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Nonlocal multiplicative anomaly in fermionic effective actions

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We discuss an ambiguity in the one-loop effective action of massive fields which takes place in massive fermionic theories. The universality of logarithmic UV divergences in different space-time dimensions leads to the nonuniversality of the finite part of effective action. This can be observed in the fermionic determinants and can be called nonlocal multiplicative anomaly. We will discuss this phenomenon in full details to the case of a Dirac fermion coupled to external scalar field by the Yukawa interaction.

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1. Introduction

The one-loop calculations have a prominent role in Quantum Field Theory (QFT) and in many of its most relevant applications. In the background field method the one-loop effective action (EA) can be always reduced to the derivation of $\text{Ln sDet}\hat{\mathscr{H}}$ of the operator $\hat{\mathscr{H}}$, which is typically a bilinear form of the classical action with respect to the quantum fields. In particular, relations such as

$$\det(\hat{\mathscr{A}}\cdot\hat{\mathscr{B}}) = \det\hat{\mathscr{A}}\cdot\det\hat{\mathscr{B}}, \qquad \text{tr}\ln\hat{\mathscr{A}} = \ln\det\hat{\mathscr{A}}\,, \tag{1.1}$$

which can be easily proved for finite dimensional matrices, may be incorrect for the functional determinants of differential operators, including the ones which are relevant for QFT applications. Considerable work has been done to prove that this equality can be violated, but in all previously known cases the difference could be reduced to renormalization ambiguity. In this manuscript we present an example where the difference between the two functional determinants is a *nonlocal* expression and therefore can not be explained by the renormalization ambiguity [1].

This contribution represents a brief version of our paper [2]. The content is not original, but the purpose is to present it in a maximally simple and qualitative form.

2. General considerations

Consider the one-loop quantum corrections to EA for a general fermionic operator

$$\bar{\Gamma}^{(1)} = -i\,\text{Ln sDet}\hat{\mathscr{H}}\,, \quad \hat{\mathscr{H}} = i\,(\gamma^\mu\nabla_\mu - im\hat{1} - i\hat{\Theta})\,. \tag{2.1}$$

Here sDet represents the functional superdeterminant (also called Berezinian) and $\hat{\Theta}$ corresponds to a condensed notation for a set of external fields.

We will define the expression (2.1) through the heat-kernel method and the Schwinger-DeWitt technique, and this requires reducing the problem to the derivation of $\text{Ln sDet}\,\hat{\xi}$, where[1]

$$\hat{\xi} = \hat{1}\Box + 2\hat{h}^\mu\nabla_\mu + \hat{\Pi}\,. \tag{2.2}$$

In order to use the Schwinger-DeWitt technique, we need to make a reduction to the second-order operator, for this end one has to multiply $\hat{\mathscr{H}}$ by an appropriate conjugate operator and arrive at $\hat{\xi} = \hat{\mathscr{H}}\cdot\hat{\mathscr{H}}^*$, and use the relation

$$\text{Ln sDet}\,\hat{\mathscr{H}} = \text{Ln sDet}\,\hat{\xi} - \text{Ln sDet}\,\hat{\mathscr{H}}^*\,. \tag{2.3}$$

Indeed, there is more than one option for choosing the conjugate operator which enables one to use the relation (2.3) in an efficient way. The simplest choice is

$$\hat{\mathscr{H}}_1^* \equiv \hat{\mathscr{H}} = i(\gamma^\mu\nabla_\mu - im\hat{1} - i\hat{\Theta}) \;\Rightarrow \text{Ln sDet}\,\hat{\mathscr{H}} = \frac{1}{2}\,\text{Ln sDet}\,\big(\hat{\mathscr{H}}\hat{\mathscr{H}}_1^*\big)\,. \tag{2.4}$$

An alternative choice of the conjugate operator is

$$\hat{\mathscr{H}}_2^* = i\,\big(\gamma^\mu\nabla_\mu - im\hat{1}\big) \;\Rightarrow \text{Ln sDet}\hat{\mathscr{H}}\Big|_{\hat{\Theta}} = \text{Ln sDet}\big(\hat{\mathscr{H}}\hat{\mathscr{H}}_2^*\big)\Big|_{\hat{\Theta}}\,, \tag{2.5}$$

[1] In this work we consider a Wick rotation of the Euclidean space to Minkowski space.

where the index $\hat{\Theta}$ means we are interested only in the $\hat{\Theta}$-dependent part of EA.

It is easy to note that if the relations (1.1) hold for the fermionic functional determinants, we are going to meet the two equal expressions,

$$\frac{1}{2}\,\mathrm{Ln\,sDet}\left(\mathscr{H}\mathscr{H}_1^*\right)\Big|_{\hat{\Theta}} = \mathrm{Ln\,sDet}\left(\mathscr{H}\mathscr{H}_2^*\right)\Big|_{\hat{\Theta}}. \tag{2.6}$$

As we shall see below, in reality the Eq. (2.6) is satisfied for divergencies, but not for the nonlocal finite parts of the two effective actions. This is nothing else, but the nonlocal version of Multiplicative Anomaly (MA) [3, 4]. The possibility of this mathematical feature of the functional determinants has been discussed for the long time on the basis of ζ-regularization (see, e.g., [5]), but it was soon realized that the difference can be just a manifestation of the different choice of μ for the distinct determinants [4]. The only safe way to obtain MA is to detect it in the nonlocal part of EA, which is qualitatively different from the local one related to divergences. In this case we will see that the MA is some new ambiguity of EA and not a particular case of the well-known μ-dependence.

Before starting practical calculations, let us make some general observations on the relation (2.6) for divergencies and for the finite part of EA. Within the heat-kernel method and Schwinger-DeWitt technique [6], the one-loop EA is given by the expression (see, e.g., [7])

$$\bar{\Gamma}^{(1)} = \frac{i}{2(4\pi)^{\omega}}\sum_{k=0}^{\infty} A_k \int_0^{\infty} ds\,(is)^{(k-\omega-1)}\exp\left(-ism^2\right), \tag{2.7}$$

where 2ω is space-time dimension and A_k are the global Schwinger-DeWitt coefficients.

The first three coefficients have the form

$$A_0 = \int d^{2\omega}x\,\sqrt{-g}\;\mathrm{tr}\,\hat{1}\,, \quad A_1 = \int d^{2\omega}x\,\sqrt{-g}\;\mathrm{tr}\,\hat{P}\,,$$
$$A_2 = \int d^{2\omega}x\,\sqrt{-g}\;\mathrm{tr}\left\{\frac{\hat{1}}{180}\left(R^2_{\mu\nu\alpha\beta} - R^2_{\mu\nu} + \Box R\right) + \frac{1}{2}\hat{P}^2 + \frac{1}{6}\left(\Box\hat{P}\right) + \frac{1}{12}\hat{S}^2_{\mu\nu}\right\}, \tag{2.8}$$

where the operators

$$\hat{P} = \hat{\Pi} + \frac{\hat{1}}{6}R - \nabla_\mu\hat{h}^\mu - \hat{h}_\mu\hat{h}^\mu\,, \quad \hat{S}_{\mu\nu} = \hat{1}[\nabla_\nu,\nabla_\mu] + \nabla_\nu\hat{h}_\mu - \nabla_\mu\hat{h}_\nu + \hat{h}_\nu\hat{h}_\mu - \hat{h}_\mu\hat{h}_\nu, \tag{2.9}$$

are defined in terms of operators (2.2).

Finally, is easy to see that only terms in the summation which can contain divergences are those with $k \leq \omega$ [6]. For example, in the 4-dimensional case the A_2 coefficient defines the logarithmic UV divergence, while the A_1 defines the quadratic divergences and A_0 is associated to quartic divergences.

3. Multiplicative anomaly in QFT

In this section we present an example in QFT in which the MA takes place in the framework of the Yukawa theory.

3.1 Calculation of A_k coefficients

In order to verify the validity of the relation (2.6), we perform the calculation of the first two Schwinger-DeWitt coefficients in two different schemes for calculating defined from Eqs. (2.4) and (2.5) and then evaluating the quantity

$$\triangle A_k(\omega,\hat{\Theta}) = A_k^{(1)}(\omega,\hat{\Theta})\Big|_{\Theta} - A_k^{(2)}(\omega,\hat{\Theta})\Big|_{\Theta}, \qquad k = 1,2, \tag{3.1}$$

which is associated with the phenomenon of MA.

In the general case of fermionic operator (2.1) with conjugate operators (2.4) and (2.5), one can take care of the most simple coefficient of A_1 to arrive at the criteria of existence for the MA. The calculations of the traces can be done by using Eq. (2.9). The difference between these two expressions can be presented as

$$\triangle A_1(\omega,\hat{\Theta}) = \frac{1}{2}\int d^{2\omega}x\sqrt{-g}\left\{(\omega-1)\,\mathrm{tr}\,(\hat{\Theta}\hat{\Theta}) - i\,\mathrm{tr}\,(\nabla_\mu\hat{\Theta}\gamma^\mu)\right\}. \tag{3.2}$$

We can see that this difference consists of two terms. The first one is proportional to $\omega-1$, exactly as we have anticipated in the previous section from general qualitative arguments. According to what we have discussed, this term does vanish in two dimensions ($\omega = 1$), where it defines the logarithmic UV divergence. However, due to the $(\omega-1)$ factor, it does not vanish in $\omega \neq 1$, and hence the quadratic divergence in $\omega = 2$ is scheme-dependent. Another part of (3.2) is the surface term, which is also quite remarkable, but for different reason. First of all, this kind of ambiguity is not related to the mass of quantum field and therefore has absolutely different origin compared to the terms of the first type as it was discussed previously in the literature on conformal anomaly [8]. Any way, the difference (3.2) includes two terms of very different origin which represent two distinct types of the QFT ambiguities and hence can not cancel.

Let us now perform complete analysis for a simplest case of Yukawa model where $\hat{\Theta} = \hat{1}h\phi$. The calculation of the second global Schwinger-DeWitt coefficient can be done in a usual way for the two calculational schemes (2.4) and (2.5) in 2ω space-time dimensions. The difference $\triangle A_2(\omega,\hat{\phi})$ can be presented in the form

$$\begin{aligned}\triangle A_2(\omega,\hat{\phi}) \;=\; & \int d^{2\omega}x\sqrt{-g}\left\{\frac{1}{3}h^2\Box\phi^2 + 2\left[m^2h^2\phi^2 + mh^3\phi^3 + \frac{7}{24}h^4\phi^4\right](\omega-2)^3\right.\\ &+\;\left[7m^2h^2\phi^2 - \frac{1}{12}Rh^2\phi^2 - \frac{1}{6}(\nabla\phi)^2h^2 + 7mh^3\phi^3 + \frac{25}{12}h^4\phi^4\right](\omega-2)^2\\ &+\;2\left.\left[3m^2h^2\phi^2 - \frac{1}{12}Rh^2\phi^2 - \frac{1}{6}(\nabla\phi)^2h^2 + 3mh^3\phi^3 + \frac{11}{12}h^4\phi^4\right](\omega-2)\right\}.\end{aligned} \tag{3.3}$$

It is easy to see that the difference consists of two kinds of terms. All but the first term do vanish in and only in the four dimensional case, exactly as the difference $\triangle A_1$ vanish in two dimensions. Obviously, the structure of these terms confirm our consideration about the universality of the dynamical terms in logarithmic UV divergences [9] and, at the same time, the non-universality of power-like divergences and finite terms in the case $\omega \neq 2$.

3.2 Calculation of form factors and β-functions to Yukawa theory

The calculation of form factors has been described in full details in [10], so we shall just give the result of the calculations in our case. The one-loop contribution to the EA can be presented in the form

$$\bar{\Gamma}^{(1)} = \frac{1}{2}\int d^4x\sqrt{-g}\left\{\nabla_\mu\phi\, k_{kin}(a)\,\nabla^\mu\phi + \phi^2\, k_{R\phi^2}(a)\,R + \phi^2\, k_{\phi^2\phi^2}(a)\,\phi^2\right\}, \tag{3.4}$$

where the form factors k are defined in terms of useful notations

$$Y = 1 - \frac{1}{a}\ln\left(\frac{2+a}{2-a}\right), \quad a^2 = \frac{4\Box}{\Box - 4m^2}. \tag{3.5}$$

We have found the following two sets of form factors corresponding to the calculational schemes (2.4) and (2.5).

$$\begin{aligned}
k^{(1)}_{R\phi^2}(a) &= -\frac{h^2(-14a^2+45Ya^2-168Y)}{9(4\pi)^2a^2}, \quad k^{(2)}_{R\phi^2}(a) = -\frac{h^2(-3a^2+10Ya^2-36Y)}{3(4\pi)^2a^2},\\
k^{(1)}_{\phi^2\phi^2}(a) &= \frac{2h^4(-8a^2+27Ya^2-96Y)}{3(4\pi)^2a^2}, \quad k^{(2)}_{\phi^2\phi^2}(a) = \frac{2h^4(6Ya^2-a^2-12Y)}{3(4\pi)^2a^2},\\
k^{(1)}_{kin}(a) &= -\frac{2h^2(a^2+12Y)}{3(4\pi)^2a^2}, \quad k^{(2)}_{kin}(a) = -\frac{h^2(a^2+3Ya^2+12Y)}{3(4\pi)^2a^2}.
\end{aligned} \tag{3.6}$$

The UV $(a \to 2)$ limits of the two sets of expressions do coincide,

$$k^{(1,2)\ UV}_{R\phi^2} = -\frac{h^2\ln(a-2)}{6(4\pi)^2}, \quad k^{(1,2)\ UV}_{\phi^2\phi^2} = \frac{h^4\ln(a-2)}{(4\pi)^2}, \quad k^{(1,2)\ UV}_{kin} = -\frac{h^2\ln(a-2)}{(4\pi)^2}, \tag{3.7}$$

and the IR limit $(a \to 0)$ for the same form factors are different,

$$\begin{aligned}
k^{(1)\ IR}_{R\phi^2} &= \frac{11h^2}{60(4\pi)^2}a^2 + \mathcal{O}(a^4), \quad k^{(2)\ IR}_{R\phi^2} = \frac{23h^2}{180(4\pi)^2}a^2 + \mathcal{O}(a^4),\\
k^{(1)\ IR}_{\phi^2\phi^2} &= -\frac{7h^4}{10(4\pi)^2}a^2 + \mathcal{O}(a^4), \quad k^{(2)\ IR}_{\phi^2\phi^2} = -\frac{7h^4}{30(4\pi)^2}a^2 + \mathcal{O}(a^4),\\
k^{(1)\ IR}_{kin} &= \frac{h^2}{10(4\pi)^2}a^2 + \mathcal{O}(a^4), \quad k^{(2)\ IR}_{kin} = \frac{2h^2}{15(4\pi)^2}a^2 + \mathcal{O}(a^4).
\end{aligned} \tag{3.8}$$

Another way to observe the MA in massive theories is though the physical β-functions. In the framework of the momentum-subtraction renormalization scheme, we obtain the corresponding β-functions

$$\begin{aligned}
\beta^{(1)}_\xi &= \frac{h^2\left\{a^2(15a^2-56)+(228a^2-672-15a^4)Y\right\}}{12(4\pi)^2a^2}, \beta^{(2)}_\xi = \frac{h^2\left\{a^2(5a^2-18)+(74a^2-216-5a^4)Y\right\}}{6(4\pi)^2a^2},\\
\beta^{(1)}_\lambda &= -\frac{h^4\left\{a^2(9a^2-32)+(132a^2-384-9a^4)Y\right\}}{2(4\pi)^2a^2}, \beta^{(2)}_\lambda = -\frac{h^4\left\{a^2(a^2-2)+(10a^2-24-a^4)Y\right\}}{(4\pi)^2a^2},\\
\gamma^{(1)}_{kin} &= \frac{2h^2\left\{a^2+(12-3a^2)Y\right\}}{(4\pi)^2a^2}, \quad \gamma^{(2)}_{kin} = \frac{h^2\left\{a^2(a^2+4)+(48-8a^2-a^4)Y\right\}}{4(4\pi)^2a^2}.
\end{aligned} \tag{3.9}$$

The UV limit in the complete β-functions (3.9) is the same for the two calculational approaches,

$$\beta^{(1,2)\,UV}_\xi = \frac{h^2}{3(4\pi)^2}, \quad \beta^{(1,2)\,UV}_\lambda = -\frac{2h^4}{(4\pi)^2}, \quad \gamma^{(1,2)\,UV}_{kin} = \frac{2h^2}{(4\pi)^2}. \tag{3.10}$$

In the opposite, IR, limit the situation is quite different, indicating an ambiguity in the Appelquist and Carazzone theorem [11],

$$\begin{aligned}
\beta_\xi^{(1)\,IR} &= \frac{11h^2}{30(4\pi)^2}a^2+O(a^4) \;\;, \beta_\xi^{(2)\,IR} = \frac{23h^2}{90(4\pi)^2}a^2+O(a^4)\,,\\
\beta_\lambda^{(1)\,IR} &= -\frac{7h^4}{5(4\pi)^2}a^2+O(a^4) \;\;, \beta_\lambda^{(2)\,IR} = -\frac{7h^4}{15(4\pi)^2}a^2+O(a^4)\,,\\
\gamma_{kin}^{(1)\,IR} &= \frac{h^2}{5(4\pi)^2}a^2+O(a^4) \;\;, \gamma_{kin}^{(2)\,IR} = \frac{4h^2}{15(4\pi)^2}a^2+O(a^4)\,.
\end{aligned} \tag{3.11}$$

We can see that in all cases the decoupling in Eqs. (3.11) is quadratic, according to the Appelquist and Carazzone theorem, but the coefficients depend on the choice of calculational scheme. Let us note that from physical viewpoint the first choice with $\hat{\mathscr{H}}_1^*$ is much better because it helps to preserve the gauge invariance [10].

4. Conclusions

We have explored in details an ambiguity which takes place in the derivation of fermionic functional determinants by means of the heat kernel method. There are two kind of ambiguities, which have essentially different origins. The first one takes place only in case of massive theories and shows the deep importance and universality of the logarithmic UV divergences. Another sort of ambiguity does not depend on whether the quantum field is massive or massless, it occurs in the divergent total-derivative terms. The common point is that both of them *can not* be compensated by the change of coefficients in the infinite local counterterms, which are introduced in the process of renormalization. It would be very interesting to find other examples of such an ambiguity for other (nonfermionic) theories, and we hope to find such examples in the future.

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