_{n;n} = S_{4;4} = \{\{1,2,3,4\}\}$ then $S_{3;4} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ and so on. For instance when $n = 4$

$$\begin{aligned}
\frac{N(q;\{p_i\})}{D_1 D_2 D_3 D_4} = & \frac{\Delta_{1234}(q;\{p_i\})}{D_1 D_2 D_3 D_4} \\
& + \frac{\Delta_{123}(q;\{p_i\})}{D_1 D_2 D_3} + \frac{\Delta_{124}(q;\{p_i\})}{D_1 D_2 D_4} + \dots \\
& + \frac{\Delta_{12}(q;\{p_i\})}{D_1 D_2} + \frac{\Delta_{13}(q;\{p_i\})}{D_1 D_3} + \dots \\
& + \frac{\Delta_{1}(q;\{p_i\})}{D_1} + \frac{\Delta_{2}(q;\{p_i\})}{D_2} + \dots
\end{aligned} \tag{2.2}$$

All these residual functions, $\Delta_{...}$ at one loop, are given by the OPP parametrization and they are in general decomposed in two terms,

$$\Delta_{...}(q;\{p_i\}) = c_{...}(\{p_i\}) + \tilde{c}_{...}(q;\{p_i\}) \tag{2.3}$$

where $c_{...}$ represent the integral coefficients and $\tilde{c}_{...}$ the so-called spurious terms, that vanish upon integration, so that the final result **after** integration is given by

$$\int d^4q \frac{N(q;\{p_i\})}{D_1 D_2 \dots D_n} = \sum_{m=1}^{4} \sum_{S_{m;n}} c_{i_1 i_2 \dots i_m}(\{p_i\}) \int d^4q \frac{1}{D_{i_1} D_{i_2} \dots D_{i_m}} \tag{2.4}$$

Eq. 2.1 can also be written in the following form

$$N(q;\{p_i\}) = \sum_{m=1}^{4} \sum_{S_{m;n}} \Delta_{i_1 i_2 \dots i_m}(q;\{p_i\}) \prod_{i \neq i_1, i_2 \dots i_m} D_i \tag{2.5}$$

In that form it is evident that Eq. 2.5 is nothing but a polynomial equation with respect to the loop momentum q. Its solution can easily be determined if we know the degree of the polynomial functions $\tilde{c}_{...}(q;\{p_i\})$. One way of solving Eq. 2.5 then is à la unitarity, namely using the 'cut' equations, by recursively putting all subsets of 4, 3, 2 and 1 denominators D_i equal to zero and recursively subtracting the computed terms. Several ways of extending the above methods in $d = 4 - 2\varepsilon$ dimensions are given in the literature [25,28]. Based on these developments, purely numerical algorithms for NLO calculations have been designed over the past years, allowing the computation of a plethora of high-multiplicity scattering processes at this level of accuracy and at the same time a few automatic software programs are nowadays available for general use [31,41,59].

Although the construction of the OPP method historically has been based on the accumulated experience from one-loop calculations, we can nowadays look at the same problem trough a different perspective, namely the algebraic geometry context. This will become evident at the two-loop level.

3. Reducibility at two loops

All two-loop amplitudes with n propagators in the loops, can be characterized by three numbers $n_1, n_2, n_3, n_1 + n_2 + n_3 = n$, as shown in the graphical representation below (iGraph) [60]. The blobs represent attachments of arbitrary tree-like structures, according to the theory at hand. The amplitude can be written as

$$\int d^4l_1 d^4l_2 \frac{N(l_1, l_2; \{p_i, q_i, r_i\})}{D_1 \dots D_n} \tag{3.1}$$

with the obvious identification, $D_i(q) = q^2 - M_i^2, i = 1 \dots n$, with $q_i = l_1 + p_i, i = 1 \dots n_1$, $q_i = l_2 + p_i, i = n_1 + 1 \dots n_1 + n_2$ and $q_i = l_1 + l_2 + p_i, i = n_1 + n_2 + 1 \dots n$.

In reference [60] we have started with the problem of reducing all scalar integrals ($N = 1$) down to integrals with the lowest possible number of denominators. In order to setup the problem

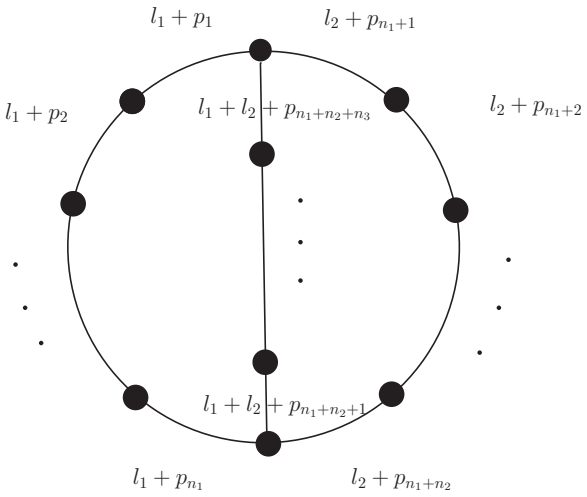

Figure 1: General iGraph at two loops.

let us consider the following equation,

$$N \equiv 1 = \sum_{j=1}^{n_1} x_j(l_1,l_2)D(l_1+p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j(l_1,l_2)D(l_2+p_j) + \sum_{j=n_1+n_2+1}^{n} x_j(l_1,l_2)D(l_1+l_2+p_j) \tag{3.2}$$

with $x_i(l_1,l_2)$ polynomials in l_1,l_2. As in the one-loop case, a generic graph of order $n \leq 2d$ (d is the space-time dimension) with $n_{1,2,3} \leq 4$ cannot be decomposed in this way, since there are $l_{1,2}$ momenta for which all propagators appearing in the above equation can be simultaneously on-shell.

The requirement for "trivial" decomposition (for x_j that are independent of l_1 and l_2 in Eq. 3.2) now reads

$$\left(\sum_{j=1}^{n_1} + \sum_{j=n_1+n_2+1}^{n}\right) x_j = \sum_{j=n_1+1}^{n} x_j = \sum_{j=n_1+n_2+1}^{n} x_j = 0 \tag{3.3}$$

$$\left(\sum_{j=1}^{n_1} + \sum_{j=n_1+n_2+1}^{n}\right) x_j p_j{}^{\mu} = \sum_{j=n_1+1}^{n} x_j p_j{}^{\mu} = 0 \ , \tag{3.4}$$

and

$$\sum_{j=1}^{n} x_j (p_j^2 - M_j^2) = 1 \ . \tag{3.5}$$

In total there are $2d+4$ conditions, so that the minimum size of a trivially decomposable iGraph is $2d+4$. In four dimensions, scalar iGraphs can therefore be decomposed down to $n = 11$. On the

other hand, since any subset of an iGraph is itself an iGraph, we only have to consider two-loop iGraphs with [60]

$$n_{1,2,3} \leq 4\ (=d)\ ,\quad n_1+n_2+n_3 \leq 11\ (=2d+3)\ . \tag{3.6}$$

Sumarising our findings, we give the following table that describes the reducibility of the iGraphs we investigated. The horizontal line is to distinguish the reducible from the non-reducible cases in all dimensions. The iGraphs of order higher than $2d$ are always reducible, while the ones with less or equal to $2d$ are not. We specify, in the cases the iGraph is reducible, the smallest degree for which the reduction can be achieved. *CU*, *Q*, *L* and *co* stand for cubic, quadratic, linear and constant terms respectively.

n	$d=6$	$d=5$	$d=4$	$d=3$	$d=2$
3					*NOT*
4					*NOT*
5				*NOT*	*Q*
6				*NOT*	*L*
7			*NOT*	*CU*	*L*
8			*NOT*	*L*	*co*
9		*NOT*	*CU*	*L*	*co*
10		*NOT*	*L*	*co*	·
11	*NOT*	*CU*	*L*	*co*	·
12	*NOT*	*L*	*co*	·	·
13	*CU*	*L*	*co*	·	
14	*L*	*co*	·	·	
15	*L*	·	·		
16	*co*	·	·		

It should be emphasized that our findings are based on generic momenta and masses configurations.

4. The Algebraic Geometry Context

Over the last year many groups have realized the relevance of the Algebraic Geometry Context (AGC) in reduction approaches [47–50]. To give an example let us consider the $n_1=4, n_2=4, n_3=1, (4,4,1)$ planar case, with a total of $n=9$ propagators. We choose the following configuration, with the nine propagators given by

$$\begin{aligned} &D_1 = l_1^2 - M_1^2, D_2 = (l_1+p_1)^2 - M_2^2, D_3 = (l_1+p_2)^2 - M_3^2, D_4 = (l_1+p_3)^2 - M_4^2, \\ &D_5 = l_2^2 - M_5^2, D_6 = (l_2+p_4)^2 - M_6^2, D_7 = (l_2+p_5)^2 - M_7^2, D_8 = (l_2+p_6)^2 - M_8^2, \\ &D_9 = (l_1+l_2)^2 - M_9^2 \end{aligned} \tag{4.1}$$

In 4 space-time dimensions we can choose a basis to express l_1, l_2 with four elements. For l_1 we can choose for instance [11]

$$v_1^\mu = \frac{\delta^{\mu p_2 p_3}_{p_1 p_2 p_3}}{\Delta}\ v_2^\mu = \frac{\delta^{p_1 \mu p_3}_{p_1 p_2 p_3}}{\Delta}\ v_3^\mu = \frac{\delta^{p_1 p_2 \mu}_{p_1 p_2 p_3}}{\Delta}\ \eta^\mu = \frac{\varepsilon^{\mu p_1 p_2 p_3}}{\sqrt{\Delta}} \tag{4.2}$$

with

$$\Delta = \delta^{p_1 p_2 p_3}_{p_1 p_2 p_3} = \varepsilon^{p_1 p_2 p_3} \varepsilon_{p_1 p_2 p_3} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 \end{vmatrix}$$

and for l_2, the same as above with p_4, p_5, p_6 replacing p_1, p_2, p_3 accordingly. The momenta $p_i, i = 1, \ldots, 6$ are arbitrary. The basis coefficients may be read as $l_1^\mu = \sum_{i=1}^3 z_i v_i^\mu + z_4 \eta^\mu$, with $z_i = l_1 \cdot p_i, i = 1 \ldots, 3$ (and similar for l_2, with w_i replacing z_i). The equation we try to solve is

$$1 = \sum_{i=1}^{9} x_i(l_1, l_2) D_i \tag{4.3}$$

This is a polynomial equation with 8 variables $(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$. According to our finding the associated scalar graph is decomposable by using $x_i(l_1, l_2)$ polynomials of degree 3. Each term x_i, being a degree 3 polynomial in these variables, consists of 165 terms. In total we need to solve for $9 \times 165 = 1485$ coefficients. As we have proved, only 831 out of 1485 are "independent". Nevertheless there is still a lot of redundancy among these 831 coefficients. In fact we could solve à la OPP, uisng 'cut' equations for all 9 subsets of 8 denominators. Since the solution of all these equations contains 4 elements in l_1, l_2 space, denoted by l_1^c, l_2^c, it is evident that $x_i(l_1^c, l_2^c)$ should be parametrized by 4 unknown coefficients, modulo terms that are zero on the cut. Each x_i can be written therefore as $x_i = V_i + R_i$ where R_i contains terms proportional to all other 8 denominators ($j \neq i$) and thus vanishes on the cut solutions. This last remark brings us the notion of reducible and irreducible scalar products, RSP and ISP respectively. RSP are those scalar products that can be written in terms of the denominators. Consider for instance the term

$$\frac{x_9(l_1, l_2)}{D_1 \ldots D_8}$$

The cut solutions in that case are directly related to the one-loop sub-graphs, and there are four of them, $(l_1^\pm l_2^\pm)$ solving

$$D_1 = \ldots = D_8 = 0$$

For this term, all variables except z_4 and w_4 are RSP, as for instance,

$$z_1 = l_1 \cdot p_1 = \frac{1}{2}\left(D_2 - D_1 - p_1^2 + M_2^2 - M_1^2\right).$$

Moreover we know that z_4^2 and w_4^2 are expressible in terms of the denominators. Therefore the only four terms surviving the criterion of highest degree 3 are $\{1, z_4, w_4, z_4 w_4\}$, exactly four as it should be. The same holds for all other 8 terms in Eq. 4.3.

The AGC context offers a systematic framework to deal with the above considerations. Owing to Hilbert's Nullstellensatz theorem, the solution of Eq. 4.3 always exist, provided that the D_i have no common root. In fact this is the case for generic external momenta. AGC does not tell us – directly – what are the polynomials $x_i(l_1, l_2)$ that solve the above equation, neither their degree. But through multivariate division, using the Groebner basis concept and algorithm we can fully reconstruct the OPP master equation at two loops,

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}. \tag{4.4}$$

Several steps towards this direction have been undertaken, and the results are very promising [61].

5. Outlook

NNLO accuracy for $2 \to n$ scattering processes demands input from three contributions: the pure virtual two-loop $n+2$-particle amplitudes, dumped as virtual-virtual (VV), the interference of NLO and real corrections for $2 \to n+1$ (real-virtual, RV) with one unresolved parton and the real corrections for $2 \to n+2$, (real-real, RR) with two unresolved partons. Great progress has been achieved in all these branches [62, 63]: the main bottleneck for the moment is the computation of VV contributions. In contrast to the one-loop case, significant part of the difficulty is related to the efficient definition and computation of master integrals (MI). Unfortunately, little is known on the analytical expression of these MI in the general case, whereas at the moment numerical approaches [64, 65] seem to be impractical. With the advent of methods for two-loop amplitude computations, a new era may be open very soon in high-precision phenomenology.

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