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Leading-order hadronic contributions to a_μ and $\alpha_{\rm QED}$ from $N_f = 2+1+1$ twisted mass fermions

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We present the first four-flavour lattice calculation of the leading-order hadronic vacuum-polarisation contribution to the anomalous magnetic moment of the muon, $a_\mu^{\rm hvp}$, and the hadronic running of the QED coupling constant, $\Delta\alpha_{\rm QED}^{\rm hvp}(Q^2)$. In the heavy sector a mixed-action setup is employed. The bare quark masses are determined from matching the K- and D-meson masses to their physical values. Several light quark masses are used in order to yield a controlled extrapolation to the physical pion mass by utilising a recently proposed improved method. We demonstrate that this method also works in the four-flavour case.

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1. Introduction

The anomalous magnetic moment of the muon, a_μ, serves as a benchmark test of the standard model (SM). It has been measured very accurately [1, 2] and can be computed precisely within the SM. A comparison between the experimental result for a_μ and the SM prediction reveals a discrepancy of more than three standard deviations which has persisted for many years now and has been confirmed by computations of a number of groups, see, for example, the review [3]. The question is whether this discrepancy originates from some effect missing in the experimental or theoretical determination of a_μ or whether it points to physics beyond the SM.

A key ingredient in the calculation of a_μ is the leading-order hadronic vacuum-polarisation contribution, $a_\mu^{\rm hvp}$, which presently is the largest source of uncertainty in the theoretical computation of a_μ, since the QED and electroweak contributions have been computed very accurately employing perturbation theory, see [4, 3] and references therein. As $a_\mu^{\rm hvp}$ is intrinsically nonperturbative, a lattice QCD computation of this observable is highly desirable. The currently accepted SM values for this quantity are obtained mainly from the investigation of e^+e^- scattering and τ decay data. In a recent study [5, 6], using two flavours of mass-degenerate quarks, a modified method to compute $a_\mu^{\rm hvp}$ has been introduced resulting in a determination of $a_{\mu,\rm light}^{\rm hvp}$ with a precision of a few percent.

Besides $a_\mu^{\rm hvp}$, the leading-order QCD contribution to the running of the QED coupling constant, $\Delta\alpha_{\rm QED}^{\rm hvp}$, also requires the hadronic vacuum-polarisation function and can likewise be investigated once this function is known. As an important input parameter to SM calculations, the QED coupling constant needs to be known very precisely in order to facilitate high precision tests of the SM at any future linear collider [7].

Below we report preliminary results of the first lattice calculation of $a_\mu^{\rm hvp}$ and $\Delta\alpha_{\rm QED}^{\rm hvp}$ with four quark flavours. The contributions of the top and bottom quarks are negligible at the current level of accuracy. So having four flavours allows for an unambiguous comparison to the dispersive analysis of $a_\mu^{\rm hvp}$ and direct use in forming the SM prediction for a_μ itself. The inclusion of the charm quark is essential because its contribution is of the order of $a_\mu^{\rm hvp,charm} \gtrsim 100\times 10^{-11}$ [9], which is larger than the currently quoted uncertainty of the difference between the experimental and the SM results. Furthermore, the order of magnitude of the charm quark contribution to $a_\mu^{\rm hvp}$ is the same as that of the hadronic light-by-light contribution [10]. Thus lattice calculations with a dynamical charm quark are necessary for computing $a_\mu^{\rm hvp}$ with a precision that matches the experimental accuracy.

The computation of $a_\mu^{\rm hvp}$ and $\Delta\alpha_{\rm QED}^{\rm hvp}$ follows closely the strategy of refs. [5, 6] using improved lattice definitions of these quantities. We demonstrate in this work that this new method continues to work well even for the four-flavour case and again results in a mild quark mass dependence leading to an accurate determination of $a_\mu^{\rm hvp}$ and $\Delta\alpha_{\rm QED}^{\rm hvp}$.

Our calculations are based on the configurations with four dynamical quark flavours generated by the European Twisted Mass Collaboration (ETMC) [11, 12]. These sets of configurations are obtained at different values of the lattice spacing and several lattice volumes, thus enabling us to estimate discretisation and finite size effects as systematic uncertainties of our lattice calculation. In addition, at each value of the lattice spacing, configurations exist at several values of the pion mass, ranging from $230\,\mathrm{MeV} \lesssim m_\pi \lesssim 450\,\mathrm{MeV}$.

2. The lattice calculation

The leading-order hadronic contribution to the muon's anomalous magnetic moment, $a_\mu^{\rm hvp}$, can be computed directly in Euclidean space-time [13]

$$a_\mu^{\rm hvp} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w\left(\frac{Q^2}{m_\mu^2}\right) \Pi_{\rm R}(Q^2) \,. \tag{2.1}$$

The renormalised vacuum-polarisation function is given by $\Pi_{\rm R}(Q^2) = \Pi(Q^2) - \Pi(0)$. $\Pi(Q^2)$ can be obtained from the hadronic vacuum-polarisation tensor knowing its Lorentz structure

$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot(x-y)} \langle J_\mu^{\rm em}(x) J_\nu^{\rm em}(y)\rangle = (Q_\mu Q_\nu - Q^2\delta_{\mu\nu})\Pi(Q^2) \,, \tag{2.2}$$

where

$$J_\mu^{\rm em}(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \frac{2}{3}\overline{c}(x)\gamma_\mu c(x) - \frac{1}{3}\overline{s}(x)\gamma_\mu s(x) \tag{2.3}$$

is the electromagnetic vector current.

On the lattice we use the conserved (point-split) vector current

$$\begin{aligned} J_\mu^C(x) &= \frac{1}{2}\Big(\overline{\chi}(x+\hat{\mu})(\mathbb{1}+\gamma_\mu)U_\mu^\dagger(x) Q_{\rm el}\chi(x) \\ &\quad - \overline{\chi}(x)(\mathbb{1}-\gamma_\mu)U_\mu(x) Q_{\rm el}\chi(x+\hat{\mu})\Big) \,. \end{aligned} \tag{2.4}$$

$\chi(x)$ denotes a fermion doublet, either the light or the heavy one, in the so-called twisted basis and $Q_{\rm el} = {\rm diag}(\frac{2}{3}, -\frac{1}{3})$ denotes the electric charge matrix. Resorting to Osterwalder-Seiler valence quarks [14, 15] in the heavy sector admits a straightforward construction of the conserved currents also for the strange and the charm quark. This is beneficial since in this way we can rely on the vector Ward-Takahashi identity for all contributions and also avoid renormalisation.

In order to have a smooth function to perform the integral in eq. (2.1), the vacuum-polarisation data obtained at discrete lattice momenta is fit for each flavour to the following functional form

$$\Pi(Q^2) = g_{\rm V}^2 \frac{m_{\rm V}^2}{Q^2 + m_{\rm V}^2} + b_0 + b_1 Q^2 \,. \tag{2.5}$$

The first term is the contribution of a narrow-width vector meson with mass $m_{\rm V}$ and electromagnetic coupling $g_{\rm V}$, which are determined directly in our calculation. The remaining terms parametrise any deviations from this form. The results reported below are obtained using uncorrelated fits to determine $g_{\rm V}$ and $m_{\rm V}$ from the zero-momentum current correlators. In these proceedings we do not perform alternative fits such as the Padé approximants suggested in [17]. We also do not provide an estimate of systematic effects by allowing additional terms in eq. (2.5). These aspects will be addressed in a forthcoming publication.

Once $\Pi_{\rm R}(Q^2)$ is known, it is straightforward to compute the leading-order hadronic contribution [7]

$$\Delta\alpha_{\rm QED}^{\rm hvp}(Q^2) = 4\pi\alpha_0 \Pi_{\rm R}\left(Q^2\right) \tag{2.6}$$

which influences the running of the fine structure constant according to [16]

$$\alpha_{\rm QED}(Q^2) = \frac{\alpha_0}{1 - \Delta\alpha_{\rm QED}(Q^2)} \,. \tag{2.7}$$

Here, α_0 is the value at $Q^2 = 0$, $\alpha_0 \approx \frac{1}{137}$.

For the lattice calculation of $a_\mu^{\rm hvp}$ discussed here, we use the modified definition from [5]

$$a_{\overline{\mu}}^{\rm hvp} = \alpha_0^2 \int_0^\infty \frac{dQ^2}{Q^2} w\left(\frac{Q^2}{H^2}\frac{H_{\rm phys}^2}{m_\mu^2}\right) \Pi_{\rm R}(Q^2)\,, \tag{2.8}$$

which goes to $a_\mu^{\rm hvp}$ for the light pseudoscalar mass m_{PS} assuming its physical value, i.e. m_π, since in this case also $H \to H_{\rm phys}$, with the choice of the hadronic scale, H, discussed below. Analogously, we determine $\Delta\alpha_{\rm QED}^{\rm hvp}(Q^2)$ from [6]

$$\Delta\overline{\alpha}_{\rm QED}^{\rm hvp}(Q^2) = 4\pi\alpha_0 \Pi_{\rm R}\left(Q^2 \frac{H^2}{H_{\rm phys}^2}\right). \tag{2.9}$$

There are several possible choices for H. Below we give results for $H = H_{\rm phys} \equiv 1$ (the standard choice) and $H = m_{\rm V}(m_{\rm PS})$, $H_{\rm phys} = m_\rho$ (improved choice). Here $m_{\rm V}(m_{\rm PS})$ is the light vector meson mass as measured on the lattice at unphysical pseudoscalar masses $m_{\rm PS}$ whereas m_ρ is the physical value of the ρ-meson mass. See refs. [5, 6] for a more detailed discussion.

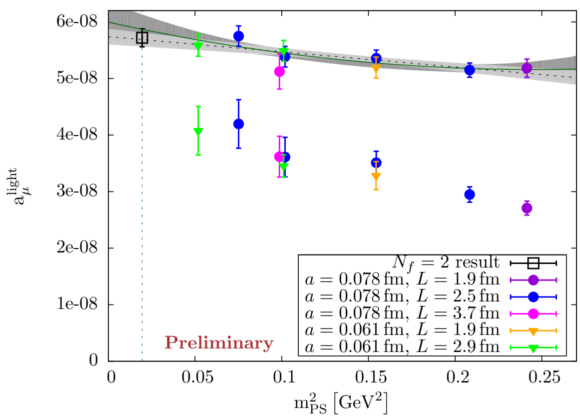

Figure 1: We show the light quark contribution to $a_\mu^{\rm hvp}$ as a function of the squared light pseudoscalar mass from our $N_f = 2+1+1$ computations. The lower set of data points corresponds to using the standard definition, $H = H_{\rm phys} = 1$ in eq. (2.8). The upper data points are obtained using the improved method, setting $H = m_{\rm V}(m_{\rm PS})$ and $H_{\rm phys} = m_\rho$. The open square represents the two-flavour result of refs. [5, 6]. The dark grey error band corresponds to the quadratic fit (solid green line) and the light grey one belongs to the linear fit (dashed black line).

The light quark contribution to $a_\mu^{\rm hvp}$ is depicted in fig. 1. In contrast to the standard choice the improved lattice definition of $a_\mu^{\rm hvp}$ shows a weaker pion mass dependence. Comparing the previous calculation with only two dynamical flavours of light, mass-degenerate quarks [5] to the one having four flavours in the sea quark sector, the results for $a_{\mu,\rm light}^{\rm hvp}$ extrapolated to the physical point are found to be in full agreement for both setups. This demonstrates that the effects of a dynamical strange and charm quark on the light quark contribution to $a_\mu^{\rm hvp}$ are small, as expected.

In fig. 2 we show the effect of adding the strange quark contribution and compare to the published results of other groups, [18, 19]. The figure demonstrates that when using the standard

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definition of a_μ^{hvp} on the lattice, all groups agree reasonably well. However, when using the improved definition, the pion mass dependence of a_μ^{hvp} is much flatter allowing for a better control of the extrapolation to the physical pion mass.

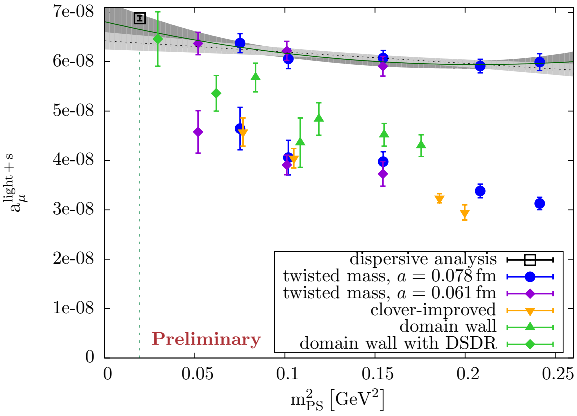

Figure 2: We compare the results for a_μ^{hvp} having the light up and down quarks as well as the strange quark in the valence sector for different collaborations. Only the upper set of data points for twisted mass fermions are obtained from the modified definition, $H = m_V(m_{\mathrm{PS}})$ and $H_{\mathrm{phys}} = m_\rho$ in eq. (2.8). Data are from [19] (clover-improved) and [18] (domain wall). The open square represents the standard model value obtained from the dispersive analysis of ref. [8]. Concerning the error bands of the fits, the same comments as in fig. 1 apply.

Finally, we show in fig. 3 the full four-flavour contribution to a_μ^{hvp}. Comparing with fig. 2, we see that including the charm quark indeed leads to a contribution of the expected order of magnitude. We perform both a linear and a quadratic extrapolation to the physical point. Within just the statistical errors, we find reasonable agreement with the dispersive result [8], as shown in fig. 3. At the current precision, we do not observe any statistically significant finite-size or lattice discretisation effects, however, the impact of disconnected contributions or unitary-violating effects due to the mixed-action setup in the heavy sector have not been examined yet. Furthermore, the effect from the ρ-meson not being able to decay to two pions for all but one ensemble has to be investigated. The details of the final extrapolation to the physical point and the associated systematic uncertainties will be presented elsewhere.

Fig. 4 shows our results for $\Delta\alpha_{\mathrm{QED}}^{\mathrm{hvp}}$ at a typical hadronic scale of $Q^2 = 1\,\mathrm{GeV}^2$ obtained from the same ensembles used to determine a_μ^{hvp}. Similar to a_μ^{hvp}, we find reasonable agreement with the dispersive result [7] to within the statistical uncertainties. A full determination of the systematic uncertainty for the extrapolated value will also be presented in a later publication.

3. Conclusion

We have pointed out that the charm quark contribution is necessary for achieving a lattice computation of a_μ^{hvp} with a precision that is comparable to the experimental one. Furthermore, we have shown that also in the case of a four-flavour calculation of the leading-order hadronic contribution to the muon anomalous magnetic moment and to the running of the electromagnetic coupling constant, the improved method of ref. [5, 6] continues to work well. We have so far not

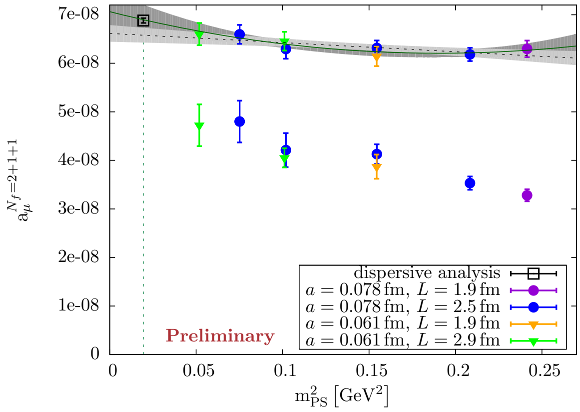

Figure 3: Full four-flavour contribution to $a_\mu^{\rm hvp}$ as a function of the squared light pseudoscalar mass from our $N_f = 2+1+1$ computations. Concerning the upper and lower data sets and the error bands of the fits, the same comments as in fig. 1 apply. The open square represents the standard model value obtained from the dispersive analysis of ref. [8]. Note that in this case of four flavours there is no ambiguity in determining the contribution to $a_\mu^{\rm hvp}$ from the dispersive analysis.

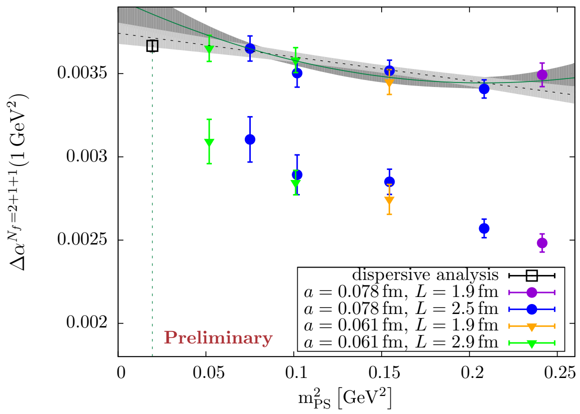

Figure 4: Full four-flavour contribution to $\Delta\alpha_{\rm QED}^{\rm hvp}(1\,{\rm GeV}^2)$ as a function of the squared light pseudoscalar mass from our $N_f = 2+1+1$ computations. The lower set of data points corresponds to the use of the standard definition, $H = H_{\rm phys} = 1$ in eq. (2.9). The upper data points are obtained using the improved method, setting $H = m_{\rm V}(m_{\rm PS})$ and $H_{\rm phys} = m_\rho$. Concerning the error bands of the fits, the same comments as in fig. 1 apply. The open square represents the standard model value obtained from the dispersive analysis of ref. [7]. Note that in this case of four flavours there is no ambiguity in determining the contribution to $\Delta\alpha_{\rm QED}^{\rm hvp}(1\,{\rm GeV}^2)$ from the dispersive analysis.

performed a comprehensive analysis of the systematic uncertainties. In particular, the disconnected contributions originating from the strange quark might be non-negligible. Additionally, it might be necessary to take isospin breaking effects into account as the precision of our computation improves.

We conclude by emphasising that the strategy followed here can also be applied to other observables such as the Adler function, the corrections to the energy levels of muonic and ordinary hydrogen, and the weak mixing angle, as discussed in [6].

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