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Properties of mesons with beauty and charm in the relativistic Hamiltonian approach

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On the basis of the investigation of the asymptotic behaviour of the correlation functions of the corresponding field currents with the necessary quantum numbers the analytic method for the determination of the mass spectrum and decay constants of mesons consisting of c and b quarks with relativistic corrections is proposed. The dependence of the constituent mass of quarks on the current mass and on the orbital and radial quantum numbers is analytically derived. The mass and wave functions of the mesons are determined via the eigenvalues of nonrelativistic Hamiltonian in which the kinetic energy term is defined by the constituent mass of the bound state forming particles and the potential energy term is determined by the contributions of every possible type of Feynman diagrams with an exchange of gauge field. In the framework of our approach the mass splitting between the singlet and triplet states is determined, and the width of $E1$ transition rates in the $(\bar{c}c)$, $(\bar{b}b)$ and $(\bar{b}c)$ systems are calculated. The obtained results are satisfactorily agree with the experimental data.

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1. Interaction

Bound state formation and description in the framework of the relativistic quantum field theory (QFT) is not yet a well-stated problem (see ref. [1, 2, 3]). The QFT describes elastic and inelastic scattering of free relativistic particles that are in a plane wave state when they are at a large distance from each other. The QFT formulation is based on the perturbation theory, i.e. on the coupling constant expansion series, where, in principal, any bound state cannot appear. Therefore, the bound state problem requires one to go beyond the framework of the perturbation theory, where the existing methods are not yet well-developed.

On the other hand, it is well known that the bound state energy spectrum can be determined with a good accuracy in the framework of nonrelativistic quantum mechanics (NQM) when an appropriate interaction potential is chosen. Nevertheless, the nonrelativistic Schrodinger equation (SE), which mathematically correctly describes the bound states, is no longer sufficient since for the description of modern experimental results, obtained in both atomic [4] and hadronic physics [5], it is necessary to take into account relativistic corrections.

Therefore, the real physics requires a creation of some mathematical solution of the bound state problem which is based on the QFT. All the efforts made in this direction could be divided into two directions.

The initial step of one direction is based on the statement that if there is a two-particle bound state with the corresponding quantum numbers, then the elastic scattering amplitude of these two-particles has a simple pole on the energy at a bound state mass point. On the basis of this idea the Bethe-Salpeter [3, 6, 7] and the so-called quasi-potential equations [8] were formulated.

The other direction is based on the statement that the nonrelativistic SE is an efficient tool for the bound state energy spectrum investigation and determination. The real relativistic corrections are small, so the theoretical problem reduces to an obtaining of the relativistic corrections to the nonrelativistic interaction potential based on the QFT formalism. This idea underlies the Breit potential [9] and the effective nonrelativistic quantum field theory of Caswell and Lepage [10]. Both these approaches use the scattering matrix as a source of required corrections. In [10,11], the nonrelativistic QED method (NRQED) for the Coulomb bound state energy spectrum determination by taking into account relativistic corrections was formulated. In hadronic physics, the hadron mass spectrum description of the orbital and radial excited states is one the fundamental problems. At the present stage, there are phenomenological quark potential models [12, 13] that well describe the hadron mass spectrum. However, most of these models consist of many parameters most of which are not physically justified or are applicable only for some particular cases.

There is another approach in the framework of the latter direction which is based on the Fock-Feynman-Schwinger representation suggested in [14]. Later, this method was improved [15, 16] and successfully applied [17] for the description of the hadron and glueball mass spectra. The present work is a direct continuation of these works. The exact quantum-field Green functions can be formally represented in the functional integral form. This functional integral evaluation technique is still in its infancy; however, the existing representations can be used for obtaining the nonrelativistic SE solution in the Feynman functional integral form with the potential consisting of necessary relativistic corrections. Our investigations continue these efforts. In [18], the energy spectrum evaluation technique is suggested which is based on the investigation of the asymptotic behavior of the vacuum averaging (of Green's functions) of the scalar charged particle currents in the external gauge field. In defining the correlation function asymptotic behavior the functional integral form representation is used so that the averaging over the external gauge field can be done precisely. The obtained representation is similar to the Feynman path integral [19] in nonrelativistic quantum mechanics. In this case, the nonlocal interaction functional (potential), which appears due to the gauge field (gluon) exchange, is defined by the Feynman diagram and contains a contribution to both self-energy of the particles and the bound state formation. Thus, the interaction potential is defined by the every possible type Feynman diagrams with exchange of gauge fields.

The paper is organized as follows. In section 2,we describe in detail the determination of the mass and constituent mass bound state system. In section 3, the mass spectrum of mesons consisting of c and b quarks, with the orbital and radial excitations is defined. The dependence of the constituent mass of the constituent particles on the mass of the initial state, as well as radial and orbital quantum numbers is determind. The obtained results are in satisfactory agreement with the available experimental data. Finally, the main results of the calculations are summarized. In Appendices the details of calculation of the energy spectrum bound state in the framework of the oscillator representation are given.

2. Determining the mass of the relativistic bound state

We now briefly discuss the details of our approach. Let us denote $J(x) = \Phi^+(x)\Phi(x)$ as the current of scalar charged particles. If we neglect the annihilation channel, then it is convenient to represent the considered correlators as the averaging over the gauge field $A_\alpha(x)$ of a product of the Green functions $G_m(x,y|A)$ of the scalar charged particles in the external gauge field:

$$\begin{aligned}\Pi(x-y) &= \langle J(x)J(y)\rangle = \langle \Phi^+(x)\Phi(x)\Phi^+(y)\Phi(y)\rangle \\ &= \langle G_{m_1}(x,y|A)G_{m_2}(y,x|A)\rangle_A \ .\end{aligned} \tag{2.1}$$

The Green function $G_m(y,x|A)$ for the scalar particle in the external gauge field is determined from the equation

$$\left[\left(i\frac{\partial}{\partial x_\alpha} + \frac{g}{c\hbar}A_\alpha(x)\right)^2 + \frac{c^2m^2}{\hbar^2}\right]G_m(x,y|A) = \delta(x-y). \tag{2.2}$$

The solution of (2.2) can be represented as a functional integral in the following way (for details see [20]):

$$G_m(x,y|A) = \int_0^\infty \frac{ds}{(4s\pi)^2}\exp\left\{-sm^2 - \frac{(x-y)^2}{4s}\right\}\int d\sigma_\beta \exp\left\{ig\int_0^1 d\xi\,\frac{\partial Z_\alpha(\xi)}{\partial\xi}A_\alpha(\xi)\right\} \ . \tag{2.3}$$

Here the following notation is used:

$$\begin{aligned} Z_\alpha(\xi) &= (x-y)_\alpha\xi + y_\alpha - 2\sqrt{s}B_\alpha(\xi); \\ d\sigma_\beta &= N\delta B_\beta \exp\left\{-\frac{1}{2}\int_0^1 d\xi\,\dot{B}^2(\xi)\right\}, \end{aligned} \tag{2.4}$$

with the normalization

$$B_\beta(0) = B_\beta(1) = 0\,; \quad \int d\sigma_\beta = 1,$$

where N is the normalization constant. In averaged over the external gauge field $A_\alpha(x)$ we limit ourselves to the lowest order, i.e. we take into account only the two-point Gaussian correlator:

$$\left\langle \exp\left\{ i\int dx A_\alpha(x) J_\alpha(x) \right\} \right\rangle_A = \exp\left\{ -\frac{1}{2} \iint dx\, dy J_\alpha(x) D_{\alpha\beta}(x-y) J_\beta(y) \right\}. \tag{2.5}$$

Here $J_\alpha(x)$ is the real current. The propagator of the gauge field has the following form:

$$D_{\alpha\beta}(x-y) = \langle A_\alpha(x) A_\beta(y) \rangle_A = \delta_{\alpha,\beta} D(x-y) + \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_d(x-y), \tag{2.6}$$

where

$$D(x) = \int \frac{dq}{(2\pi)^4} \frac{\exp\{iqx\}}{q^2}\,, \quad D_d(x) = \int \frac{dq}{(2\pi)^4} \frac{\exp\{iqx\}}{q^2} \frac{d(q^2)}{q^2}\,. \tag{2.7}$$

So the external field exists only in a virtual state. The mass of the bound state is usually defined through the correlation function in the following way:

$$M = -\lim_{|x-y|\to\infty} \frac{\ln \Pi(x-y)}{|x-y|}\,. \tag{2.8}$$

Thus, if we know the correlation function, then we can determine the bound state mass.

From (2.8) one can see that for determination of the mass M one needs to calculate correlation function $\Pi(x)$ in the asymptotics $|x| \to \infty$. Substituting (2.3) into (2.1) and averaging over the external gauge field we obtain:

$$\Pi(x) = \iint\limits_0^\infty \frac{d\mu_1\, d\mu_2}{(8\pi^2 x)^2} J(\mu_1,\mu_2) \exp\left\{ -\frac{|x|}{2}\left(\frac{m_1^2}{\mu_1} + \mu_1 \right) - \frac{|x|}{2}\left(\frac{m_2^2}{\mu_2} + \mu_2 \right) \right\}. \tag{2.9}$$

Here

$$\begin{aligned} J(\mu_1,\mu_2) &= N_1 N_2 \iint \delta r_1\, \delta r_2 \exp\left\{ -\frac{1}{2}\int\limits_0^x d\tau \left[\mu_1 \dot{r}_1^2(\tau) + \mu_2 \dot{r}_2^2(\tau) \right] \right\} \exp\{-W\}, \\ W &= W_{1,1} + W_{2,2} - 2W_{1,2}, \end{aligned} \tag{2.10}$$

and following notation is used:

$$W_{i,j} = \frac{g^2}{2}(-1)^{i+j} \int\limits_0^x \int\limits_0^x d\tau_1\, d\tau_2 Z'^{(i)}{}_\alpha(\tau_1) D_{\alpha\beta} \left(Z^{(i)}(\tau_1) - Z^{(j)}(\tau_2) \right) Z'^{(j)}{}_\beta(\tau_2). \tag{2.11}$$

Representation (2.10) is analogous to the quantum Green function in the Feynman functional integral, when two particles with masses μ_1 Í μ_2 interacts via the nonlocal potential $W_{i,j}$. Therefore, we call mass m_1 and m_2 the current, and parameters μ_1 and μ_2 constituent masses. We emphasize that in (2.10) the functional integration is made over the four-vectors $r_1 = (\vec{r}_1, r_1^{(4)})$, $r_2 = (\vec{r}_2 and r_2^{(4)})$.

The term $W_{i,j}$, in this case, is defined by all kinds of Feynman diagrams. There are two types of interactions: the first is the interaction of the constituent particle via the gauge field the contribution of which is defined by the term $W_{i,j}, (i \neq j)$; the second is the interaction of the constituent particles with each other, i.e. the self-energy diagram the contribution of which is defined by the terms $W_{1,1}$ and $W_{2,2}$. In the nonrelativistic limit the terms $W_{i,j}$ correspond to the potential interactions, whereas the terms $W_{j,j}$ correspond to the nonpotential interactions which define the renormalization mass contribution.

In the asymptotics $|x| \to \infty$ the integral (2.10) behaves like:

$$\lim_{|x|\to\infty} J(\mu_1,\mu_2) \Longrightarrow \exp\{-xE(\mu_1,\mu_2)\}, \tag{2.12}$$

where the function $E(\mu_1,\mu_2)$ depends on the coupling constant g and parameters μ_1, μ_2, and is independent of the masses m_1, m_2. If $|x| \to \infty$ the integral (2.9) is calculated by the saddle point method. The bound state mass is determined by the saddle -point:

$$M = \frac{1}{2} \min_{\mu_1,\mu_2} \left\{ \frac{m_1^2}{\mu_1} + \mu_1 + \frac{m_2^2}{\mu_2} + \mu_2 + 2E(\mu_1,\mu_2) \right\}. \tag{2.13}$$

Thus, the problem reduced to calculation of the functional integral in (2.10). However, this integral cannot be evaluated in a general form and is defined in various framework approaches. At present, there are no exact mathematical methods for the evaluation of this integral. Therefore, we have to apply some physical assumptions or approaches in order to somehow perform the integration over the fourth components of $r_j^{(4)}$. The integration over the fourth components effectively corresponds to the transition to the nonrelativistic limit. In other words, we define the interaction potential with the corrections connected with the nonperturbative, relativistic and nonlocal characters of the interaction. In particular, if we neglect the dependence of the functional $W_{i,j}$ in (2.11) on $r_1^{(4)}$ and $r_2^{(4)}$, then the system (2.10) is reduced to the Feynman path integral of the scalar particles with the masses μ_1 and μ_2 in NRQM [19] with the local potential. In this approximation, according to (2.10), the interaction Hamiltonian of the scalar particles with the masses μ_1 and μ_2 reads:

$$H = \frac{1}{2\mu_1}\mathbf{P}_1^2 + \frac{1}{2\mu_2}\mathbf{P}_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2), \tag{2.14}$$

where $V(\mathbf{r}_1 - \mathbf{r}_2)$ interaction potential, which is expressed in terms of $W_{i,j}$, then $E(\mu_1,\mu_2)$ is the eigenvalue of the interaction Hamiltonian (2.14), i.ě.

$$H\Psi(\mathbf{r}_1,\mathbf{r}_2) = E(\mu_1,\mu_2)\Psi(\mathbf{r}_1,\mathbf{r}_2). \tag{2.15}$$

Then, from the minimum condition of (2.13) one obtains the equation for μ_j:

$$\mu_j - \frac{m_j^2}{\mu_j} + 2\mu_j \frac{dE(\mu_1,\mu_2)}{d\mu_j} = 0\,; \quad j = 1,\ 2. \tag{2.16}$$

The parameters μ_1, μ_2 have the dimension of mass. In further calculations we introduce a new parameter

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2}. \tag{2.17}$$

Then equation (2.13)takes the form:

$$M = \mu_1 + \mu_2 + \mu \frac{dE}{d\mu} + E(\mu) \; ; \qquad E(\mu_1, \mu_2) = E(\mu), \tag{2.18}$$

where

$$\mu_1 = \sqrt{m_1^2 - 2\mu^2 \frac{dE}{d\mu}} \; ; \quad \mu_2 = \sqrt{m_2^2 - 2\mu^2 \frac{dE}{d\mu}} \; . \tag{2.19}$$

In our approach, the energy spectrum and the wave function bound state are determined by the SE with the constituent mass μ_1 and μ_2. The corrections connected with the relativistic character of the interaction are taken into account not only by the corrections to the interaction potential, but also by the parameters μ_1 and μ_2 (constituent masses), which are defined in (2.16). Therefore, from the SE with the constituent mass, we will determine the energy spectrum and wave function of the atomic and hadronic bound states system by taking into account the relativistic correction.

Now we apply our approach for determining the mass and energy spectrum, as well as to determine the decay width of mesons consisting of b and c quarks with orbital and radial excitations.

3. Mess spectrum of mesons with orbital excitations

3.1 The Hamiltonian of the interaction.

Let us, we determine the mass spectrum of charmonium, bottom and B_c mesons with spin-spin and spin-orbit interactions is determined from the SE with the constituent mass. The total interaction Hamiltonian of quarks is represented as:

$$H = H_c + H_{spin}, \tag{3.1}$$

where H_c is the central Hamiltonian

$$H_c = \frac{1}{2\mu} \vec{P}^2 + \sigma \cdot r - \frac{4}{3} \frac{\alpha_s}{r} \; . \tag{3.2}$$

The second part of the Hamiltonian describes the spin-orbit interaction and is written in the standard form (for details see [21, 22]):

$$H_{spin} = H_{SS} + H_{LS} + H_{TT}. \tag{3.3}$$

Here H_{SS} is the spin-spin interaction Hamiltonian:

$$H_{SS} = \frac{2}{3\mu_1 \, \mu_2} \, (\mathbf{S_1 S_2}) \, \Delta V_{\text{v}} = \frac{32 \, \pi \, \alpha_s \, (\mathbf{S_1 S_2})}{9\mu_1 \mu_2} \cdot \delta(\mathbf{r}) \; , \tag{3.4}$$

also H_{LS} is the describing the spin-orbital interaction:

$$\begin{aligned} H_{LS} \; = \; & \frac{1}{4 \, \mu_1^2 \mu_2^2} \frac{1}{r} \left\{ \left[\left((\mu_1 + \mu_2)^2 + 2\mu_1\mu_2 \right) (\mathbf{L} \cdot \mathbf{S}_+) + (\mu_1^2 - \mu_2^2)(\mathbf{L} \cdot \mathbf{S}_-) \right] \frac{\partial}{\partial r} V_{\text{v}} \right. \\ & - \left. \left[(\mu_1^2 + \mu_2^2)(\mathbf{L} \cdot \mathbf{S}_+) + (\mu_1^2 - \mu_2^2)(\mathbf{L} \cdot \mathbf{S}_-) \right] \frac{\partial}{\partial r} V_s \right\} \; , \end{aligned} \tag{3.5}$$

and finally the tensor Hamiltonian of the interaction is

$$H_{TT} = \frac{1}{12\mu_1\mu_2}\left[\frac{1}{r}\frac{\partial}{\partial r}V_{\rm v} - \frac{\partial^2}{\partial r^2}V_{\rm v}\right] S_{12}\ . \tag{3.6}$$

Here $V_{\rm v}$ is the vector potential corresponding to the one-gluon exchange:

$$V_{\rm v} = -\frac{4\alpha_s}{3}\frac{1}{r}\ ; \tag{3.7}$$

and V_s is the confinement potential

$$V_s = r\sigma\ ; \tag{3.8}$$

and we introduced the following notation:

$$\mathbf{S}_+ = \mathbf{S_1} + \mathbf{S_2}\ ; \quad \mathbf{S}_- = \mathbf{S_1} - \mathbf{S_2}\ ; \tag{3.9}$$
$$S_{12} = \frac{4}{(2\ell+3)(2\ell-1)}\left[\mathbf{L}^2\mathbf{S}^2 - \frac{3}{2}(\mathbf{LS}) - 3(\mathbf{LS})^2\right]\ .$$

Using expressions (3.1-3.9) for the interaction Hamiltonian we calculate the mass spectrum of mesons.

3.2 The energy spectrum of quarkonium

Now, using the explicit form of the total Hamiltonian let us determine the energy spectrum of quarkonium. We determine the energy spectrum and wave functions from the SE

$$H\Psi = E\Psi\ . \tag{3.10}$$

we will apply the oscillator-representation (OR) method [23] for determination of eigenvalues and the wave functions(WF) from the SE (3.10). Before determining the energy spectrum and WF of the SE by means of the OR method [23] it should be recalled that this method is based on the ideas and techniques of quantum field theory. One of the essential differences of QFT from quantum mechanics is that quantized fields, which represent an assembly of an infinite number of oscillators for the ground state (or vacuum), keep their oscillatory nature in the quantum-field interaction. In QM eigenfunctions of most potentials differ from the Gauss behaviour of the oscillatory wave function. Therefore, the variables in the original SE must be changed so that the modified equation should have solutions with the oscillator behavior at large distances. Since this transformation is not a canonical one, after the transformation we have a new system with another set of quantum numbers and wave functions which contains, however, a subset of the original wave functions. The transformation of variables leading to the Gaussian asymptotic behavior in the expanded space is one of the basic elements of the ORM. Let us note that a similar idea was discussed by Fock in the solution of the problem about the hydrogen spectrum using the transformation into the four-dimensional momentum space [24]. According to the statements above, let us change the variables in the following way (see details in refs.[23, 25]):

$$r = q^{2\rho}, \qquad \Psi \Rightarrow \Psi(q^2) = q^{2\rho\ell}\Phi(q^2). \tag{3.11}$$

Using the atomic system of units ($\hbar = c = 1$), considering (3.1)-(3.9) and after some standard simplifications from (3.10) obtain for the modified Schrodinger equation:

$$\Big\{ -\frac{1}{2}\left(\frac{\partial^2}{\partial q^2}+\frac{d-1}{q}\frac{\partial}{\partial q}\right)-4\rho^2\,\mu E\,q^{2(2\rho-1)}+4\rho^2\,\mu\,\sigma\,q^{2(3\rho-1)}- \tag{3.12}$$

$$-\frac{16\,\rho^2\,\mu\,\alpha_s}{3}\cdot q^{2(\rho-1)}+\frac{64\alpha_s\mu\rho^2}{9\pi\mu_1\mu_2}\cdot(\vec{S}_1\vec{S}_2)\lim_{\Lambda\to\infty}\int_0^\Lambda dt\,q^{2(\rho-1)}t\sin(tq^{2\rho})-$$

$$-\frac{\sigma\rho^2\mu}{\mu_1^2\mu_2^2}q^{2(\rho-1)}\left[\left((\mu_1^2+\mu_2^2)\right)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]+\frac{4\mu\rho^2\alpha_s}{3\mu_1\mu_2}\cdot\frac{S_{12}}{q^{2(\rho+1)}}+$$

$$+\frac{4\mu\rho^2\alpha_s}{3\,\mu_1^2\mu_2^2q^{2(\rho+1)}}\left[\left((\mu_1+\mu_2)^2+2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\Big\}\Phi(q^2)=0\,,$$

where d is the dimension of the auxiliary space:

$$d = 2+2\rho+4\rho\ell. \tag{3.13}$$

As a result of the change of variables, we get the modified SE in the d-dimensional auxiliary space R^d. From (3.12) and (3.13) it follows that the orbital quantum number ℓ has entered into the dimension d of the space. This technique allows us to determine all characteristics we are interested in the spectrum and WF by solving the modified SE only for the ground state in the d-space. The wave function $\Psi_m(q^2)$ of the ground state depends only on the q^2 variable. Thus, the operator

$$\frac{\partial^2}{\partial q^2}+\frac{d-1}{q}\frac{\partial}{\partial q}\equiv\triangle_q, \tag{3.14}$$

can be identified with the Laplacian in the R^d space which acts on the ground state wave function depending only on the radius q. The modified SE written as

$$H\Phi(q) = \varepsilon(E)\,\Phi(q), \tag{3.15}$$

can be seen, according to (3.12), $\varepsilon(E)$ is to be equal to zero in R^d

$$\varepsilon(E) = 0. \tag{3.16}$$

We will consider this equation as the condition for determination of the energy spectrum E of the initial system. Following the OR method, let us represent the canonical variables in terms of the creation (a^+) and annihilation (a) operators in the R^d space

$$q_j = \frac{a_j+a_j^+}{\sqrt{2\omega}},\quad P_j = \sqrt{\frac{\omega}{2}}\cdot\frac{a_j-a_j^+}{i},\quad j=1,...,d,\quad [a_i,a_j^+]=\delta_{i,j}, \tag{3.17}$$

where ω is the oscillator frequency which has been unknown yet. Substituting (3.17) into (3.15) and carrying out ordering by the creation and annihilation operators we obtain

$$H = H_0+\varepsilon_0(E)+H_I. \tag{3.18}$$

Here H_0 is the Hamiltonian of the free oscillators:

$$H_0 = \omega(a_j^+a_j) \tag{3.19}$$

and ε_0 is the energy of the ground state in the zero approximation of OR:

$$\begin{aligned}\varepsilon_0(E) \;=\; & \frac{d\omega}{4} - \frac{4\rho^2 E\ \mu}{\omega^{2\rho-1}} \frac{\Gamma(d/2+2\rho-1)}{\Gamma(d/2)} - \frac{16\alpha_s \mu\rho^2}{3\omega^{\rho-1}} \frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)} + \\ & + \frac{4\rho^2\sigma\ \mu}{\omega^{3\rho-1}} \cdot \frac{\Gamma(d/2+3\rho-1)}{\Gamma(d/2)} + \frac{32\alpha_s\mu\rho}{9\mu_1\mu_2} \cdot \frac{(\vec{S}_1\vec{S}_2)\ \omega^{d/2}}{\Gamma(d/2)} \cdot \delta_{\ell,0} - \\ & - \frac{\rho^2\sigma\ \mu}{M_1^2\ \omega^{\rho-1}} \cdot \frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)} + \frac{4\alpha_s\mu\rho^2 S_{12}}{3\ \mu_1\mu_2} \cdot \frac{\omega^{\rho+1}\Gamma(d/2-\rho-1)}{\Gamma(d/2)} + \\ & + \frac{4\alpha_s\mu\rho^2}{3\ M_2^2} \cdot \frac{\omega^{\rho+1}\Gamma(d/2-\rho-1)}{\Gamma(d/2)}\ . \end{aligned} \tag{3.20}$$

Here the following notation is used:

$$\begin{aligned}\frac{1}{M_1^2} \;&=\; \frac{1}{\mu_1^2\ \mu_2^2}\left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\ ; \\ \frac{1}{M_2^2} \;&=\; \frac{1}{\mu_1^2\ \mu_2^2}\left[\left((\mu_1+\mu_2)^2+2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\ . \end{aligned} \tag{3.21}$$

The interaction Hamiltonian H_I can be represented in the normal form of the creation a^+ and a operators and it does not contain the quadratic terms of the canonical variables

$$H_I = \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d \exp\left\{-\eta^2(1+x)\right\} : e_2^{-i\sqrt{x\omega}(q\eta)} : \tag{3.22}$$

$$\begin{aligned}&\left[-\frac{4\rho^2\mu}{\omega^{2\rho-1}}\frac{Ex^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\mu}{\omega^{3\rho-1}}\frac{\sigma x^{-3\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} - \right.\\ &- \frac{\sigma\rho^2\mu}{M_1^2\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} + \frac{4\rho^2\mu\alpha_s S_{12}}{3\mu_1\mu_2}\frac{\omega^{\rho+1}x^\rho}{\Gamma(1+\rho)} + \frac{4\rho^2\mu\alpha_s}{3M_2^2}\frac{\omega^{\rho+1}x^\rho}{\Gamma(1+\rho)} + \\ &\left. + \frac{16\rho^2\mu\alpha_s(\vec{S}_1\vec{S}_2)}{9\pi\mu_1\mu_2}\lim_{\Lambda\to\infty}\sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)!}\frac{\Lambda^{2j+3}}{2j+3}\frac{x^{-2\rho-2\rho j}}{\omega^{2\rho+2\rho j-1}\Gamma(1-2\rho-2\rho j)}\right].\end{aligned}$$

Here $:\star:$ is a symbol of normal ordering, and we used the notation:

$$e_2^{-x} = e^{-x} - 1 + x - \frac{1}{2}x^2.$$

Some details of the representation in the normal form of the interaction Hamiltonian H_I are given in Appendix A. The contribution of the interaction Hamiltonian is considered as small perturbation. In quantum field theory, after the representation of the canonical variables in terms of the creation and annihilation operators and after transformation of the interaction Hamiltonian into the normal form, the requirement of the absence of the second order field operators is equivalent, in essence, to the renormalization of the coupling constant and the wave function [26]-[28]. Moreover, such a procedure permits one to take the main contribution into consideration in terms of the mass renormalization and in terms of the vacuum energy. In other words, all quadratic terms are completely included in the free oscillator Hamiltonian. This requirement allows formulate the following condition, according to the OR [23]

$$\frac{\partial\varepsilon_0(E)}{\partial\omega} = 0\ , \tag{3.23}$$

in order to find the frequency ω of the oscillator, which determines the main quantum contribution. Taking into account (3.20), from equations (3.16) and (3.23) we can calculate the energy spectrum of the initial system E. In the framework of the OR method for various potentials [25] it has been shown that the corrections connected with the interaction Hamiltonian are the first order corrections identically equal to zero and the second order corrections are less than one per cent. So let us restrict ourselves only to the consideration of the zeroth order approximation.

3.3 The mass spectrum of the mesons for the ground state.

Let us determine the mass spectrum and wave functions of mesons consisting of the b and c quarks. First of all, we consider the basic state, i.e, determine the properties of η_c, J/ψ, η_b, Υ and B_c - mesons taking into account the spin-spin interaction. From (3.20) we get for the ground state

$$\begin{aligned}\varepsilon_0(E) = \frac{d\omega}{4} - \frac{4\rho^2 E\mu}{\omega^{2\rho-1}}\frac{\Gamma(3\rho)}{\Gamma(1+\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{\Gamma(2\rho)}{\Gamma(1+\rho)} \\ + \frac{4\rho^2\sigma\mu}{\omega^{3\rho-1}}\frac{\Gamma(4\rho)}{\Gamma(1+\rho)} + \frac{16\alpha_s\mu\rho\omega^{\rho+1}}{3\mu_1\mu_2}\frac{[s(s+1)-3/2]}{\Gamma(1+\rho)}\,,\end{aligned} \tag{3.24}$$

where s is the spin of mesons. In the this case, the parameter ω is defined from the following equation:

$$\begin{aligned}\omega^{3\rho} - \frac{16\alpha_s\rho^2\omega^{2\rho}\mu}{3}\frac{\Gamma(2\rho)}{\Gamma(2+\rho)} - \frac{4\rho^2\mu\sigma\Gamma(4\rho)}{\Gamma(2+\rho)} \\ + \frac{16\alpha_s\rho\mu}{3\mu_1\mu_2}\frac{\omega^{4\rho}[s(s+1)-3/2]}{\Gamma(1+\rho)} = 0\end{aligned} \tag{3.25}$$

and for the ground state energy we obtain:

$$\begin{aligned}E = \min_\rho\Bigg\{ \frac{\omega^{2\rho}\Gamma(2+\rho)}{8\rho^2\mu\Gamma(3\rho)} - \frac{4\alpha_s\omega^\rho\Gamma(2\rho)}{3\Gamma(3\rho)} + \\ + \frac{\sigma\Gamma(4\rho)}{\omega^\rho\Gamma(3\rho)} + \frac{4\alpha_s[s(s+1)-3/2]\omega^{3\rho}}{9\rho\mu_1\mu_2\Gamma(3\rho)}\Bigg\}\,.\end{aligned} \tag{3.26}$$

According to (2.16), mass of singlet triplet states are determined by the system of equations

$$\begin{aligned}\mu_1 - \frac{m_1^2}{\mu_1} + 2\mu_1\frac{dE}{d\mu_1} = 0\,; \\ \mu_2 - \frac{m_2^2}{\mu_2} + 2\mu_2\frac{dE}{d\mu_2} = 0\,.\end{aligned} \tag{3.27}$$

Here m_1 and m_2 are the current masses of the quarks. As follow the experimentally [5] fitted value of the current masses c and b quarks is:

$$\begin{aligned}m_c = 1.275 \pm 0.025\ GeV; \\ m_b(1S) = 4.65 \pm 0.03\ GeV.\end{aligned} \tag{3.28}$$

The value of the running coupling constant of the quark-gluon interactions is represented as:

$$\alpha_s = \frac{4\pi}{\beta_0\ln(\frac{\mu_{12}^2}{\Lambda^2})}; \quad \beta_0 = 11 - \frac{2}{3}n_f; \quad \mu_{12} = \frac{2\mu_1\mu_2}{\mu_1+\mu_2}, \tag{3.29}$$

where n_f is the flavor quantum number, and $\Lambda = 0.169\ GeV$ is the scale of confinement for heavy quarks. Then the mass of mesons consisting of these quarks is defined as:

$$M = \frac{1}{2}\left(\mu_1 + \frac{m_1^2}{\mu_1} + \mu_2 + \frac{m_2^2}{\mu_2}\right) + E. \qquad (3.30)$$

The results of numerical calculations are introduced in Table 1. According to (3.28), for the current quark masses use the values $m_c = 1.275\ GeV$ and $m_b = 4.62\ GeV$. The oscillator frequency ω and the constituent quark masses μ_q are determined from the equation presented in (3.25) and (3.28), respectively. The numerical results for the this parameters also given in Table 1. In this case, the accuracy of calculations are: $\delta_\mu \sim 7.2\cdot 10^{-10}$ and $\delta_\omega \sim 1.8\cdot 10^{-9}$, for charmonium.

Table 1. *The mass spectrum of mesons consisting of b and c quarks for the ground state. The experimental are data from work [5].*

		$\bar{c}c$	$\bar{b}b$	bc
$S=0$	m_c GeV	1.275	-	1.275
	m_b GeV	-	4.62	4.62
	α_s	0.30366	0.194679	0.248935
	$\sigma\ GeV^2$	0.195	0.153	0.195
	E GeV	0.413530	0.157253	0.363173
	ρ	0.526448	0.651103	0.46495
	ω^ρ GeV	0.652	1.164	0.648335
	μ_c GeV	1.42862	-	1.51306
	μ_b GeV	-	4.73493	4.68082
	M_{our} MeV	2980.05	9400.04	6.2773
	M_{exp} MeV	2980.3 ± 1.2	9390.9 ± 2.8	6277 ± 4
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.047003	0.196457	0.0525517
	$f_\eta\ \ GeV$	0.435053	0.500795	0.316955
$S=1$	α_s	0.299085	0.194459	0.247683
	E	0.519023	0.216613	0.412532
	ρ	1.03926	1.24871	1.11493
	ω^ρ GeV	1.4311	3.4511	2.0512
	μ_c GeV	1.47617	-	1.53652
	μ_b GeV	-	4.75281	4.71302
	M_{our} MeV	3096.44	9.4603	6.33071
	M_{exp} MeV	3096.916 ± 0.11	9460.3 ± 0.26	-
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.1004	0.5973	0.219078
	$f_\eta\ \ GeV$	0.62372	0.8704	0.644412
	Γ_{our} keV	6.135	1.330	-
	Γ_{exp} keV	5.55 ± 0.14	1.340 ± 0.018	-

From Table 1 we can see that the constituent quark mass is greater than the current masses. According to (3.29), with changing of the constituent quark masses the running coupling constant of quark gluon interactions also changed the values are also given in Table 1. WF in the OR method is defined by two parameters ρ and ω, the value of these parameters is also presented in Table 1. From Table 1, we see that our results for the meson masses are in good agreement with experimental data. The value of WF at the origin $\Psi(0)$ is in Table 1. The calculation details of $\Psi(0)$, with the orbital and radial excitation are presented in Appendix C. From (C.8) for the ground state we have:

$$|\Psi(0)|^2 = \frac{1}{4\pi}\frac{\omega^{3\rho}}{\rho\Gamma(3\rho)}. \tag{3.31}$$

Using $|\Psi_n(0)|^2$ leptonic decay constant is determined by the vector and pseudoscalar mesons:

$$f_p^{NR} = f_v^{NR} = \sqrt{\frac{12}{M_{p,v}}}|\Psi_{p,v}(0)|, \tag{3.32}$$

where $M_{p,v}$ mass of vector and pseudoscalar mesons. The leptonic decay width of vector mesons is determined as follows:

$$\Gamma(V\to\ell\ell) = \frac{16\pi\alpha_{em}^2 e_Q^2}{M_V^2}|\Psi(0)|^2(1-\frac{16\alpha_s}{3\pi}) \tag{3.33}$$

where $\alpha_{em} = 1/137$ is the electromagnetic coupling constant; e_Q is the quark current, and M_V - is the vector meson mass. The obtained numerical value for the ground state are in Table 1.

3.4 The mass spectrum mesons with orbital excitation

Let us calculate the energy and the mass spectrum of mesons consisting of c and b quarks with orbital excitation. From (3.4) we see that for $\ell \neq 0$ spin interactions are determined only by the spin-orbit interaction. In this case the interaction Hamiltonian H_I does not give the contribute. First of all, we consider the case $S = 0$. Taking into account (3.20) from (3.23) we obtain the equation for determining the frequency ω:

$$\omega^{3\rho} - \omega 2\rho\cdot\frac{16\alpha_s\mu\rho^2}{3}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} + \frac{4\rho^2\mu\sigma\Gamma(4\rho+2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} = 0 \tag{3.34}$$

and for the energy spectrum we have

$$E = \min_\rho\left\{\frac{\omega^{2\rho}\Gamma(2+\rho+2\rho\ell)}{8\rho^2\mu\Gamma(3\rho+2\rho\ell)} - \frac{\alpha\omega^\rho\Gamma(2\rho+2\rho\ell)}{3\Gamma(3\rho+2\rho\ell)} + \frac{\sigma}{\omega^\rho}\cdot\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\right\} \tag{3.35}$$

Taking into account (3.34), (3.35) and (3.27) from (3.30) we determine the mass spectrum of mesons with orbital excitation. The numerical results are given in Table 2 and 3 for the charmonium and bottomium, respectively.

Let us calculate the energy spectrum of mesons spin triplet state $S = 1$ with orbital excitations. First of all we define the contribution of the standard spin-orbit interaction to the energy spectrum. Taking into account (3.9) and after some simplifications, we determine the contributions spin orbit interactions to triplet state. The our results represented in Table 4. In the this case the parameter ω is determined from the next equation

$$\omega^{3\rho} - \omega 2\rho\cdot\frac{16\alpha_s\mu\rho^2}{3}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} + \frac{4\sigma\rho^2\mu\Gamma(4\rho+2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} - \omega^{2\rho}\frac{\sigma\rho^2\mu}{M_1^2}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} +$$
$$+\frac{4\rho^2\alpha_s\mu}{\mu_1\mu_2}\cdot\frac{S_{12}\omega^{4\rho}\Gamma(2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} + \frac{\rho^2\mu\alpha_s\omega^{4\rho}}{M_2^2}\cdot\frac{\Gamma(2\rho\ell)}{\Gamma(2+\rho+2\rho\ell)} = 0 \tag{3.36}$$

and for the energy we get:

$$E = \min_{\rho}\left\{ \frac{\omega^{2\rho}\Gamma(2+\rho+2\rho\ell)}{8\rho^2\mu\Gamma(3\rho+2\rho\ell)} - \frac{\alpha\omega^{\rho}\Gamma(2\rho+2\rho\ell)}{3\Gamma(3\rho+2\rho\ell)} + \frac{\sigma}{\omega^{\rho}}\cdot\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \right.$$
$$\left. - \frac{\sigma}{4M_1^2}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \frac{\alpha_s S_{12}}{3\mu_1\mu_2}\frac{\omega^{3\rho}\Gamma(2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \frac{\alpha_s\omega^{3\rho}}{M_2^2}\frac{2\rho\ell}{\Gamma(3\rho+2\rho\ell)} \right\} \tag{3.37}$$

First of all, for the specific values of the orbital quantum number ℓ in which given in Table 2, from (3.21) we define the value of M_1^2 Í M_2^2. Numerical results for the P Í D states shown in Table 2 and 3, respectively.

Table 2. *The mass spectrum of charmonium with orbital excitations. The experimental data are from [5].*

		$J=\ell-1$ S=1	$J=\ell$ S=1	$J=\ell+1$ S=1	$J=\ell$ S=0
$\ell=1$	α_s	0.3013	0.2981	0.2987	0.2978
	E GeV	0.923955	0.960388	0.976759	0.945799
	ρ	0.808694	0.613677	0.775801	0.230383
	ω^{ρ} GeV	1.14386	0.618518	0.851913	0.276542
	μ_c GeV	1.45188	1.48592	1.47997	1.48936
	M_{our} Mev	3495.5	3540.33	3555.15	3526.6
	M_{exp} Mev	3416.75±.31	3510.66±.07	3556.2±.09	3525.41±.16
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.116538	0.0325412	0.0534605	0.00557
	f GeV	0.632513	0.332113	0.424794	0.13763
$\ell=2$	α_s	0.2987	0.2944	0.2962	0.2936
	E GeV	1.2229	1.22267	1.22909	1.21638
	ρ	0.612313	0.595989	0.366686	1.39076
	ω^{ρ} GeV	0.595536	0.560571	0.323594	5.59744
	μ_c GeV	1.53846	1.5278	1.50771	1.5371
	M_{our} Mev	3.81728	3.8145	3.81501	3.81107
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.1165385	0.025338	0.0077298	1.34144
	f GeV	0.632505	0.282333	0.155929	2.05519

Table 3. *Bottomonium mass spectrum with orbital excitations. The experimental data are from [5].*

		$J=\ell-1$ S=1	$J=\ell$ S=1	$J=\ell+1$ S=1	$J=\ell$ S=0
$\ell=1$	α_s	0.1944	0.1943	0.1943	0.1946
	E GeV	0.635856	0.6479	0.669121	0.657241
	ρ	0.628027	0.780369	0.312187	0.0915
	ω^ρ GeV	0.985258	1.36504	0.49	0.273495
	μ_b GeV	4.7567	4.76124	4.76007	4.76134
	M_{our} Mev	9879.78	9892.09	9913.24	9901.44
	M_{exp} Mev	9859.44±0.42	9892.78±0.26	9912.21±0.26	–
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.0514891	0.09099	0.002881	0.03731
	f GeV	0.250077	0.332242	0.186755	0.21496
$\ell=2$	α_s	0.1939	0.1939	0.1939	0.1939
	E GeV	0.906587	0.911645	0.916257	0.911824
	ρ	0.184697	0.177198	0.169101	0.0634526
	ω^ρ GeV	0.327369	0.321223	0.315	0.2129
	μ_b GeV	4.79492	4.79433	4.7926	4.7967
	M_{our} Gev	10.153	10.158	10.1625	10.1583
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.000194	0.0089142	0.000156	0.0000124
	f GeV	0.015147	0.102619	0.0136138	0.0038378

Table 4. *Contribution of the spin-orbit interaction to the triplet state*

J	$\ell+1$	ℓ	$\ell-1$
$(\vec{L}\vec{S}_+)$	ℓ	-1	$-(\ell+1)$
$(\vec{L}\vec{S}_-)$	-1	$-(\ell+1)$	ℓ
S_{12}	$-\frac{2\ell}{2\ell+1}$	2	$-\frac{2(\ell+1)}{2\ell-1}$

3.5 The mass spectrum mesons with radial excitation.

In this section, we will determine the mass and energy spectrum of mesons with only radial excitation. In this case, the energy $\varepsilon_0(E)$ of the zeroth approximation in the OR are given by (3.24), and for the interaction Hamiltonian of (3.22) we have:

$$H_I = \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} : e_2^{2i\sqrt{x\omega}(q\eta)} :$$
$$\Big\{ -\frac{4\rho^2\mu}{\omega 2\rho-1}\frac{Ex^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\mu}{\omega^{3\rho-1}}\frac{\sigma x^{-3\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)}$$
$$+ \frac{16\rho^2\mu\alpha_s(\vec{S}_1\vec{S}_2)}{9\pi\mu_1\mu_2}\lim_{\Lambda\to\infty}\sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)!}\frac{\Lambda^{2j+3}}{(2j+1)}\frac{x^{-2\rho-2\rho j}}{\omega^{2\rho+2\rho j-1}\Gamma(1-2\rho-2\rho j)}\Big\} \tag{3.38}$$

In this case, the energy spectrum has the following form [20]:

$$\varepsilon_n(E) = \varepsilon_0(E) + 2n_2\omega + \langle n_2|H_I^c|n_2\rangle + \langle n_2|H_I^s|n_2\rangle \tag{3.39}$$

where H_I^c is the Hamiltonian of the interaction of the central part, H_I^s is the spin part of the interaction Hamiltonian. The explicit form of the WF with the radial excitation is presented in (A.13). The details of the calculation of the matrix element $\langle n|H_I|n\rangle$ given in Appendix B.

After some simplification, the energy spectrum with the radial excitations we obtain:

$$E_{n_2} = \frac{\omega^{2\rho}\Gamma(2+\rho)}{8\rho^2\mu\Gamma(3\rho)}\frac{1+\frac{n_2}{1+\rho}}{1+W_1} - \frac{\sigma\Gamma(4\rho)}{\omega^\rho\Gamma(3\rho)}\frac{1+W_3}{1+W_1} + \frac{4\alpha_s[S(S+1)-3/2]\omega^{3\rho}}{9\rho\mu_1\mu_3\Gamma(3\rho)}\frac{1+W_s}{1+W_1} \tag{3.40}$$

In this case, the oscillator frequency is determined from the following equation:

$$\omega^{3\rho} - \frac{16\alpha_s\mu\rho^2\omega^{2\rho}\Gamma(2\rho)}{3\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_2-(\rho-1)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{4n_2}{1+\rho})} + \frac{4\rho^2\mu\sigma\Gamma(4\rho)}{\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_3-(3\rho)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{n_2}{1+\rho})} +$$
$$+\frac{16\alpha_s\mu\rho[S(S+1)-3/2]\omega^{4\rho}}{9\mu_1\mu_2\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_s+(1+\rho)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{n_2}{1+\rho})} \tag{3.41}$$

where the following notation is used; $\tilde{W}_j = 1+W_j, \quad j=1,2,3$ Í s: Using (3.40) from (2.18) and (2.19) we determine the meson mass and the constituent mass of quarks, and the numerical results are shown in Table 5.

Table 5. *The mass spectrum of mesons consisting of b and c quarks with radial excitation. Experimental data are from [5].*

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
$S=0$	α_s	0.2745	0.19027	0.22974
	E GeV	0.939195	0.704855	0.79797
	ρ	0.504507	0.45040495	0.537141
	ω^ρ GeV	0.61426	0.913661	0.732053
	μ_c GeV	1.79312	-	2.01377
	μ_b GeV	-	5.115	4.841
	M_{our} MeV	3638.9	9992.76	6833.53
	M_{exp} MeV	3638.9 ± 1.3	-	-
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.0409795	0.161118	0.0649499
	f_η GeV	0.367611	0.439865	0.33772
$S=1$	α_s	0.27479	0.18989	0.22996
	E	1.01391	0.728737	0.836904
	ρ	0.644051	0.452765	0.571577
	ω^ρ GeV	0.629255	0.905397	0.73425
	μ_c GeV	1.7888	-	2.03489
	μ_b GeV	-	5.1501	4.896
	M_{our} MeV	3711.48	10023.3	6881.57
	M_{exp} MeV	3686.109 ± 0.012	10023.26 ± 3.1	-
	$\lvert\Psi(0)\rvert^2$ GeV^3	0.025839	0.15568	0.0604622
	f GeV	0.289038	0.43172	0.324705
	Γ_{our} keV	0.691849	0.260597	-
	Γ_{exp} keV	2.35 ± 0.04	0.612 ± 0.011	-

4. The width of the radiative decay.

Let us we determine the width of the radiative decay or $E1$ – transition. Matrix element $E1$ the transition from state $(n^{2s+1}J),i$, to the state $(n'^{2s'+1}J'),f$, written as:

$$M(i\to f)_\mu = \delta_{s,s'}(-1)^{s+J+J'+1m'}k\sqrt{(2J+1)(2J'+1)(2\ell+1)(2\ell'+1)}\times \\ \times\begin{pmatrix} J' & 1 & J \\ -M' & \mu & M\end{pmatrix}\begin{pmatrix} \ell' & 1 & l \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} \ell & s & J \\ J' & 1 & \ell'\end{pmatrix} e_Q\, I_{i,f}\ , \tag{4.1}$$

where the usual notation in brackets $3j$ – symbol, and e_Q is the quark charge, and $I_{i,f}$ is radial matrix element $i\to f$ transition:

$$I_{if} = \int_0^\infty dr\, r^2\Psi^*_{n'\ell'}(r)r\Psi_{nl}(r) \tag{4.2}$$

where $\Psi_{i,f}$ is the radial wave function of the initial and final state. Then the width of the radiative decay is defined as follows:

$$\Gamma(i\to f+\gamma) = \frac{4\alpha_{em}\, e_Q^2}{3}(2J'+1)S^E_{if}\, k^3\, |I_{i,f}|^2 \tag{4.3}$$

where k is the photon momentum and it is equal to

$$k = \frac{m_i^2 - m_f^2}{2m_i} \tag{4.4}$$

and m_i, m_f mass of the initial and final state. Statistical factor $S^E_{if} = S^E_{fi}$ is:

$$S^E_{if} = \max(\ell,\ell')\begin{Bmatrix} J & 1 & J' \\ \ell' & s & \ell\end{Bmatrix}^2 \tag{4.5}$$

Thus for determine we need to calculate the transition of matrix elements, which are represented in (4.1). Details of calculation $I_{i\to f}$ given in Appendix C. At specific transitions $I_{i\to f}$ determined from (C.14), and the numerical results of the decay width shown in the Table 6.

Table 6. *The E1 radiative decay rates.*

Transition $i \to f$	k MeV	I_{if} GeV^{-1}	$\Gamma_{our}(i \to f)$ keV	$\Gamma_{exp}(i \to f)$ keV
$\chi_{c_0} \to \gamma + J/\psi$	376.3	2.33	139.312	-
$\chi_{c_1} \to \gamma + J/\psi$	416.06	1.73	310.3	295.84
$\chi_{c_2} \to \gamma + J/\psi$	429.12	2.18	450.5	~ 500
$1^3D_1 \to \gamma + 1^1P_0$	308.22	1.78	267.92	~ 299
$1^3D_1 \to \gamma + 1^1P_1$	266.90	3.274	146.9	~ 99
$1^3D_1 \to \gamma + 1^1P_2$	253.13	2.751	3.54	~ 3.88
$\chi_{c_0} \to \gamma + \Upsilon$	410.57	1.422	16.81	-
$\chi_{c_1} \to \gamma + \Upsilon$	422.366	1.57	66.9	-
$\chi_{c_2} \to \gamma + \Upsilon$	442.592	0.6644	22.97	–
$1^3D_1 \to \gamma + 1^1P_0$	269.544	0.1526	0.33	–
$1^3D_1 \to \gamma + 1^1P_1$	257.56	0.135	0.06	–
$1^3D_1 \to \gamma + 1^1P_2$	236.929	0.4988	0.024	–

4.1 Conclusion

On the basis of the obtained results, the following conclusions can be made:

- Our approach is based on the investigation of the asymptotic behaviour of the correlation functions for scalar charged particles in an external gauge field and we determined the interaction Hamiltonian including the relativistic corrections. The kinetic energy term of interaction Hamiltonian is expressed in terms of the constituent mass of bound-state forming particles and the potential energy term is determined by the contributions of a every possible type of Feynman diagrams with exchange of gauge fields. The mass spectrum of the bound state is analytically derived. The mechanism for arising of the constituent mass of the relativistic bound state forming particles is explained.

- In our approach, constituent quark masses are not free parameters, are determined for each quarkonium separately and differ from the mass of a free state, i.e., from the valence quark masses. In this case, the constant α_s of the strong interaction differents from each other for meson. Free parameter in the our approach is the string tension σ and for quarkonium consisting of c quarks is $\sigma = 19.5\ GeV^2$ and for bottomonium consisting of b is $\sigma = 15.3\ GeV^2$.

- In the framework of our approach the mass splitting between the singlet and triplet states is determined and the radiative decay widths of the $\bar{c}c$, $\bar{b}b$ and $\bar{b}c$ systems are calculated.

Appendix A.

The crucial point of calculations in OR [23] is the representation of the canonical variables in the normal form. Therefore, we give examples of this representation for various potentials. We will give here the details of representation in the normal form for the additional potential. Taking into account the relations

$$e^{i\mathbf{ka}}e^{i\mathbf{pa}^+} = e^{i\mathbf{pa}^+}e^{i\mathbf{ka}} \cdot e^{-(kp)}, \tag{A.1}$$

where **k** and **p** are vectors in d-dimensional space. Let us consider the expression:

$$Y_j(\mathbf{k}) = \mathbf{e^{ika}a_j^+e^{-ika}}. \tag{A.2}$$

When $\vec{k} = 0$ from (A.2) we get

$$Y_j(0) = a_j^+. \tag{A.3}$$

Taking into account (A.3) from (A.2) for $dY_j(\mathbf{k})/\mathbf{dk_l}$ we have:

$$\frac{dY_j(\mathbf{k})}{dk_l} = e^{i\mathbf{ka}}i\left[a_l, a_j^+\right]e^{-i\mathbf{ka}} = i\delta_{jl}. \tag{A.4}$$

Integrating over k_l and taking into account (A.3), we get:

$$Y_j(\mathbf{k}) = \mathbf{e^{ika}a_j^+e^{-ika}} = \mathbf{a_j^+} + \mathbf{ik_j}. \tag{A.5}$$

Similarly, we can establish the relation:

$$\begin{aligned} &e^{-i\mathbf{pa}^+}a_je^{i\mathbf{pa}^+} = a_j + ip_j, \\ &e^{\alpha\mathbf{a}^+\mathbf{a}}a_je^{-\alpha\mathbf{a}^+\mathbf{a}} = a_je^{-\alpha} \\ &e^{\alpha\mathbf{a}^+\mathbf{a}}a_j^+e^{-\alpha\mathbf{a}^+\mathbf{a}} = a_j^+e^{\alpha}. \end{aligned} \tag{A.6}$$

Using these relations, we represent the normal form of various types of interaction potentials. Let us consider the detail of the specific potentials in normal form:
a) the increasing potential:

$$\begin{aligned} q^{2n} &= (-1)^n\frac{d^n}{dx^n}e^{-xq^2}{}_{|x=0} \\ &= (-1)^n\frac{d^n}{dx^n}\int\left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x/\omega)} : e^{-2i\sqrt{x}(q\eta)} :_{|x=0} \\ &= \frac{1}{\omega^n}\frac{\Gamma(\frac{d}{2}+n)}{\Gamma(\frac{d}{2})} + :q^2: \frac{n}{\omega^{n-1}}\frac{\Gamma(\frac{d}{2}+n)}{\Gamma(\frac{d}{2}+1)} \\ &+ \frac{(-1)^n}{\omega^n}\frac{d^n}{dx^n}\int\left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x\omega}(q\eta)} :_{|x=0}, \end{aligned} \tag{A.7}$$

where $n = 1, 2, ...$ is a integer and positive,

b) the power potential:

$$
\begin{aligned}
q^{2\tau} &= \int_0^\infty \frac{dx}{\Gamma(-\tau)} x^{-1-\tau} e^{-xq^2} \\
&= \int_0^\infty \frac{dx}{\Gamma(-\tau)} x^{-1-\tau} \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x/\omega)} : e^{-2i\sqrt{x}(q\eta)} := \\
&= \frac{1}{\omega^\tau} \frac{\Gamma(\frac{d}{2}+\tau)}{\Gamma(\frac{d}{2})} + : q^2 : \frac{\tau}{\omega^{\tau-1}} \cdot \frac{\Gamma(\frac{d}{2}+\tau)}{\Gamma(\frac{d}{2}+1)} + \\
&+ \frac{1}{\omega^\tau} \int_0^\infty \frac{dx}{\Gamma(-\tau)} x^{-1-\tau} \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x\omega}(q\eta)} :,
\end{aligned}
\tag{A.8}
$$

where $\tau \neq n$ and the notation: $e_2^x = e^x - 1 - x - x^2/2$ is used.

c) In the general case we use the capacity of the Fourier transform:

$$
\begin{aligned}
W(q^2) &= \int \left(\frac{dk}{2\pi}\right)^d \widetilde{W}(k^2) e^{i(kq)} \\
&= \int \left(\frac{dk}{2\pi}\right)^d \widetilde{W}(k^2) \exp\left(ik\frac{a+a^+}{\sqrt{2\omega}}\right) \\
&= \int \left(\frac{dk}{2\pi}\right)^d \widetilde{W}(k^2) \exp\left(-\frac{k^2}{4\omega}\right) \exp\left(ik\frac{a^+}{\sqrt{2\omega}}\right) \exp\left(ik\frac{a}{\sqrt{2\omega}}\right) \\
&= \int \left(\frac{dk}{2\pi}\right)^d \widetilde{W}(k^2) \exp\left(-\frac{k^2}{4\omega}\right) : e^{ikq} : .
\end{aligned}
\tag{A.9}
$$

Here $(kq) = \sum k_j q_j$ and

$$
\widetilde{W}(k^2) = \int (dx)^d W(x^2) e^{i(kx)} .
$$

Using these relations, which are given in equations (A.7-A.9), normal forms of different potentials are defined. In particular, the (A.7) obtain for $n = 1,\ 2,\ 3$:

$$
\begin{aligned}
q^2 &= \frac{d}{2\omega} + : q^2 :, \\
q^4 &= \frac{d(d+2)}{4\omega^2} + \frac{d+2}{\omega} : q^2 : + : q^4 :, \\
q^6 &= \frac{d(d+2)(d+4)}{8\omega^3} + \frac{3(d+2)(d+4)}{4\omega^2} : q^2 : \\
&+ \frac{3(d+4)}{2\omega} : q^4 : + : q^6 : .
\end{aligned}
\tag{A.10}
$$

In determining the energy spectrum of different potentials with radial excitation need to define the operators: $(a^+a^+)^n$ or $(aa)^n$. For these operators, we use the following representation:

$$
\left(a^+a^+\right)^n = (-1)^n \frac{d^n}{d\beta^n} \exp\{-\beta(a^+a^+)\}\Big|_{\beta=0} \times
\tag{A.11}
$$

$$
\times\ (-1)^n \frac{d^n}{d\beta^n} \int \left(\frac{d\xi}{\sqrt{\pi}}\right)^d \exp\{-\xi^2 - 2i\sqrt{\beta}(a^+\xi)\}\Big|_{\beta=0} ,.
$$

Will also use the following relation:

$$e_2^{i(kq)} = P_v e^{iv(kq)},$$

where P_v –operator, working only on the variable v, and the action of the operator is defined as follows:

$$P_v = const = 0, \quad P_v v^n = 0, \quad n \leq 2, \quad P_v v^n = 1 \;\; n > 2.$$

These relations are often found in the calculation of matrix elements of the various physical processes in the OR. Using these representations we have an expression for $\varepsilon_0(E)$ – the ground state energy, as well as for H_I – the interaction Hamiltonian.

d) Determination of the normalization constant of the wave function.
In the OR the wave function with radial excitation is defined as:

$$|n_r\rangle = C_{n_r} \left(a_j^+ a_j^+\right)^{n_r} |0\rangle \;, \quad j = 1, \; \ldots, d, \tag{A.12}$$

where C_n is the normalization constant and is determined by the conditions:

$$1 \equiv \langle n|n\rangle = C_n^2 \langle 0| \left(a_i a_i\right)^n \left(a_j^+ a_j^+\right)^n |0\rangle \;. \tag{A.13}$$

Considering(A.11) and (A.1), after some simplifications of (A.13) we have:

$$\begin{aligned} 1 &= C_n^2 \frac{\partial^{2n}}{\partial\alpha^n \partial\beta^n} \int \left(\frac{d\xi}{\sqrt{\pi}}\right)^d \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\xi^2-\eta^2} \times \\ &\times \langle 0| e^{-2i\sqrt{\alpha}(a\xi)} \cdot e^{-2i\sqrt{\beta}(a^+\eta)} |0\rangle \Big|_{\alpha,\beta=0} = \\ &= C_n^2 \frac{\partial^{2n}}{\partial\alpha^n \partial\beta^n} \int \left(\frac{d\xi}{\sqrt{\pi}}\right)^d \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\xi^2-\eta^2-4\sqrt{\alpha\beta(\xi\eta)}} \Big|_{\alpha,\beta=0} = \\ &= C_n^2 \frac{\partial^{2n}}{\partial\alpha^n \partial\beta^n} \frac{1}{(1-4\alpha\beta)^{d/2}} \Big|_{\alpha,\beta=0} . \end{aligned} \tag{A.14}$$

Finally, from (A.14) we obtain:

$$C_n = \left(\frac{\Gamma(d/2)}{4^n n! \Gamma(d/2+n)}\right)^{1/2} . \tag{A.15}$$

Appendix B

In this section, we present some details of the calculation of the matrix element $\langle n_r|H_I^0|n_r\rangle$.The interaction Hamiltonian is presented in (3.24), and the correct matrix element can be written as:

$$\begin{aligned} \langle n_r|H_I|n_r\rangle &= \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} \langle n_r| : e_2^{-2i\sqrt{x\omega}(qn)} : |n_r\rangle \times \\ &\times \left[-\frac{4\rho^2 E\mu}{\omega^{2\rho-1}} \frac{x^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\sigma\mu}{\omega^{3\rho-1}} \frac{x^{-3\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}} \frac{x^{-\rho}}{\Gamma(1-\rho)}\right] . \end{aligned} \tag{B.1}$$

From (B.1) it is seen that for the calculation of the matrix element $\langle n_r|H_I|n_r\rangle$ we need to determine:

$$T_n(x) = \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} \langle n| : e_2^{-2i\sqrt{x\omega}(qn)} : |n\rangle. \tag{B.2}$$

Considering (A.11) and (A.12), after some simplifications from (B.2) we obtain:

$$\begin{aligned} T_n(x) &= \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} \langle n| : e_2^{-2i\sqrt{x\omega}(qn)} : |n\rangle \\ &= P_v C_n^2 \frac{d^{2n}}{d\alpha^n d\beta^n} \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d \iint \left(\frac{d\xi_1}{\sqrt{\pi}}\right)^d \left(\frac{d\xi_2}{\sqrt{\pi}}\right)^d \times \\ &\times \quad e^{-\eta^2(1+x)-\xi_1^2-\xi_2^2} \langle 0| e^{-2i\sqrt{\alpha}(a\xi_1)} e^{-iv\sqrt{2x}(a^+\eta)} \times \\ &\times \quad e^{-iv\sqrt{2x}(a\eta)} \cdot e^{-2i\sqrt{\beta}(a^+\xi_2)} 0\rangle \Big|_{\beta,\alpha=0} \end{aligned} \tag{B.3}$$

and finally:

$$\begin{aligned} T_n(k) &= \sum_{k=2}^{2n} \sum_{s=0}^{n} (-1)^k \frac{x^k}{(1+x)^{k+d/2}} \cdot \frac{\Gamma(1+n)}{\Gamma(n+d/2)} \times \\ &\times \frac{2^{2s-k}}{\Gamma(n-s+1)} \cdot \frac{\Gamma(k+n-s+d/2)}{\Gamma^2(k-s+1)\Gamma(2s-k+1)}. \end{aligned} \tag{B.4}$$

Substituting (B.4) into (B.1) and integrating over x, from (B.1) we have:

$$\begin{aligned} \langle n_r|H_I^c|n_r\rangle &= -\frac{4\rho^2 E\mu}{\omega^{2\rho-1}} \frac{\Gamma(d/2+2\rho-1)}{\Gamma(d/2)} W_1 - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}} \frac{\Gamma d/2+\rho-1}{\Gamma(d/2)} W_2 + \\ &+ \frac{4\rho^2\sigma\mu}{\omega^{3\rho-1}} \frac{\Gamma(d/2+3\rho-1)}{\Gamma(d/2)} W_3. \end{aligned} \tag{B.5}$$

Here we use the notation:

$$\begin{aligned} W_1 &= \frac{\Gamma(1+n_r)}{\Gamma(n_r+d/2)} \frac{\Gamma(d/2)}{\Gamma(1-2\rho)} \sum_{k=2}^{2n_r} (-1)^k A_{n_r}(k) \frac{\Gamma(1+k-2\rho)}{\Gamma(k+d/2)}; \\ W_2 &= \frac{\Gamma(1+n_r)}{\Gamma(n_r+d/2)} \frac{\Gamma(d/2)}{\Gamma(1-\rho)} \sum_{k=2}^{2n_r} (-1)^k A_{n_r}(k) \frac{\Gamma(1+k-\rho)}{\Gamma(k+d/2)} \end{aligned} \tag{B.6}$$

and

$$W_3 = \frac{\Gamma(1+n_r)}{\Gamma(n_r+d/2)} \frac{\Gamma(d/2)}{\Gamma(1-3\rho)} \sum_{k=2}^{2n_r} (-1)^k A_{n_r}(k) \frac{\Gamma(1+k-3\rho)}{\Gamma(k+d/2)}, \tag{B.7}$$

where

$$A_{n_r}(k) = \sum_{s=1}^{n_r} \frac{2^{2s-k}}{\Gamma(n_r-s+1)} \frac{\Gamma(k+n_r-s+d/2)}{\Gamma^2(k-s+1)\Gamma(2s-k+1)}. \tag{B.8}$$

Similarly, the matrix element is calculated $\langle n_r|H_I^s|n_r\rangle$ for the spin interaction Hamiltonian .

$$\langle n_r|H_I^s|n_r\rangle = \frac{32\rho\mu\alpha_s[S(S+1)-3/2]}{9\ \mu_1\ \mu_2} \cdot \frac{\omega^{d/2} W_s}{\Gamma(d/2)}\ \delta_{\ell,0}\ , \tag{B.9}$$

where

$$W_s = \sum_{k=2}^{2n_r} (-1)^k A_{n_r}(k). \tag{B.10}$$

Using (B.5) and (B.9) the energy spectrum of the radial excitationis determined.

Appendix C

In this section, we give some details of computing the value of the wave function at the origin. For this we define WF, i.e. the normalization coefficient

$$1 = C_{n\ell}^2 \int d\vec{r} \cdot \Psi_{n\ell}^*(\vec{r})\Psi_{n\ell}(\vec{r}) = 4\pi \cdot C_{n\ell}^2 \int_0^\infty dr \cdot r^2 \Psi_{n\ell}^*(r)\Psi_{n\ell}(r)\ , \tag{C.1}$$

where ℓ is the orbital, n is the radial quantum number, and $\Psi_{n\ell}(r)$ is the radial wave function. To calculate the integral (C.1), apply the OR method, and give the substitution

$$r = q^{2\rho};\ \ \Psi_{nl} \to q^{2\rho\ell}\Phi_n(q^2) \tag{C.2}$$

Considering (C.2) after some of the (C.1) we have:

$$1 = 4\cdot \pi C_{n\ell}^2 2\rho \int_0^\infty dq \cdot q^{d-1}\Phi_{n\ell}^* q^{2(2\rho-1)}\Phi_{n\ell} = 8\pi\rho C_{n\ell}^2 \cdot \langle n|q^{2(2\rho-1)}|n\rangle \tag{C.3}$$

In further calculations we use the representation:

$$q^{2(2\rho-1)} = \frac{1}{\omega^{2\rho-1}} \int_0^\infty \frac{dx \cdot x^{-2\rho}}{\Gamma(1-2\rho)} \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} : e^{2i\sqrt{x\omega}(q\eta)} : \tag{C.4}$$

as well as the explicit form of the radial wave function, and after some calculations from (C.3) for $C_{n\ell}^2$ we obtain:

$$C_{n\ell}^2 = \frac{1}{4\pi} \cdot \frac{\omega^{\frac{d}{2}+2\rho-1}}{\rho\Gamma(\frac{d}{2}+2\rho-1)} \frac{1}{S_n} \tag{C.5}$$

where

$$\begin{aligned} S_n \ =& \ \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2}-2\rho-1)} \frac{\Gamma(1+n)}{\Gamma(\frac{d}{2}+n)} \cdot \frac{1}{\Gamma(1-2\rho)} \times \\ & \times \sum_{k=0}^{n} \frac{1}{\Gamma^2(n-k+1)} \cdot \sum_{k=0}^{k} \frac{(-1)^s}{s!(k-s)!} \times \\ & \times \Gamma(2n-2k+s-2\rho+1)\Gamma(\frac{d}{2}+k-s+2\rho-1). \end{aligned} \tag{C.6}$$

In particular,

$$S_0 = 1; \quad S_1 = 1 - \frac{2\rho(1-2\rho)}{1+\rho+2\rho\ell}. \tag{C.7}$$

From (C.5) for the wave function at the origin we have:

$$|\Psi_n(0)|^2 = \frac{1}{4\pi} \cdot \frac{(\omega^{\rho})^{(3+2\ell)}}{\rho\ \Gamma(3\rho+2\rho\ell)} \cdot \frac{1}{S_n} \tag{C.8}$$

Now, here given some details of the calculation of the integrals presented in (3.37), which defines the matrix element $E1$ – transition. Details of the calculation of this integral, similar to the calculation of the integral presented in (C.1). After similar calculations from the (3.37) we have:

$$I^{n_1\ell_1}_{n_2\ell_2} = \frac{2\rho}{\sqrt{1\rho_1\langle n_1|q^{2(2\rho_1-1)}|n_1\rangle}} \cdot \frac{1}{\sqrt{1\rho_2\langle n_2|q^{2(2\rho_2-1)}|n_2\rangle}} B_{n_1n_2} \tag{C.9}$$

where we introduced the notation:

$$B_{n_1n_2} = \int_0^{\infty} dq \cdot q^{d-1} \langle 0|_{d_2} (aa)^{n_2} q^{2(2\rho-1)} (a^+a^+)^{n_1}|0\rangle_{d_1} \tag{C.10}$$

Using expression (C.4) for the normal view from (C.10) we obtain:

$$\begin{aligned} B_{n_1n_2} \ &= A(\rho_1,\rho_2) \cdot \int_0^{\infty} \frac{dx \cdot x^{-2\rho}}{\Gamma(1-2\rho)} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} \times \\ &\times \ \langle 0|(aa)^{n_2} : e^{-2i\sqrt{x\omega}(q\eta)} : (a^+a^+)^{n_1}|0\rangle \end{aligned} \tag{C.11}$$

Performing integration and considering the action of operators a and a^+ from (C.11) we have:

$$\begin{aligned} B_{n_1n_2} \ &= A(\rho_1,\rho_2)(-1)^{n_1+n_2} \frac{\partial^{n_1+n_2}}{\partial\alpha^{n_1}\partial\beta^{n_2}} \int_0^{\infty} \frac{dx \cdot x^{-2\rho}}{\Gamma(1-2\rho)} \times \\ &\times \ \frac{1}{[(1+x-2x\alpha)(1-\alpha\beta)-2x\beta(1-2\alpha)^2]^{\frac{d}{2}}}\Bigg|_{\beta,\alpha=0}, \end{aligned} \tag{C.12}$$

where used the notation:

$$A(\rho_1,\rho_2) = \left(\frac{2\omega_1}{\omega_1+\omega_2}\right)^{\frac{d_1}{4}} \left(\frac{2\omega_2}{\omega_1+\omega_2}\right)^{\frac{d_2}{4}} \frac{2^{\rho_1+\rho_2-1}}{(\omega_1+\omega_2)^{\rho_1+\rho_2-1}}\ . \tag{C.13}$$

This integral for specific values n_1 and n_2 calculated analytically.

The considering (C.12) from (C.9) we obtain:

$$I^{n_1\ell_1}_{n_2\ell_2} = \frac{\rho_1+\rho_2}{\sqrt{\rho_1\rho_2}} \frac{(\omega_1\rho_1)^{\frac{3}{2}+\ell_1}(\omega_2\rho_2)^{\frac{3}{2}+\ell_2}}{\sqrt{f_{n_1}f_{n_2}}\sqrt{\Gamma(3\rho_1+2\rho_1\ell_1)}} \frac{B_{n_1n_2}}{\Gamma(3\rho_2+2\rho_2\ell_2)} \tag{C.14}$$

This analytical expression is used to calculate the width $E1$ –transitions.

References

[1] Bogolyubov N.N. and Shirkov D.V.: Introduction to the theory of quantized fields, Moscow, "Nauka", 1976.

[2] Weinberg S.: The quantum theory of fields. Cambridge: Cambridge Univ. Press, 1995.

[3] Itzykson C., Zuber J.B.: Quantum field theory. N. Y.: McGraw-Hill, 1980.

[4] Eides M.I. et al.: Phys. Rep. **342**,61 (2001).

[5] Beringer J. et al.: Review of Particle Physics.// Phys. Rev. **D 86** ,010001 (2012).

[6] Greiner W., Reinhard J.: Quantum Electrodynamics. Berlin: Springer, 1992.

[7] Wick G.C.: Phys. Rev. **96**,1124(1954) ;
Cutkosky R.E. 1135(Ibid);
Efimov G.V.: Few-Body Syst.**33**,199 (2003).

[8] Logunov A.A., Tavkhelidze A.N.: Nuovo Cim. **29**,380(1963) ;
Faustov R.N., Galkin V.O., Mishurov A.Yu.: Phys.Lett. **B356**, P.516(1995); Phys. Rev. **D53**, 6302(1996).

[9] Berestetskii V.B., Lifshitz E.M., Pitaevskii L.P.: Quantum Electrodynamics 2nd Ed.Oxford: Pergamon Press, 1982.

[10] Caswell W.E., Lepage G.P.: Phys. Let. **B167**,437(1986).

[11] *Kinoshita T., Nio M.*// Phys. Rev. D. 1996. V.53. P.4909.

[12] *Quigg C., Rosner J.L.*// Phys. Rev. 1990.V. 56. P.167

[13] *Godfrey S., Isgur N.*// Phys. Rev. D. 1985. V.32. P.189; *Isgur N., Wise M.*// Phys. Lett. B. 1984. V. 232.P. 113; 1990. V. 237. P.527.

[14] Dosch, H. G.: Phys. Lett. **B190**, 177 (1987); Dosch, H. G., Simonov, Yu. A.: Phys.Lett. **B205**, 339 (1998); Simonov, Yu. A.: Nucl. Phys. **B307**, 512 (1988)

[15] Simonov, Yu. A.: Phys. Lett. **B249**, 514 (1990)

[16] Dubin, A. Yu., Kaidalov, A. B., Simonov, Yu. A.: Yad. Fiz. **56**, 213 (1993)

[17] Kaidalov, A. B., Simonov, Yu. A.: Yad. Fiz **63**, 1428 (2000)

[18] *Dineykhan M. et. al.* //J. Phys. B: At. Mol. Opt. Phys. 2009. V.42. P. 145001.

[19] *Feynman R.P., Hibbs A.P.* Quantum Mechanics and Path Integrals. N. Y.: Me Graw-Hill, 1963.

[20] *Dineykhan M., Efimov G.V., Namsrai Kh.*// Fortschr. Phys. 1991. V.39. P. 259.

[21] Lucha W., Schoberl F.F., Gromes D. //Phys. Rep. 1991. V. 200. P.127.

[22] Eichten E., Feinberg F. // Phys. Rev. D. 1981. V. 23. P.2724.

[23] *Dineykhan M. et al.* Oscillator Representation in Quantum Physics. Lecture Notes in Physics. Berlin: Springer-Verlag. 1995. V. 26.

[24] *Fock V.A.* Principles of Quantum Mechanics. M.: Nauka 1976.

[25] Dineykhan M., Efimov G. V. // Rep. Math. Phys. 1995. V.36, P.287; Yad. Fiz. 1996. V.59. 862;
Dineykhan M.// Z. Phys. 1997. V. D41. P.77;
Dineykhan M., Nazmitdinov R.G.// Yad. Fiz. 1999. V. 62. P.143;
Dineykhan M., Zhaugasheva, S.A., Nazmitdinov R.G.//JETP. 2001. V.119. P.1210.

[26] Fradkin E. S. // Nucl. Phys. V.49. P.624.

[27] Hayashi K. et al.// Fortsh. Phys. 1967. V.15. P.625.

[28] *Salam A.* Nonpolynomial Lagrangians. Renormalization and Gravity. N.Y.: Gordon and Breach Sci. Publ., 1971.