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On the perturbative expansion of τ hadronic spectral function moments

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In the determination of α_s from tau decays several different moments of the hadronic spectral functions have been used. In a recent work, we performed an analysis of their perturbative behaviour under two different assumptions for the higher-order coefficients of the Adler function. We showed that the various moments can be divided in a small number of classes. We concluded that some of the moments commonly employed in α_s extractions should be avoided due to their bad perturbative behaviour. Furthermore, for the moments that have a good perturbative behaviour, and under reasonable assumptions for the higher-order behaviour of the Adler function, fixed-order perturbation theory (FOPT) provides the superior framework for the renormalization group improvement. Here we discuss an extension of this analysis where we consider the perturbative series for values of the hadronic invariant mass squared $s_0 \leq m_\tau^2$. Our conclusions are not altered within a reasonable s_0 window.

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1. Introduction

In the last 20 years, hadronic tau decays have been an important source of empirical information on fundamental parameters of QCD. Notably, the strong coupling, α_s, can be extracted with a good precision at relatively low energies, close to the edge of the validity of perturbative QCD. After the measurement of the spectral functions at LEP, other parameters such as the strange quark mass, the CKM matrix element V_{us}, as well as non-perturbative condensates could be extracted (see e.g. [1]). The extraction of these parameters relies on sum rules. Quark-hadron duality and the optical theorem allow to express the decay rate as weighted integrals of the vector and axial-vector spectral functions running over the hadronic invariant mass squared s from threshold up to m_τ^2.

Using the analytic properties of the quark-antiquark correlators, the theoretical counter-part of the experimental quantities are expressed as contour-integrals in the complex energy plane with fixed $|s| = m_\tau^2$. However, in the theoretical description of τ decays two main obstacles remain. The first is referred to as *duality violations* (DVs). They are related to the break-down of local quark-hadron duality in the vicinity of the Minkowski axis (where resonance effects become important). In the past, they have been neglected due to a fortuitous kinematical suppression of the problematic region in the contour integration. Recently, thanks to the progress in modelling DVs realistically [3, 4], they have been included self-consistently in a full-fledged α_s analysis [5, 6]. The second important obstacle is the prescription for the renormalization group (RG) improvement of the perturbative series. The most widely employed prescriptions are fixed-order perturbation theory (FOPT) [7, 8] and contour improved perturbation theory (CIPT) [9, 10]. When used in practice, they lead to different α_s results. With the recently computed α_s^4 correction [11], the difference became even more pronounced. Several works have dealt with this discrepancy [8, 12, 13, 14, 15, 16] in the light of the α_s^4 term. The conclusions in favour of FOPT or CIPT (or a third prescription) are based on (implicit or explicit) assumptions on the yet unknown higher order α_s corrections. In this context, the goal of Ref. [8] was to construct a plausible model for the higher-order corrections of the Adler function from the leading renormalon singularities of its Borel transform, using only general RG arguments to describe the structure of the singularities in the Borel plane. After matching the model to the known coefficients in QCD, the main conclusion of Ref. [8] was that FOPT is to be preferred over CIPT, since FOPT provides a closer approach to the Borel resummed results — in the spirit of an asymptotic series.

This conclusion was based solely on the analysis of the weight $w_\tau(x)$, obtained from the kinematics of the decay. This is not entirely satisfactory since realistic determinations of α_s employ (and often require) several different weight functions $w_i(x)$. In fact, any analytical $w_i(x)$ gives rise to a valid sum-rule that emphasises a given part of the spectral functions, as well as different contributions in the theoretical description. In the literature, several weight-functions have been employed and yet little attention has been paid to the moment dependence of the convergence properties of the perturbative series. We have addressed this question in Ref. [17] and pursued the FOPT/CIPT comparison for several weight functions. We showed that the different moments employed in the literature can be divided in a small number of categories. The characteristics of their perturbative series could be linked to generic features of the moment weight function and the dominant renormalon singularities of the Adler function. We concluded that some of the moments currently employed in some α_s extractions should be avoided due to the poor convergence of their

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perturbative expansions. Additionally, for all moments that display good perturbative behaviour — and under reasonable assumptions for the higher-order corrections — FOPT provided the best framework to the RG improvement.

A point that was not discussed in [17] is the stability of the conclusions with respect to s_0 variations (s_0 being the upper limit of integration in the sum-rule, see Eq. (2.1)). The relevance of this issue lies in the fact that several α_s analyses use sum-rules where the data are integrated up to $s_0 < m_\tau^2$. Here we show that the conclusions of [17] remain valid when s_0 is varied away from m_τ^2.

2. Theoretical framework, model, and results

We work with generalized sum-rules, where the weight function in the integrals can be any analytical function $w_i(x)$ and the upper limit of integration is taken to be any point $s_0 \leq m_\tau^2$. The experimental side of the sum-rules are then written as integrals over the spectral functions as

$$R^{w_i}_{\tau,V/A}(s_0) = 12\pi S_{\rm EW}|V_{ud}|^2 \int_0^{s_0} \frac{ds}{s_0}\left(1-\frac{s}{s_0}\right)^2\left(1+2\frac{s}{s_0}\right)\left[\mathrm{Im}\,\Pi^{(1+0)}_{V/A}(s) - \frac{2s}{s_0+2s}\mathrm{Im}\,\Pi^{(0)}_{V/A}(s)\right]. \tag{2.1}$$

The two point functions are defined as $\Pi^{\mu\nu}_{V/A}(p) \equiv i\int dx\, e^{ipx}\,\langle\Omega|\,T\{J^\mu_{V/A}(x)J^\nu_{V/A}(0)^\dagger\}|\Omega\rangle$ and they assume the usual decomposition into longitudinal and transversal components. The V and A currents are given by $J^\mu_{V/A}(x) = (\bar{u}\gamma^\mu(\gamma_5)d)(x)$.

The theoretical counter-part of Eq. (2.1) is obtained from the counter-clock wise contour integration of the correlators with $|s| = s_0$. The contributions on the theory side can be organized as

$$R^{w_i}_{V/A}(s_0) = \frac{N_c}{2}S_{EW}|V_{ud}|^2\left[\delta^{\rm tree}_{w_i} + \delta^{(0)}_{w_i}(s_0) + \sum_{D\geq 2}\delta^{(D)}_{w_i,V/A}(s_0) + \delta^{\rm DV}_{w_i,V/A}(s_0)\right]. \tag{2.2}$$

The perturbative contribution is contained in $\delta^{\rm tree}_{w_i}$ and $\delta^{(0)}_{w_i}$, of which $\delta^{(0)}_{w_i}$ contains the loop corrections. In the chiral limit they are the same for V and A correlators, and correspond to the perturbative series of $\Pi^{(1+0)}_{V/A}(s)$. The quark-mass corrections, as well as contributions from operators with $D > 2$ in the OPE, are encoded in the terms $\delta^{(D)}_{w_i,V/A}$; DV contributions are represented by $\delta^{\rm DV}_{w_i,V/A}$.

Our focus is on the behaviour of the perturbative correction and it is convenient to write it in terms of the RG invariant Adler function, whose expansion in α_s can be written as

$$D^{(1+0)}(s) \equiv -s\frac{d}{ds}\Pi^{(1+0)}(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty}a_\mu^n\sum_{k=1}^{n+1}k\,c_{n,k}\left(\log\frac{-s}{\mu^2}\right)^{k-1}, \tag{2.3}$$

where $a_\mu = \alpha(\mu)/\pi$. RG invariance implies that only the coefficients $c_{n,1}$ are independent. The other $c_{n,k}$ can be expressed in terms of $c_{n,1}$ and β-function coefficients. The perturbative contribution to the theory side of the sum-rules is then

$$\delta^{(0)}_{w_i} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}k\,c_{n,k}\,\frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}W_i(x)\log^{k-1}\left(\frac{-s_0 x}{\mu^2}\right)a_\mu^n\,, \tag{2.4}$$

with $x = s/s_0$ and $W_i(x) = 2\int_x^1 dz\, w_i(z)$. Due to the RG invariance of $D^{(1+0)}(s)$ one has the freedom of setting the scale μ. The FOPT prescription corresponds to the choice $\mu^2 = s_0$. In this case, the coupling $a(s_0)$ is taken out-side the integrals and one is left with the integration of powers of $\log(-x)$. The CIPT choice correspond to $\mu^2 = -s_0 x$, which resums the logarithms but, in turn, the integrals are done (numerically) over the running coupling $a(-s_0 x)$.

In order to compare FOPT and CIPT as well as understand the perturbative behaviour of spectral function moments, one must have an ansatz for the unknown higher-order Adler function coefficients $c_{n,1}$. Here we follow the method introduced in Ref. [8] which makes use of the available knowledge of the renormalon structure of the Borel transformed Adler function. The idea is to construct a realistic model for the Borel transform using the leading singularities. We work with the function $\widehat{D}(s)$ and its Borel transform, $B[\widehat{D}](t)$, defined as

$$\frac{12\pi^2}{N_c} D_V^{(1+0)}(s) \equiv 1 + \widehat{D}(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \, \alpha_s(\sqrt{s})^{n+1} \,, \qquad B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} . \tag{2.5}$$

The original series can be understood as an asymptotic expansion of the inverse of $B[\widehat{D}](t)$,

$$\widehat{D}(\alpha) \equiv \int_0^{\infty} dt \, \mathrm{e}^{-t/\alpha} B[\widehat{D}](t) \,, \tag{2.6}$$

when the integral exists. Singularities of $B[\widehat{D}](t)$ on the positive real axis (infra red (IR) renormalons) give rise to fixed-sign asymptotic series and obstruct the Borel summation, Eq. (2.6). This introduces an ambiguity in the integral that is expected to be cancelled against exponentially small terms in α_s, or power corrections(due to the logarithmic running of the coupling). Singularities on the negative real axis (ultra violet renormalons (UV)) give rise to sign-alternating series.

General RG arguments and the structure of the OPE allow one to determine the position and strength of the renormalon singularities in the t plane, though not their residues [18]. The fixed-sign nature of the exactly known coefficients of the Adler function suggest that at low and intermediate orders the series is dominated by IR singularities. The reference model (RM) of [8] contains the first two IR and the leading UV singularities. The Borel transform is given by

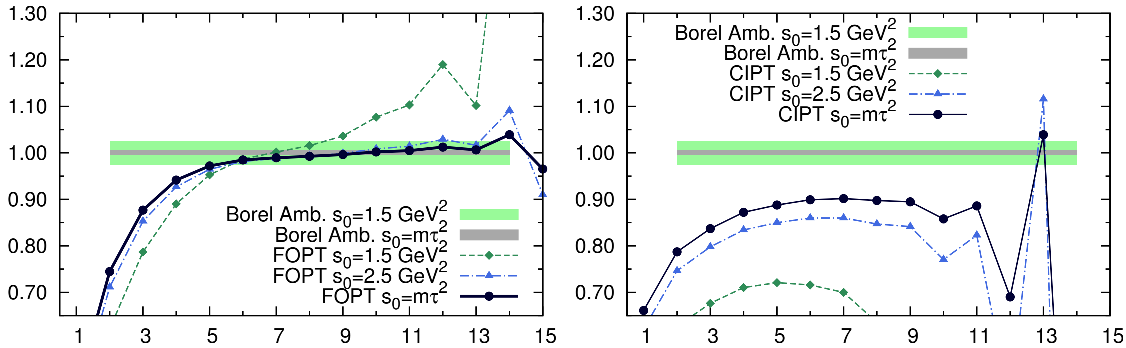
$$B[\widehat{D}](u) = B[\widehat{D}_1^{\mathrm{UV}}](u) + B[\widehat{D}_2^{\mathrm{IR}}](u) + B[\widehat{D}_3^{\mathrm{IR}}](u) + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u. \tag{2.7}$$

The structure of the branch-cut singularities can be found in [8]. The residues and the coefficients $d_{0,1}^{\mathrm{PO}}$ are fixed by matching to the exactly known $c_{1,1}$ to $c_{4,1}$ (augmented by an estimate for $c_{5,1}$).

Within this model, the conclusion of Ref. [8] in favour of FOPT has been corroborated and extended in our recent work [17]. All moments that display a good perturbative behaviour favour the FOPT prescription within the RM. This conclusion can be traced back to the contribution of the leading IR singularity, related to the $D = 4$ corrections in the OPE. If this singularity is arbitrarily suppressed, one generates a model — less realistic, in our opinion — in which CIPT is the preferred prescription. To realize this scenario in practice, and assess possible model dependencies in our conclusions, we introduced the following alternative model (AM) where the leading singularity is absent whereas the sub-leading one at $u = 4$ is explicitly taken into account:

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\mathrm{UV}}](u) + B[\widehat{D}_3^{\mathrm{IR}}](u) + B[\widehat{D}_4^{\mathrm{IR}}](u) + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u \,. \tag{2.8}$$

Within the AM, moments with good perturbative behaviour favour CIPT.



(a) $w_\tau(x) = (1-x)^2(1+2x)$, FOPT.

(b) $w_\tau(x) = (1-x)^2(1+2x)$, CIPT.

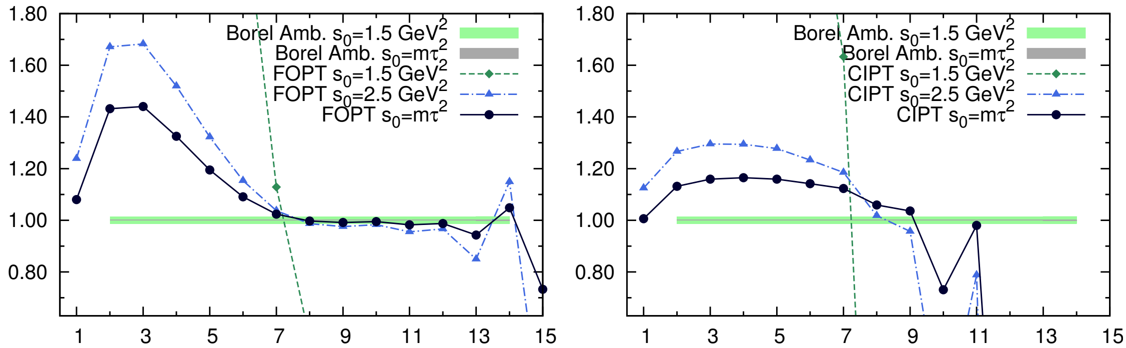
Figure 1: Reference model. $\delta^{(0)}_{w_\tau}(s_0)$ order by order in α_s normalised to the Borel sum for FOPT (left) and CIPT (right) with three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. Bands give the Borel ambiguities.

The models represent two quite different situations regarding the interplay of the Adler function coefficients and the running coupling effects. In the RM, there are cancellations between the contribution from the high-order coefficients $c_{n,1}$ and the running coupling effects, at a given order in α_s. In this case FOPT is superior since it treats these contributions on an equal footing, while CIPT misses the cancellations due to the resummation of the running effects to all orders. On the other hand, the AM represents a situation where the running effects are dominant and should be resummed. In this case, the high-order coefficients can be neglected and CIPT is a better prescription. Since there is no known mechanism that would naturally suppress the leading IR singularity in QCD, we believe the scenario of Eq, (2.7) to be more realistic.

Using these two models, we compared in Ref. [17] the perturbative series in FOPT and CIPT generated from 17 polynomial weight-functions $w_i(x)$. We showed that they can be divided into a small number of categories regarding the behaviour of their perturbative series. Generic features of the functions $w_i(x)$ (such as starting or not with the unity), together with the assumptions upon the Adler function, suffice to determine whether they are suitable for α_s extractions and whether FOPT or CIPT is more suitable for the RG improvement. We showed that some of the weight functions used in the literature, e.g. polynomials containing solely powers of x^i with $i \geq 2$, should be avoided due to their bad perturbative behaviour. We also provided further arguments that support the plausibility of the RM of [8] and concluded that for well-behaved moments FOPT is preferred.

An aspect that was not considered in [17] was the s_0 dependence of these conclusions. This is important because sum-rules with different values of $s_0 \leq m_\tau^2$ are used in extractions of α_s [5, 6, 19]. Here we show explicit results for the FOPT/CIPT comparison for two moments within the two models given in Eqs. (2.7) and (2.8) and considering three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. The interval [1.5 GeV^2: m_τ^2] spans the values used in the fits of [5, 6]. Since we intent to compare the perturbative series at different values of s_0 a normalisation procedure is in order. For better comparison, we normalise the series generated for each value of s_0 by its corresponding Borel sum, Eq. (2.6). Hence, in the plots, meaningful series should be asymptotic to the unity.

We start by considering the case of moments that have good perturbative behaviour for $s_0 = m_\tau^2$. As a representative we choose to use the kinematic moment w_τ. In Fig. 1, we consider the FOPT and CIPT series within the RM. On the left-hand side, Fig. 1(a), one observes that the normalised



(a) $w_{17}(x) = (1-x)^2x^3(1+2x)$, FOPT. (b) $w_{17}(x) = (1-x)^2x^3(1+2x)$, CIPT.

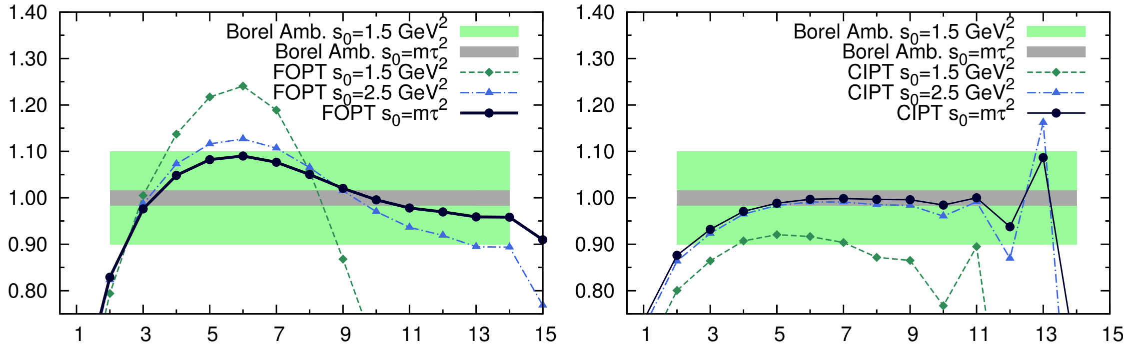
Figure 2: Reference model. $\delta^{(0)}_{w_{17}}(s_0)$ order by order in α_s normalised to the Borel sum for FOPT (left) and CIPT (right) with three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. Bands give the Borel ambiguities.

FOPT series still behaves as a good asymptotic series even for s_0 significantly smaller than m_τ^2. As expected, for lower s_0, the larger values of α_s amplify the divergent behaviour above the 8th order. Nevertheless, the first few terms of the series approach the Borel resummed value. Note also that the Borel sum has a larger ambiguity for smaller s_0 due to larger α_s. On the right-hand side, in Fig. 1(b), one sees that the poor performance of CIPT is amplified by the larger values of the coupling at lower s_0. That is, within the RM, CIPT is not a good approximation to the Borel resummed values, and even less so for smaller s_0.

We now turn to a moment with bad perturbative behaviour: $w_{17}(x) = (1-x)^2x^3(1+2x)$ (to employ the notation of [17]). In Ref. [17] we showed that moments starting with powers of x (that do not contain the unity) tend to have bad perturbative behaviour and are largely dominated by power corrections. In Fig. 2 we address the s_0 dependence of this conclusion. On the left, Fig. 2(a) shows that for higher values of s_0 FOPT can approach the Borel result only at high orders (not available exactly). At low s_0 the series displays a wild behaviour and cannot be considered a good approximation to the Borel sum. In CIPT, Fig. 2(b), the bad behaviour already observed for $s_0 = m_\tau^2$ is amplified at lower s_0. The series are erratic and cannot be consider suitable asymptotic approximations to the Borel sum. Note that this moment, despite of its bad perturbative behaviour, enters several determinations of α_s from τ decays (e.g. Refs. [20, 21]).

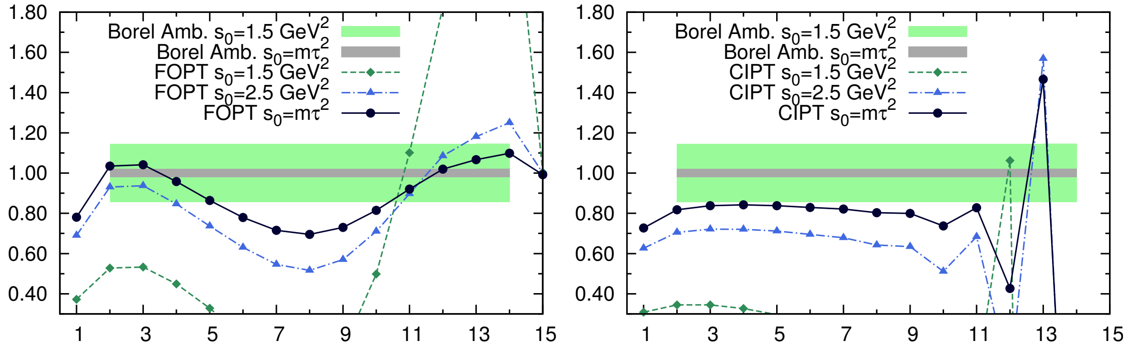
We can perform the same analysis in the alternative model, Eq. (2.8), which receives no contribution from the leading IR singularity. In Fig. 3, we show the series normalised to their respective Borel resummed values within the AM for $w_\tau(x)$. In this model, CIPT provides the better framework also for lower values of s_0, as shown in Fig. 3(b). The series remains very stable for $s_0 = 2.5$ GeV^2 and still approaches the Borel result well. For $s_0 = 1.5$ GeV^2 CIPT can still be considered a good approximation taking into account the amplified Borel ambiguity. The oscillations of FOPT, already present at $s_0 = m_\tau^2$, are much amplified for lower s_0 (see Fig. 3(a)). Within the AM, the FOPT series are not a good approximation to the Borel resummed values.

To conclude we examine the case of w_{17} in the context of the AM. The results are shown in Fig. 4. The bad perturbative behaviour of FOPT and CIPT remains for all values of s_0. This is an indication of the model independence of the conclusion that w_{17} (and a number of other moments also discussed in Ref. [17]) should be avoided in determinations of α_s.



(a) $w_\tau(x) = (1-x)^2(1+2x)$, FOPT. (b) $w_\tau(x) = (1-x)^2(1+2x)$, CIPT.

Figure 3: Alternative model. $\delta^{(0)}_{w_\tau}(s_0)$ order by order in α_s normalised to the Borel sum for FOPT (left) and CIPT (right) with three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. Bands give the Borel ambiguities.



(a) $w_{17}(x) = (1-x)^2x^3(1+2x)$, FOPT. (b) $w_{17}(x) = (1-x)^2x^3(1+2x)$, CIPT.

Figure 4: Alternative model. $\delta^{(0)}_{w_{17}}(s_0)$ order by order in α_s normalised to the Borel sum for FOPT (left) and CIPT (right) with three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. Bands give the Borel ambiguities.

3. Conclusions

Recently, we have analysed the perturbative behaviour of several moments often employed in analyses of α_s from τ decays under different assumptions for the large-order behaviour of the Adler function [17]. We have shown that some of these moments should be avoided due to their bad perturbative behaviour. Furthermore, under reasonable assumptions for the Borel transformed Adler function, we showed that FOPT provides the preferred framework for the RG improvement of moments that display good perturbative behaviour.

Here we showed, for the first time, that these conclusions are still valid if one considers the perturbative series generated by FOPT and CIPT for $s_0 \leq m_\tau^2$. This is a relevant question, since in α_s extractions one often considers sum-rules with $s_0 \leq m_\tau^2$. We have shown explicitly the results for two representative moments previously investigated in Ref. [17] for $s_0 = m_\tau^2$. The s_0 dependence analysis was also carried out for the remaining moments studied in [17] with similar conclusions; they are not shown here for the sake of brevity.

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