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Dimensional Reduction of $\mathcal{N}=1$, E_8 SYM over $SU(3)/U(1)\times U(1)\times\mathbb{Z}_3$ and its four-dimensional effective action

Nikos Irges*
National Technical University of Athens Zografou Campus, GR-15780 Athens Greece
E-mail: irges@mail.ntua.gr

Georgios Orfanidis†
Theory Group, Physics Department CERN, Geneva, Switzerland
E-mail: orfanidi@mail.ntua.gr

George Zoupanos †
Theory Group, Physics Department CERN, Geneva, Switzerland
E-mail: George.Zoupanos@cern.ch

We present an extension of the Standard Model inspired by the $E_8\times E_8$ Heterotic String. In order that a reasonable effective Lagrangian is presented we neglect everything else other than the ten-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills sector associated with one of the gauge factors and certain couplings necessary for anomaly cancellation. We consider a compactified space-time $M_4\times B_0/\mathbb{Z}_3$, where B_0 is the nearly-Kähler manifold $SU(3)/U(1)\times U(1)$ and $\mathbb{Z}_3$ is a freely acting discrete group on B_0. Then we reduce dimensionally the E_8 on this manifold and we employ the Wilson flux mechanism leading in four dimensions to an $SU(3)^3$ gauge theory with the spectrum of a $\mathcal{N}=1$ supersymmetric theory. We compute the effective four-dimensional Lagrangian and demonstrate that an extension of the Standard Model is obtained with interesting features including a conserved baryon number and fixed tree level Yukawa couplings and scalar potential. The spectrum contains new states such as right handed neutrinos and heavy vector-like quarks.

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*Speaker.
†On leave from Physics Department, National Technical University, Zografou Campus: Heroon Polytechniou 9, 15780 Zografou, Athens, Greece

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1. Introduction

Superstring Theory is often regarded as the best candidate for a quantum theory of gravitation, or more generally as a unified theory of all fundamental interactions. On the other hand the main goal expected from a unified description of interactions by the Particle Physics community is to understand the present day large number of free parameters of the Standard Model (SM) in terms of a few fundamental ones. Indeed the celebrated SM had so far outstanding successes in all its confrontations with experimental results. However its apparent success is spoiled by the presence of a plethora of free parameters mostly related to the ad-hoc introduction of the Higgs and Yukawa sectors in the theory.

It is worth recalling that various dimensional reduction schemes, with the Coset Space Dimensional Reduction (CSDR) [1, 2, 3] and the Scherk-Schwarz reduction [4] being pioneers, suggest that a unification of the gauge and Higgs sectors can be achieved in higher than four dimensions. The four-dimensional gauge and Higgs fields are simply the surviving components of the gauge fields of a pure gauge theory defined in higher dimensions, while the addition of fermions in the higher-dimensional gauge theory leads naturally after CSDR to Yukawa couplings in four dimensions. The last step in this unified description in high dimensions is to relate the gauge and fermion fields, which can be achieved by demanding that the higher-dimensional gauge theory is $\mathcal{N} = 1$ supersymmetric, i.e. the gauge and fermion fields are members of the same vector supermultiplet. Furthermore a very welcome additional input coming from Superstring Theory (for instance the heterotic string [5]) is the suggestion of the space-time dimensions and the gauge group of the higher-dimensional supersymmetric theory [6].

Superstring Theory is consistent only in ten dimensions and therefore the following crucial issues have to be addressed, (a) distinguish the extra dimensions from the four observable ones which are experimentally approachable, i.e. determine a suitable compactification which is a solution of the theory (b) reduce the higher-dimensional theory to four dimensions and determine the corresponding four-dimensional theory, which may subsequently be compared to observations. Among superstring theories the heterotic string [5] has always been considered as the most promising version in the prospect to find contact with low-energy physics studied in accelerators, mainly due to the presence of the ten-dimensional $\mathcal{N} = 1$ gauge sector. Upon compactification of the ten-dimensional space-time and subsequent dimensional reduction the initial $E_8 \times E_8$ gauge theory can break to phenomenologically interesting Grand Unified Theories (GUTs), where the SM could in principle be accommodated [7]. Moreover, the presence of chiral fermions in the higher-dimensional theory serves as an advantage in view of the possibility to obtain chiral fermions also in the four-dimensional theory. Finally, the original $\mathcal{N} = 1$ supersymmetry can survive and not get enhanced in four dimensions, provided that appropriate compactification manifolds are used. In order to find contact with the minimal supersymmetric standard model (MSSM), the non-trivial part of this scenario was to invent mechanisms of supersymmetry breaking within the string framework.

The task of providing a suitable compactification and reduction scheme which would lead to a realistic four-dimensional theory has been pursued in many diverse ways for more than twenty years. The realization that Calabi-Yau (CY) threefolds serve as suitable compact internal spaces in order to maintain an $\mathcal{N} = 1$ supersymmetry after dimensional reduction from ten dimensions to four [8] led from the beginning to pioneering studies in the dimensional reduction of superstring models [9, 10]. However, in CY compactifications the resulting low-energy field theory in four dimensions contains a number of massless chiral fields, known as moduli, which correspond to flat directions of the effective potential and therefore their values are left undetermined. The attempts to resolve the moduli stabilization problem led to the study of compactifications with fluxes (for reviews see e.g. [11]). In the context of flux compactifications the recent developments suggested the use of a wider class of internal spaces, called manifolds with $SU(3)$-structure, which contains CYs. Admittance of an $SU(3)$-structure is a milder condition as compared to $SU(3)$-holonomy, which is the case for CY manifolds, in the sense that a nowhere-vanishing, globally-defined spinor can be defined such that it is covariantly constant with respect to a connection with torsion and not with respect to the Levi-Civita connection as in the CY case. An interesting class of manifolds admitting an $SU(3)$-structure is that of nearly-Kähler manifolds. The homogeneous nearly-Kähler manifolds in six dimensions have been classified in [12] and they are the three non-symmetric six-dimensional coset spaces (see table 1 Appendix B) and the group manifold $SU(2) \times SU(2)$. In the studies of heterotic compactifications the use of non-symmetric coset spaces was introduced in [13, 15, 14] and recently developed further in [16, 17]. Particularly, in [17] it was shown that supersymmetric compactifications of the heterotic string theory of the form $AdS_4 \times S/R$ exist when background fluxes and general condensates are present. Moreover, the effective theories resulting from dimensional reduction of the heterotic string over nearly-Kähler manifolds were studied in [18].

Last but not least it is worth noting that the dimensional reduction of ten-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories over non-symmetric coset spaces led in four dimensions to softly broken $\mathcal{N} = 1$ theories [19, 20].

Here we would like to present the significant progress that has been made recently concerning the dimensional reduction of the N=1 supersymmetric E_8 gauge theory resulting in the field theory limit of the heterotic string over the nearly-Kähler manifold $SU(3)/U(1) \times U(1)$. Specifically an extension of the Standard Model (SM) inspired by the $E_8 \times E_8$ heterotic string was derived [21][1]. In addition in order to make the presentation self-contained we present first a short review of the CSDR.

2. The Coset Space Dimensional Reduction.

Given a gauge theory defined in higher dimensions the obvious way to dimensionally reduce it is to demand that the field dependence on the extra coordinates is such that the Lagrangian is independent of them. A crude way to fulfill this requirement is to discard the field dependence on the extra coordinates, while an elegant one is to allow for a non-trivial dependence on them, but impose the condition that a symmetry transformation by an element of the isometry group S of the space formed by the extra dimensions B corresponds to a gauge transformation. Then the Lagrangian will be independent of the extra coordinates just because it is gauge invariant. This is the basis of the CSDR scheme [1, 2, 3], which assumes that B is a compact coset space, S/R.

In the CSDR scheme one starts with a Yang-Mills-Dirac Lagrangian, with gauge group G, defined on a D-dimensional spacetime M^D, with metric g^{MN}, which is compactified to $M^4 \times S/R$

[1] For earlier attempts to obtain realistic models by CSDR see ref [2, 31, 38, 50].

with S/R a coset space. The metric is assumed to have the form

$$g^{MN} = \begin{bmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{bmatrix}, \tag{2.1}$$

where $\eta^{\mu\nu} = diag(1,-1,-1,-1)$ and g^{ab} is the coset space metric. The requirement that transformations of the fields under the action of the symmetry group of S/R are compensated by gauge transformations lead to certain constraints on the fields. The solution of these constraints provides us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. Therefore a potential unification of all low energy interactions, gauge, Yukawa and Higgs is achieved, which was the first motivation of this framework.

It is interesting to note that the fields obtained using the CSDR approach are the first terms in the expansion of the D-dimensional fields in harmonics of the internal space B [2, 49]. The effective field theories resulting from compactification of higher dimensional theories might contain also towers of massive higher harmonics (Kaluza-Klein) excitations, whose contributions at the quantum level alter the behaviour of the running couplings from logarithmic to power [22]. As a result the traditional picture of unification of couplings may change drastically [23]. Higher dimensional theories have also been studied at the quantum level using the continuous Wilson renormalization group [24] which can be formulated in any number of space-time dimensions with results in agreement with the treatment involving massive Kaluza-Klein excitations. However we should stress that in ref [25] the CSDR has been shown to be a consistent scheme.

Before we proceed with the description of the CSDR scheme we need to recall some facts about coset space geometry needed for subsequent discussions. Complete reviews can be found in [2, 26].

2.1 Coset Space Geometry.

Assuming a D-dimensional spacetime M^D with metric g^{MN} given in (2.1) it is instructive to explore further the geometry of all coset spaces S/R.

We can divide the generators of S, Q_A in two sets : the generators of R, Q_i $(i = 1,\ldots,dimR)$, and the generators of S/R, Q_a($a = dimR+1\ldots,dimS$), and $dimS/R = dimS - dimR = d$. Then the commutation relations for the generators of S are the following:

$$\begin{aligned} [Q_i,Q_j] &= f_{ij}{}^{k}Q_k, \\ [Q_i,Q_a] &= f_{ia}{}^{b}Q_b, \\ [Q_a,Q_b] &= f_{ab}{}^{i}Q_i + f_{ab}{}^{c}Q_c. \end{aligned} \tag{2.2}$$

So S/R is assumed to be a reductive but in general non-symmetric coset space. When S/R is symmetric, the $f_{ab}{}^{c}$ in (2.2) vanish. Let us call the coordinates of $M^4 \times S/R$ space $z^M = (x^m, y^\alpha)$, where α is a curved index of the coset, a is a tangent space index and y defines an element of S which is a coset representative, $L(y)$. The vielbein and the R-connection are defined through the Maurer-Cartan form which takes values in the Lie algebra of S :

$$L^{-1}(y)dL(y) = e^A_\alpha Q_A dy^\alpha. \tag{2.3}$$

Using (2.3) we can compute that at the origin $y=0$, $e^a_\alpha = \delta^a_\alpha$ and $e^i_\alpha = 0$. A connection on S/R which is described by a connection-form θ^a_b, has in general torsion and curvature. In the general case where torsion may be non-zero, we calculate first the torsionless part ω^a_b by setting the torsion form T^a equal to zero,

$$T^a = de^a + \omega^a_b \wedge e^b = 0, \tag{2.4}$$

while using the Maurer-Cartan equation,

$$de^a = \frac{1}{2} f^a_{bc} e^b \wedge e^c + f^a_{bi} e^b \wedge e^i, \tag{2.5}$$

we see that the condition of having vanishing torsion is solved by

$$\omega^a_b = -f^a_{ib} e^i - \frac{1}{2} f^a_{bc} e^c - \frac{1}{2} K^a_{bc} e^c, \tag{2.6}$$

where K^a_{bc} is symmetric in the indices b,c, therefore $K^a_{bc} e^c \wedge e^b = 0$. The K^a_{bc} can be found from the antisymmetry of ω^a_b, $\omega^a_b g^{cb} = -\omega^b_c g^{ca}$, leading to

$$K^a{}_{bc} = g^{ad}(g_{be} f_{dc}{}^e + g_{ce} f_{db}{}^e). \tag{2.7}$$

In turn ω^a_b becomes

$$\omega^a_b = -f^a_{ib} e^i - D^a{}_{bc} e^c, \tag{2.8}$$

where

$$D^a{}_{bc} = \frac{1}{2} g^{ad} [f_{db}{}^e g_{ec} + f_{cb}{}^e g_{de} - f_{cd}{}^e g_{be}].$$

The D's can be related to f's by a rescaling [2]:

$$D^a_{bc} = (\lambda^a \lambda^b / \lambda^c) f^a_{bc},$$

where the λ's depend on the coset radii. Note that in general the rescalings change the antisymmetry properties of f's, while in the case of equal radii $D^a_{bc} = \frac{1}{2} f^a_{bc}$. Note also that the connection-form ω^a_b is S-invariant. This means that parallel transport commutes with the S action [26]. Then the most general form of an S-invariant connection on S/R would be

$$\omega^a_b = f^a{}_{ib} e^i + J^a{}_{cb} e^c, \tag{2.9}$$

with J an R-invariant tensor, i.e.

$$\delta J_{cb}{}^a = -f_{ic}{}^d J_{db}{}^a + f_{id}{}^a J_{cb}{}^d - f_{ib}{}^d J_{cd}{}^a = 0.$$

This condition is satisfied by the D's as can be proven using the Jacobi identity.

In the case of non-vanishing torsion we have

$$T^a = de^a + \theta^a_b \wedge e^b, \tag{2.10}$$

where

$$\theta^a_b = \omega^a_b + \tau^a_b,$$

with

$$\tau^a_b = -\frac{1}{2}\Sigma^a{}_{bc}e^c, \tag{2.11}$$

while the contorsion $\Sigma^a{}_{bc}$ is given by

$$\Sigma^a{}_{bc} = T^a{}_{bc} + T_{bc}{}^a - T_{cb}{}^a \tag{2.12}$$

in terms of the torsion components $T^a{}_{bc}$. Therefore in general the connection-form θ^a_b is

$$\theta^a_b = -f^a_{ic}e^i - (D^a{}_{bc} + \frac{1}{2}\Sigma^a{}_{bc})e^c = -f^a_{ic}e^i - G^a{}_{bc}e^c. \tag{2.13}$$

The natural choice of torsion which would generalize the case of equal radii [15, 27, 28], $T^a_{bc} = \eta f^a_{bc}$ would be $T^a_{bc} = 2\tau D^a{}_{bc}$ except that the D's do not have the required symmetry properties. Therefore we must define Σ as a combination of D's which makes Σ completely antisymmetric and S-invariant according to the definition given above. Thus we are led to the definition

$$\Sigma_{abc} \equiv 2\tau(D_{abc} + D_{bca} - D_{cba}). \tag{2.14}$$

In this general case the Riemann curvature two-form is given by [2], [28]:

$$R^a{}_b = [-\frac{1}{2}f_{ib}{}^a f^i_{de} - \frac{1}{2}G^{\ a}_{cb}f^{\ c}_{de} + \frac{1}{2}(G^{\ a}_{dc}G^{\ c}_{eb} - G^{\ a}_{ec}G^{\ c}_{db})]e^d \wedge e^e, \tag{2.15}$$

whereas the Ricci tensor $R_{ab} = R^d{}_{adb}$ is

$$R_{ab} = G^{\ c}_{ba}G^{\ d}_{dc} - G^{\ d}_{bc}G^{\ c}_{da} - G^{\ d}_{ca}f^{\ c}_{db} - f^{\ d}_{ia}f^{\ i}_{db}. \tag{2.16}$$

By choosing the parameter τ to be equal to zero we can obtain the *Riemannian connection* $\theta_R{}^a{}_b$. We can also define the *canonical connection* by adjusting the radii and τ so that the connection form is $\theta_C{}^a{}_b = -f^a_{bi}e^i$, i.e. an R-gauge field [15]. The adjustments should be such that $G_{abc} = 0$. In the case of $G_2/SU(3)$ where the metric is $g_{ab} = a\delta_{ab}$, we have $G_{abc} = \frac{1}{2}a(1+3\tau)f_{abc}$ and in turn $\tau = -\frac{1}{3}$. In the case of $Sp(4)/(SU(2)\times U(1))_{non-max.}$, where the metric is $g_{ab} = diag(a,a,b,b,a,a)$, we have to set $a = b$ and then $\tau = -\frac{1}{3}$ to obtain the canonical connection. Similarly in the case of $SU(3)/(U(1)\times U(1))$, where the metric is $g_{ab} = diag(a,a,b,b,c,c)$, we should set $a = b = c$ and take $\tau = -\frac{1}{3}$. By analogous adjustments we can set the Ricci tensor equal to zero [15], thus defining a *Ricci flattening connection*.

2.2 Reduction of a D-dimensional Yang-Mills-Dirac Lagrangian.

The group S acts as a symmetry group on the extra coordinates. The CSDR scheme demands that an S-transformation of the extra d coordinates is a gauge transformation of the fields that are defined on $M^4 \times S/R$, thus a gauge invariant Lagrangian written on this space is independent of the extra coordinates.

To see this in detail we consider a D-dimensional Yang-Mills-Dirac theory with gauge group G defined on a manifold M^D which as stated will be compactified to $M^4 \times S/R$, $D = 4+d$, $d = dimS - dimR$:

$$A = \int d^4x d^d y \sqrt{-g}\Big[-\frac{1}{4}Tr\left(F_{MN}F_{K\Lambda}\right)g^{MK}g^{N\Lambda} + \frac{i}{2}\overline{\psi}\Gamma^M D_M\psi\Big], \tag{2.17}$$

where

$$D_M = \partial_M - \theta_M - A_M, \tag{2.18}$$

with

$$\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{N\Lambda} \tag{2.19}$$

the spin connection of M^D, and

$$F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N], \tag{2.20}$$

where M, N run over the D-dimensional space. The fields A_M and ψ are, as explained, symmetric in the sense that any transformation under symmetries of S/R is compensated by gauge transformations. The fermion fields can be in any representation F of G unless a further symmetry such as supersymmetry is required. So let ξ_A^α, $A = 1,\dots,dimS$, be the Killing vectors which generate the symmetries of S/R and W_A the compensating gauge transformation associated with ξ_A. Define next the infinitesimal coordinate transformation as $\delta_A \equiv L_{\xi_A}$, the Lie derivative with respect to ξ, then we have for the scalar,vector and spinor fields,

$$\begin{aligned}
\delta_A\phi &= \xi_A^\alpha\partial_\alpha\phi = D(W_A)\phi, \\
\delta_A A_\alpha &= \xi_A^\beta\partial_\beta A_\alpha + \partial_\alpha\xi_A^\beta A_\beta = \partial_\alpha W_A - [W_A, A_\alpha], \\
\delta_A\psi &= \xi_A^\alpha\psi - \frac{1}{2}G_{Abc}\Sigma^{bc}\psi = D(W_A)\psi.
\end{aligned} \tag{2.21}$$

W_A depend only on internal coordinates y and $D(W_A)$ represents a gauge transformation in the appropriate representation of the fields. G_{Abc} represents a tangent space rotation of the spinor fields. The variations δ_A satisfy, $[\delta_A, \delta_B] = f_{AB}^C\delta_C$ and lead to the following consistency relation for W_A's,

$$\xi_A^\alpha\partial_\alpha W_B - \xi_B^\alpha\partial_\alpha W_A - [W_A, W_B] = f_{AB}{}^C W_C. \tag{2.22}$$

Furthermore the W's themselves transform under a gauge transformation [2] as,

$$\widetilde{W}_A = gW_Ag^{-1} + (\delta_A g)g^{-1}. \tag{2.23}$$

Using (2.23) and the fact that the Lagrangian is independent of y we can do all calculations at $y = 0$ and choose a gauge where $W_a = 0$.

The detailed analysis of the constraints (2.21) given in refs.[1, 2] provides us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. Here we give the results. The components $A_\mu(x,y)$ of the initial gauge field $A_M(x,y)$ become, after dimensional reduction, the four-dimensional gauge fields and furthermore they are independent of y. In addition one can find that they have to commute with the elements of the R_G subgroup of G. Thus the four-dimensional gauge group H is the centralizer of R in G, $H = C_G(R_G)$. Similarly, the $A_\alpha(x,y)$ components of $A_M(x,y)$ denoted by $\phi_\alpha(x,y)$ from now on, become scalars at four dimensions. These fields transform under R as a vector v, i.e.

$$\begin{aligned}
S &\supset R \\
adjS &= adjR + v.
\end{aligned} \tag{2.24}$$

Moreover $\phi_\alpha(x,y)$ act as an intertwining operator connecting induced representations of R acting on G and S/R. This implies, exploiting Schur's lemma, that the transformation properties of the fields $\phi_\alpha(x,y)$ under H can be found if we express the adjoint representation of G in terms of $R_G \times H$:

$$\begin{aligned} G &\supset R_G \times H \\ adjG &= (adjR, 1) + (1, adjH) + \sum (r_i, h_i). \end{aligned} \tag{2.25}$$

Then if $v = \sum s_i$, where each s_i is an irreducible representation of R, there survives an h_i multiplet for every pair (r_i, s_i), where r_i and s_i are identical irreducible representations of R.

Turning next to the fermion fields [2, 30, 31, 29] similarly to scalars, they act as intertwining operators between induced representations acting on G and the tangent space of S/R, $SO(d)$. Proceeding along similar lines as in the case of scalars to obtain the representation of H under which the four-dimensional fermions transform, we have to decompose the representation F of the initial gauge group in which the fermions are assigned under $R_G \times H$, i.e.

$$F = \sum (t_i, h_i), \tag{2.26}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j. \tag{2.27}$$

Then for each pair t_i and σ_i, where t_i and σ_i are identical irreducible representations there is an h_i multiplet of spinor fields in the four-dimensional theory. In order however to obtain chiral fermions in the effective theory we have to impose further requirements. We first impose the Weyl condition in D dimensions. In $D = 4n+2$ dimensions which is the case at hand, the decomposition of the left handed, say spinor under $SU(2) \times SU(2) \times SO(d)$ is

$$\sigma_D = (2, 1, \sigma_d) + (1, 2, \overline{\sigma}_d). \tag{2.28}$$

So we have in this case the decompositions

$$\sigma_d = \sum \sigma_k, \ \overline{\sigma}_d = \sum \overline{\sigma}_k. \tag{2.29}$$

Let us start from a vector-like representation F for the fermions. In this case each term (t_i, h_i) in (2.26) will be either self-conjugate or it will have a partner $(\bar{t}_i, \bar{h}_i)$. According to the rule described in eqs.(2.26), (2.27) and considering σ_d we will have in four dimensions left-handed fermions transforming as $f_L = \sum h_k^L$. It is important to notice that since σ_d is non self-conjugate, f_L is non self-conjugate too. Similarly from $\overline{\sigma}_d$ we will obtain the right handed representation $f_R = \sum \overline{h}_k^R$ but as we have assumed that F is vector-like, $\overline{h}_k^R \sim h_k^L$. Therefore there will appear two sets of Weyl fermions with the same quantum numbers under H. This is already a chiral theory, but still one can go further and try to impose the Majorana condition in order to eliminate the doubling of the fermionic spectrum. We should remark now that if we had started with F complex, we should have again a chiral theory since in this case $\overline{h}_k^R$ is different from h_k^L (σ_d non self-conjugate). Nevertheless starting with F vector-like is much more appealing and will be used in the following along with the Majorana condition. Majorana and Weyl conditions are compatible in $D = 4n+2$ dimensions.

Then in our case if we start with Weyl-Majorana spinors in $D = 4n+2$ dimensions we force f_R to be the charge conjugate to f_L, thus arriving in a theory with fermions only in f_L.

An important requirement is that the resulting four-dimensional theories should be anomaly free. Starting with an anomaly free theory in higher dimensions, Witten [32] has given the condition to be fulfilled in order to obtain anomaly free four-dimensional theories. The condition restricts the allowed embeddings of R into G by relating them with the embedding of R into $SO(6)$, the tangent space of the six-dimensional cosets we consider [2, 33]. To be more specific if L_a are the generators of R into G and T_a are the generators of R into $SO(6)$ the condition reads

$$Tr(L_a L_b) = 30\, Tr(T_a T_b)\,. \tag{2.30}$$

According to ref. [33] the anomaly cancellation condition (2.30) is automatically satisfied for the choice of embedding

$$E_8 \supset SO(6) \supset R\,, \tag{2.31}$$

which we adopt here. Furthermore concerning the abelian group factors of the four-dimensional gauge theory, we note that the corresponding gauge bosons surviving in four dimensions become massive at the compactification scale [32, 34] and therefore, they do not contribute in the anomalies; they correspond only to global symmetries.

2.3 The Four-Dimensional Theory.

Next let us obtain the four-dimensional effective action. Assuming that the metric is block diagonal, taking into account all the constraints and integrating out the extra coordinates we obtain in four dimensions the following Lagrangian :

$$A = C \int d^4x \left(-\frac{1}{4} F^t_{\mu\nu} F^{t\,\mu\nu} + \frac{1}{2} (D_\mu \phi_\alpha)^t (D^\mu \phi^\alpha)^t + V(\phi) + \frac{i}{2} \overline{\psi} \Gamma^\mu D_\mu \psi - \frac{i}{2} \overline{\psi} \Gamma^a D_a \psi \right), \tag{2.32}$$

where $D_\mu = \partial_\mu - A_\mu$ and $D_a = \partial_a - \theta_a - \phi_a$ with $\theta_a = \frac{1}{2}\theta_{abc}\Sigma^{bc}$ the connection of the coset space, while C is the volume of the coset space. The potential $V(\phi)$ is given by:

$$V(\phi) = -\frac{1}{4} g^{ac} g^{bd} Tr(f^C_{ab}\phi_C - [\phi_a, \phi_b])(f^D_{cd}\phi_D - [\phi_c, \phi_d]), \tag{2.33}$$

where, $A = 1, \ldots, dimS$ and f ' s are the structure constants appearing in the commutators of the generators of the Lie algebra of S. The expression (2.33) for $V(\phi)$ is only formal because ϕ_a must satisfy the constraints coming from eq.(2.21),

$$f^D_{ai}\phi_D - [\phi_a, \phi_i] = 0, \tag{2.34}$$

where the ϕ_i generate R_G. These constraints imply that some components ϕ_a's are zero, some are constants and the rest can be identified with the genuine Higgs fields. When $V(\phi)$ is expressed in terms of the unconstrained independent Higgs fields, it remains a quartic polynomial which is invariant under gauge transformations of the final gauge group H, and its minimum determines the vacuum expectation values of the Higgs fields [35, 36, 37]. The minimization of the potential is in general a difficult problem. If however S has an isomorphic image S_G in G which contains R_G in a consistent way then it is possible to allow the ϕ_a to become generators of S_G. That is

$\overline{\phi}_a =< \phi^i > Q_{ai} = Q_a$ with $< \phi^i > Q_{ai}$ suitable combinations of G generators, Q_a a generator of S_G and a is also a coset-space index. Then

$$\begin{aligned}\overline{F}_{ab} &= f_{ab}^{\ \ i}Q_i + f_{ab}^{\ \ c}\overline{\phi}_c - [\overline{\phi}_a, \overline{\phi}_b] \\ &= f_{ab}^{\ \ i}Q_i + f_{ab}^{\ \ c}Q_c - [Q_a, Q_b] = 0\end{aligned}$$

because of the commutation relations of S. Thus we have proven that $V(\phi = \overline{\phi}) = 0$ which furthermore is the minimum, because V is positive definite. Furthermore, the four-dimensional gauge group H breaks further by these non-zero vacuum expectation values of the Higgs fields to the centralizer K of the image of S in G, i.e. $K = C_G(S)$ [2, 35, 36, 37]. This can been seen if we examine a gauge transformation of ϕ_a by an element h of H. Then we have

$$\phi_a \to h\phi_a h^{-1}, h \in H$$

We note that the v.e.v. of the Higgs fields is gauge invariant for the set of h's that commute with S. That is h belongs to a subgroup K of H which is the centralizer of S_G in G.

In the fermion part of the Lagrangian the first term is just the kinetic term of fermions, while the second is the Yukawa term [38]. Note that since ψ is a Majorana-Weyl spinor in ten dimensions the representation in which the fermions are assigned under the gauge group must be real. The last term in (2.32) can be written as

$$L_Y = -\frac{i}{2}\overline{\psi}\Gamma^a(\partial_a - \frac{1}{2}f_{ibc}e^i_\Gamma e^\Gamma_a \Sigma^{bc} - \frac{1}{2}G_{abc}\Sigma^{bc} - \phi_a)\psi = \frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi + \overline{\psi}V\psi,$$

where

$$\nabla_a = -\partial_a + \frac{1}{2}f_{ibc}e^i_\Gamma e^\Gamma_a \Sigma^{bc} + \phi_a, \tag{2.35}$$

$$V = \frac{i}{4}\Gamma^a G_{abc}\Sigma^{bc}, \tag{2.36}$$

and we have used the full connection with torsion [28] given by

$$\theta^a{}_{cb} = -f^a{}_{ib}e^i_\alpha e^\alpha_c - (D^a{}_{cb} + \frac{1}{2}\Sigma^a{}_{cb}) = -f^a{}_{ib}e^i_\alpha e^\alpha_c - G^a{}_{cb} \tag{2.37}$$

with

$$D^a{}_{cb} = g^{ad}\frac{1}{2}[f_{db}^{\ \ e}g_{ec} + f_{cb}^{\ \ e}g_{de} - f_{cd}^{\ \ e}g_{be}] \tag{2.38}$$

and

$$\Sigma_{abc} = 2\tau(D_{abc} + D_{bca} - D_{cba}). \tag{2.39}$$

We have already noticed that the CSDR constraints tell us that $\partial_a\psi = 0$. Furthermore we can consider the Lagrangian at the point $y = 0$, due to its invariance under S-transformations, and as we mentioned $e^i_\Gamma = 0$ at that point. Therefore (2.35) becomes just $\nabla_a = \phi_a$ and the term $\frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi$ in (2.35) is exactly the Yukawa term.

Let us examine now the last term appearing in (2.35). One can show easily that the operator V anticommutes with the six-dimensional helicity operator [2]. Furthermore one can show that V commutes with the $T_i = -\frac{1}{2}f_{ibc}\Sigma^{bc}$ (T_i close the R-subalgebra of $SO(6)$). In turn we can draw the conclusion, exploiting Schur's lemma, that the non-vanishing elements of V are only those which appear in the decomposition of both $SO(6)$ irreps 4 and $\overline{4}$, e.g. the singlets. Since this term is of pure geometric nature, we reach the conclusion that the singlets in 4 and $\overline{4}$ will acquire large geometrical masses, a fact that has serious phenomenological implications. In supersymmetric theories defined in higher dimensions, it means that the gauginos obtained in four dimensions after dimensional reduction receive masses comparable to the compactification scale. However as we shall see in the next section this result changes in presence of torsion. We note that for symmetric coset spaces the V operator is absent because f_{ab}^{c} are vanishing by definition in that case.

3. Dimensional Reduction of E_8 over $SU(3)/U(1)\times U(1)$ and soft supersymmetry breaking

In this model we consider the coset space $B = SU(3)/U(1)\times U(1)$ on which we reduce the ten-dimensional theory. To determine the four-dimensional gauge group, the embedding of $R = U(1)\times U(1)$ in E_8 is suggested by the decomposition

$$E_8 \supset E_6 \times SU(3) \supset E_6 \times U(1)_A \times U(1)_B. \tag{3.1}$$

Then, the surviving gauge group in four dimensions is

$$H = C_{E_8}(U(1)\times U(1)) = E_6 \times U(1)_A \times U(1)_B.$$

The 248 of E_8 decomposes under $E_6 \times U(1)_A \times U(1)_B$ in the following way:

$$\begin{aligned} 248 = {} & 1_{(0,0)} + 1_{(0,0)} + 1_{(3,\frac{1}{2})} + 1_{(-3,\frac{1}{2})} + \\ & 1_{(0,-1)} + 1_{(0,1)} + 1_{(-3,-\frac{1}{2})} + 1_{(3,-\frac{1}{2})} + \\ & 78_{(0,0)} + 27_{(3,\frac{1}{2})} + 27_{(-3,\frac{1}{2})} + 27_{(0,-1)} + \\ & \overline{27}_{(-3,-\frac{1}{2})} + \overline{27}_{(3,-\frac{1}{2})} + \overline{27}_{(0,1)}. \end{aligned} \tag{3.2}$$

The $R = U(1)\times U(1)$ decomposition of the vector and spinor representations of $SO(6)$ (see table 1, Appendix B) is

$$6_v = (3,\frac{1}{2}) + (-3,\frac{1}{2}) + (0,-1) + (-3,-\frac{1}{2}) + (3,-\frac{1}{2}) + (0,1)$$

and

$$4_s = (0,0) + (3,\frac{1}{2}) + (-3,\frac{1}{2}) + (0,-1)$$

respectively. Thus applying the CSDR rules we find that the surviving fields in four dimensions are three $\mathcal{N}=1$ vector multiplets $V^{\alpha}, V_{(1)}, V_{(2)}$, (where α is an E_6, 78 index and the other two refer to the two $U(1)'s$) containing the gauge fields of $E_6 \times U(1)_A \times U(1)_B$. The matter content consists of three $\mathcal{N}=1$ chiral multiplets (A^i, B^i, C^i) with i an E_6, 27 index and three $\mathcal{N}=1$ chiral multiplets (A, B, C) which are E_6 singlets and carry only $U(1)_A \times U(1)_B$ charges.

To determine the potential we examine further the decomposition of the adjoint of the specific $S = SU(3)$ under $R = U(1)\times U(1)$, i.e.

$$SU(3) \supset U(1)\times U(1)$$

$$8 = (0,0) + (0,0) + 6_{\upsilon}. \tag{3.3}$$

Then according to the decomposition (3.3) the generators of $SU(3)$ can be grouped as

$$Q_{SU(3)} = \{Q_0, Q'_0, Q_1, Q_2, Q_3, Q^1, Q^2, Q^3\}. \tag{3.4}$$

The non trivial commutator relations of $SU(3)$ generators (3.4) are given in the table 2 given in the Appendix B. The decomposition (3.4) suggests the following change in the notation of the scalar fields,

$$(\phi_I, I = 1, \ldots, 8) \longrightarrow (\phi_0, \phi'_0, \phi_1, \phi^1, \phi_2, \phi^2, \phi_3, \phi^3). \tag{3.5}$$

The potential of any theory reduced over $SU(3)/U(1) \times U(1))$ is given in terms of the redefined fields in (3.5) by

$$\begin{aligned}
V(\phi) = {} & (3\Lambda^2 + \Lambda'^2)\left(\frac{1}{R_1^4} + \frac{1}{R_2^4}\right) + \frac{4\Lambda'^2}{R_3^2} \\
& + \frac{2}{R_2^2R_3^2}Tr(\phi_1\phi^1) + \frac{2}{R_1^2R_3^2}Tr(\phi_2\phi^2) + \frac{2}{R_1^2R_2^2}Tr(\phi_3\phi^3) \\
& + \frac{\sqrt{3}\Lambda}{R_1^4}Tr(Q_0[\phi_1,\phi^1]) - \frac{\sqrt{3}\Lambda}{R_2^4}Tr(Q_0[\phi_2,\phi^2]) - \frac{\sqrt{3}\Lambda}{R_3^4}Tr(Q_0[\phi_3,\phi^3]) \\
& + \frac{\Lambda'}{R_1^4}Tr(Q'_0[\phi_1,\phi^1]) + \frac{\Lambda'}{R_2^4}Tr(Q'_0[\phi_2,\phi^2]) - \frac{2\Lambda'}{R_3^4}Tr(Q'_0[\phi_3,\phi^3]) \\
& + \left[\frac{2\sqrt{2}}{R_1^2R_2^2}Tr(\phi_3[\phi_1,\phi_2]) + \frac{2\sqrt{2}}{R_1^2R_3^2}Tr(\phi_2[\phi_3,\phi_1]) + \frac{2\sqrt{2}}{R_2^2R_3^2}Tr(\phi_1[\phi_2,\phi_3]) + h.c\right] \\
& + \frac{1}{2}Tr\left(\frac{1}{R_1^2}([\phi_1,\phi^1]) + \frac{1}{R_2^2}([\phi_2,\phi^2]) + \frac{1}{R_3^2}([\phi_3,\phi^3])\right)^2 \\
& - \frac{1}{R_1^2R_2^2}Tr([\phi_1,\phi_2][\phi^1,\phi^2]) - \frac{1}{R_1^2R_3^2}Tr([\phi_1,\phi_3][\phi^1,\phi^3]) \\
& - \frac{1}{R_2^2R_3^2}Tr([\phi_2,\phi_3][\phi^2,\phi^3]),
\end{aligned} \tag{3.6}$$

where R_1, R_2, R_3 are the coset space radii[2]. In terms of the radii the real metric[3] of the coset is

$$g_{ab} = diag(R_1^2, R_1^2, R_2^2, R_2^2, R_3^2, R_3^2). \tag{3.7}$$

Next we examine the commutation relations of E_8 under the decomposition (3.2). Under this decomposition the generators of E_8 can be grouped as

$$\begin{aligned}
Q_{E_8} = \{ & Q_0, Q'_0, Q_1, Q_2, Q_3, Q^1, Q^2, Q^3, Q^\alpha, \\
& Q_{1i}, Q_{2i}, Q_{3i}, Q^{1i}, Q^{2i}, Q^{3i}\},
\end{aligned} \tag{3.8}$$

where, $\alpha = 1, \ldots, 78$ and $i = 1, \ldots, 27$. The non-trivial commutation relations of the E_8 generators (3.8) are given in Appendix B in the table 3.

[2]To bring the potential into this form we have used (A.22) of ref.[2] and relations (7),(8) of ref.[9].

[3]The complex metric that was used is $g^{1\bar{1}} = \frac{1}{R_1^2}, g^{2\bar{2}} = \frac{1}{R_2^2}, g^{3\bar{3}} = \frac{1}{R_3^2}$.

Now the constraints (2.34) for the redefined fields in (3.5) are,

$$\begin{aligned}
[\phi_1,\phi_0] = \sqrt{3}\phi_1 \ , \ [\phi_1,\phi_0'] = \phi_1, \\
[\phi_2,\phi_0] = -\sqrt{3}\phi_2 \ , \ [\phi_2,\phi_0'] = \phi_2, \\
[\phi_3,\phi_0] = 0 \ , \ [\phi_3,\phi_0'] = -2\phi_3.
\end{aligned} \tag{3.9}$$

The solutions of the constraints (3.9) in terms of the genuine Higgs fields and of the E_8 generators (3.8) corresponding to the embedding (3.2) of $R = U(1)\times U(1)$ in the E_8 are, $\phi_0 = \Lambda Q_0$ and $\phi_0' = \Lambda Q_0'$,with $\Lambda = \Lambda' = \frac{1}{\sqrt{10}}$, and

$$\begin{aligned}
\phi_1 &= R_1\alpha^i Q_{1i} + R_1\alpha Q_1, \\
\phi_2 &= R_2\beta^i Q_{2i} + R_2\beta Q_2, \\
\phi_3 &= R_3\gamma^i Q_{3i} + R_3\gamma Q_3,
\end{aligned} \tag{3.10}$$

where the unconstrained scalar fields transform under $E_6\times U(1)_A\times U(1)_B$ as

$$\begin{aligned}
\alpha_i \sim 27_{(3,\frac{1}{2})} \ , \ \alpha \sim 1_{(3,\frac{1}{2})}, \\
\beta_i \sim 27_{(-3,\frac{1}{2})} \ , \ \beta \sim 1_{(-3,\frac{1}{2})}, \\
\gamma_i \sim 27_{(0,-1)} \ , \ \gamma \sim 1_{(0,-1)}.
\end{aligned} \tag{3.11}$$

The potential (3.6) becomes

$$\begin{aligned}
V(\alpha^i,\alpha,\beta^i,\beta,\gamma^i,\gamma) = const. &+ \left(\frac{4R_1^2}{R_2^2R_3^2} - \frac{8}{R_1^2}\right)\alpha^i\alpha_i + \left(\frac{4R_1^2}{R_2^2R_3^2} - \frac{8}{R_1^2}\right)\overline{\alpha}\alpha \\
&+ \left(\frac{4R_2^2}{R_1^2R_3^2} - \frac{8}{R_2^2}\right)\beta^i\beta_i + \left(\frac{4R_2^2}{R_1^2R_3^2} - \frac{8}{R_2^2}\right)\overline{\beta}\beta \\
&+ \left(\frac{4R_3^2}{R_1^2R_2^2} - \frac{8}{R_3^2}\right)\gamma^i\gamma_i + \left(\frac{4R_3^2}{R_1^2R_2^2} - \frac{8}{R_3^2}\right)\overline{\gamma}\gamma \\
&+ \Bigg[\sqrt{280}\left(\frac{R_1}{R_2R_3} + \frac{R_2}{R_1R_3} + \frac{R_3}{R_2R_1}\right)d_{ijk}\alpha^i\beta^j\gamma^k \\
&+ \sqrt{280}\left(\frac{R_1}{R_2R_3} + \frac{R_2}{R_1R_3} + \frac{R_3}{R_2R_1}\right)\alpha\beta\gamma + h.c\Bigg] \\
&+ \frac{1}{6}\left(\alpha^i(G^\alpha)_i^j\alpha_j + \beta^i(G^\alpha)_i^j\beta_j + \gamma^i(G^\alpha)_i^j\gamma_j\right)^2 \\
&+ \frac{10}{6}\left(\alpha^i(3\delta_i^j)\alpha_j + \overline{\alpha}(3)\alpha + \beta^i(-3\delta_i^j)\beta_j + \overline{\beta}(-3)\beta\right)^2 \\
&+ \frac{40}{6}\left(\alpha^i(\frac{1}{2}\delta_i^j)\alpha_j + \overline{\alpha}(\frac{1}{2})\alpha + \beta^i(\frac{1}{2}\delta_i^j)\beta_j + \overline{\beta}(\frac{1}{2})\beta + \gamma^i(-1\delta_i^j)\gamma_j + \overline{\gamma}(-1)\gamma\right)^2 \\
&+ 40\alpha^i\beta^j d_{ijk}d^{klm}\alpha_l\beta_m + 40\beta^i\gamma^j d_{ijk}d^{klm}\beta_l\gamma_m + 40\alpha^i\gamma^j d_{ijk}d^{klm}\alpha_l\gamma_m \\
&+ 40(\overline{\alpha}\overline{\beta})(\alpha\beta) + 40(\overline{\beta}\overline{\gamma})(\beta\gamma) + 40(\overline{\gamma}\overline{\alpha})(\gamma\alpha).
\end{aligned} \tag{3.12}$$

From the potential (3.12) we read the F-, D- and scalar soft terms. The F-terms are obtained from the superpotential

$$\mathscr{W}(A^i,B^j,C^k,A,B,C) = \sqrt{40}d_{ijk}A^iB^jC^k + \sqrt{40}ABC. \tag{3.13}$$

The D-terms have the structure

$$\frac{1}{2}D^{\alpha}D^{\alpha}+\frac{1}{2}D_1D_1+\frac{1}{2}D_2D_2, \tag{3.14}$$

where

$$D^{\alpha}=\frac{1}{\sqrt{3}}\Big(\alpha^i(G^{\alpha})_i^j\alpha_j+\beta^i(G^{\alpha})_i^j\beta_j+\gamma^i(G^{\alpha})_i^j\gamma_j\Big),$$

$$D_1=\sqrt{\frac{10}{3}}\Big(\alpha^i(3\delta_i^j)\alpha_j+\overline{\alpha}(3)\alpha+\beta^i(-3\delta_i^j)\beta_j+\overline{\beta}(-3)\beta\Big)$$

and

$$D_2=\sqrt{\frac{40}{3}}\Big(\alpha^i(\frac{1}{2}\delta_i^j)\alpha_j+\overline{\alpha}(\frac{1}{2})\alpha+\beta^i(\frac{1}{2}\delta_i^j)\beta_j+\overline{\beta}(\frac{1}{2})\beta+\gamma^i(-1\delta_i^j)\gamma_j+\overline{\gamma}(-1)\gamma\Big),$$

which correspond to the $E_6\times U(1)_A\times U(1)_B$ structure of the gauge group. The rest terms are the trilinear and mass terms which break supersymmetry softly and they form the scalar SSB part of the Lagrangian,

$$\begin{aligned}\mathscr{L}_{scalarSSB}=&\left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\alpha^i\alpha_i+\left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\overline{\alpha}\alpha\\&+\left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\beta^i\beta_i+\left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\overline{\beta}\beta+\left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\gamma^i\gamma_i+\left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\overline{\gamma}\gamma\\&+\Big[\sqrt{2}80\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\right)d_{ijk}\alpha^i\beta^j\gamma^k\\&+\sqrt{2}80\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\right)\alpha\beta\gamma+h.c\Big].\end{aligned} \tag{3.15}$$

Note that the potential (3.12) belongs to the case analyzed in subsection 2.3 where S has an image in G. Here $S=SU(3)$ has an image in $G=E_8$ [39] so we conclude that the minimum of the potential is zero. Finally in order to determine the gaugino mass, we calculate the V operator of the eq. (2.36) in the case of $SU(3)/U(1)\times U(1)$ using Appendix C and using the Γ-matrices given in the Appendix A we calculate $\Sigma^{ab}=\frac{1}{4}[\Gamma^a,\Gamma^b]$ and then $G_{abc}\Gamma^a\Sigma^{bc}$. The combination of all leads to the gaugino mass

$$M=V=(1+3\tau)\frac{(R_1^2+R_2^2+R_3^2)}{8\sqrt{R_1^2R_2^2R_3^2}}. \tag{3.16}$$

Note again that the chosen embedding satisfies the condition (2.30) and the absence in the four-dimensional theory of any other term that does not belong to the supersymmetric $E_6\times U(1)_A\times U(1)_B$ gauge theory or to its SSB sector. The gaugino mass (3.16) has a contribution from the torsion of the coset space. A final remark concerning the gaugino mass is that the adjustment required to obtain the *canonical connection* leads also to vanishing gaugino masses. Contrary to the gaugino mass term the soft scalar terms of the SSB does not receive contributions from the torsion. This is due to the fact that gauge fields, contrary to fermions, do not couple to torsion.

Concluding the present section, we would like to note that the fact that, starting with a $\mathcal{N}=1$ supersymmetric theory in ten dimensions, the CSDR leads to the field content of an $\mathcal{N}=1$ supersymmetric theory in the case that the six-dimensional coset spaces used are non-symmetric, can been seen by inspecting the table 1. More specifically, one notices in table 1 that when the coset spaces are non-symmetric the decompositions of the spinor 4 and antispinor $\overline{4}$ of $SO(6)$ under R contain a singlet, i.e. have the form $1+r$ and $1+\bar{r}$, respectively, where r is possibly reducible. The singlet under R provides the four-dimensional theory with fermions transforming according to the adjoint as was emphasized in subsection 2.3 and correspond to gauginos, which obtain geometrical and torsion mass contributions as we have seen in the present section 3. Next turning the decomposition of the vector 6 of $SO(6)$ under R in the non-symmetric cases, we recall that the vector can be constructed from the tensor product 4×4 and therefore has the form $r+\bar{r}$. Then the CSDR constraints tell us that the four-dimensional theory will contain the same representations of fermions and scalars since both come from the adjoint representation of the gauge group G and they have to satisfy the same matching conditions under R. Therefore the field content of the four-dimensional theory is, as expected, $\mathscr{N}=1$ supersymmetric. To find out that furthermore the $\mathscr{N}=1$ supersymmetry is softly broken, requires the lengthy and detailed analysis that was done above.

4. Wilson flux breaking

Clearly, we need to further reduce the gauge symmetry. We will employ the Wilson flux breaking mechanism [51, 52, 53]. Let us briefly recall the Wilson flux mechanism for breaking spontaneously a gauge theory.

4.1 The Wilson flux mechanism

Instead of considering a gauge theory on $M^4\times B_0$, with B_0 a simply connected manifold, and in our case a coset space $B_0=S/R$, we consider a gauge theory on $M^4\times B$, with $B=B_0/F^{S/R}$ and $F^{S/R}$ a freely acting discrete symmetry of B_0. It turns out that B becomes multiply connected, which means that there will be contours not contractible to a point due to holes in the manifold. For each element $g\in F^{S/R}$, we pick up an element U_g in H, i.e. in the four-dimensional gauge group of the reduced theory, which can be represented as the Wilson loop (WL)

$$U_g=\mathscr{P}exp\left(-i\int_{\gamma_g}T^aA^a_M(x)dx^M\right), \tag{4.1}$$

where $A^a_M(x)$ are vacuum H fields with group generators T^a, γ_g is a contour representing the abstract element g of $F^{S/R}$, and $\mathscr{P}$ denotes the path ordering. Now if γ_g is chosen not to be contractible to a point, then $U_g\neq 1$ although the vacuum field strength vanishes everywhere. In this way an homomorphism of $F^{S/R}$ into H is induced with image T^H, which is the subgroup of H generated by $\{U_g\}$. A field $f(x)$ on B_0 is obviously equivalent to another field on B_0 which obeys $f(g(x))=f(x)$ for every $g\in F^{S/R}$. However in the presence of the gauge group H this statement can be generalized to

$$f(g(x))=U_gf(x)\,. \tag{4.2}$$

The discrete symmetries $F^{S/R}$, which act freely on coset spaces $B_0=S/R$ are the center of S, $\mathrm{Z}(S)$ and $\mathrm{W}=\mathrm{W}_S/\mathrm{W}_R$, where W_S and W_R are the Weyl groups of S and R, respectively. The case of our interest here is

$$F^{S/R}=\mathbb{Z}_3\subseteq\mathrm{W}\,. \tag{4.3}$$

4.2 $SU(3)^3$ due to Wilson flux

In order to derive the projected theory in the presence of the WL, one has to keep the fields which are invariant under the combined action of the discrete group $\mathbb{Z}_3$ on the geometry and on the gauge indices. The discrete symmetry acts non-trivially on the gauge fields and on the matter in the 27 and the singlets. The action on the gauge indices is implemented via the matrix [40] $\mathrm{diag}(\mathbf{1}_9, \omega\mathbf{1}_9, \omega^2\mathbf{1}_9)$ with $\omega = e^{2i\pi/3}$. Thus, the gauge fields that survive the projection are those that satisfy

$$A_\mu = \gamma_3 A_\mu \gamma_3^{-1}, \tag{4.4}$$

while the surviving components of the matter fields in the 27's are those that satisfy

$$\vec{\alpha} = \omega\gamma_3\vec{\alpha}, \quad \vec{\beta} = \omega^2\gamma_3\vec{\beta}, \quad \vec{\gamma} = \omega^3\gamma_3\vec{\gamma}. \tag{4.5}$$

Finally, the projection on the complex scalar singlets is

$$\alpha = \omega\alpha, \quad \beta = \omega^2\beta, \quad \gamma = \omega^3\gamma. \tag{4.6}$$

It is easy to see then that after the $\mathbb{Z}_3$ projection the gauge group reduces to

$$A_\mu^A, \qquad A \in SU(3)_c \times SU(3)_L \times SU(3)_R \tag{4.7}$$

(the first of the $SU(3)$ factors is the SM colour group) and the scalar matter fields are in the bi-fundamental representations

$$\alpha_3 \sim H_1 \sim (\bar{3},1,3)_{(3,1/2)}, \quad \beta_2 \sim H_2 \sim (3,\bar{3},1)_{(-3,1/2)}, \quad \gamma_1 \sim H_3 \sim (1,3,\bar{3})_{(0,-1)}. \tag{4.8}$$

There are also fermions in similar representations. Note that with three families this is a finite theory [45, 42]. Clearly, the Higgs is identified with the 9-component vector $H_{3a}, a = 1,\cdots,9$. Among the singlets, only $\gamma_{(0,-1)}$ survives. In the following we will be using indices $a,b,c\cdots$ to count the complex components of a given bi-fundamental representation and $i,j,k,\cdots = 1,2,3$ the different bifundamental representations.

Before we write the explicit scalar potential, we take appropriate actions such that there are 3 identical flavours from each of the bifundamental fields. This can, in general, be achieved by introducing non-trivial windings in R. We denote the resulting three copies of the bifundamental fields as (we will be using the index $l = 1,2,3$ to specify the flavours)

$$\begin{aligned}
3\cdot H_1 &\longrightarrow {H_1}^{(l)} \sim 3\cdot(\bar{3},1,3)_{(3,1/2)}\\
3\cdot H_2 &\longrightarrow {H_2}^{(l)} \sim 3\cdot(3,\bar{3},1)_{(-3,1/2)}\\
3\cdot H_3 &\longrightarrow {H_3}^{(l)} \sim 3\cdot(1,3,\bar{3})_{(0,-1)}\,.
\end{aligned} \tag{4.9}$$

Similarly we denote the three copies of the scalar as

$$3\cdot\gamma_{(0,-1)} \longrightarrow \theta^{(l)}_{(0,-1)}\,. \tag{4.10}$$

The scalar potential gets accordingly three copies of each contribution.

In the following when it does not cause confusion we denote a chiral superfield and its scalar component with the same letter. Also, it is clear that the potential after the projection will have the same form as before the projection with the only difference that only $\theta^{(l)}$ is non-vanishing among the singlets and that the sums over components now run only over the even under the projection components.

We can now rewrite the scalar potential as [21]

$$V_{\rm sc} = 3(3\Lambda^2 + \Lambda'^2)\left(\frac{1}{R_1^4} + \frac{1}{R_2^4}\right) + \frac{3\cdot 4\Lambda'^2}{R_3^4} + \sum_{l=1,2,3} V^{(l)} \tag{4.11}$$

where

$$V^{(l)} = V_{\rm susy} + V_{\rm soft} \tag{4.12}$$

with $V_{\rm susy} = V_D + V_F$. Since there are three identical contributions to the potential, at least until we give vevs to the Higgses (which in general can be different for each l) we can drop the flavour superscript (l) from most of the fields. Then, the explicit form of the D and F terms are

$$\begin{aligned} V_D &= \frac{1}{2}\sum_A D^A D^A + \frac{1}{2} D_1 D_1 + \frac{1}{2} D_2 D_2 \\ V_F &= \sum_{i=1,2,3} |F_{H_i}|^2 + |F_\theta|^2\,, \qquad F_{H_i} = \frac{\partial \mathscr{W}}{\partial H_i}, \qquad F_\theta = \frac{\partial \mathscr{W}}{\partial \theta}\,. \end{aligned} \tag{4.13}$$

The F-terms derive from

$$\mathscr{W} = \sqrt{40}\, d_{abc} H_1^a H_2^b H_3^c \tag{4.14}$$

and the D-terms are

$$\begin{aligned} D^A &= \frac{1}{\sqrt{3}} \langle H_i | G^A | H_i \rangle \\ D_1 &= 3\sqrt{\frac{10}{3}} \left(\langle H_1|H_1\rangle - \langle H_2|H_2\rangle\right) \\ D_2 &= \sqrt{\frac{10}{3}} \left(\langle H_1|H_1\rangle + \langle H_2|H_2\rangle - 2\langle H_3|H_3\rangle - 2|\theta|^2\right)\,, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \langle H_i | G^A | H_i \rangle &= \sum_{i=1,2,3} H_i^a (G^A)_a^b H_{ib} \\ \langle H_i | H_i \rangle &= \sum_{i=1,2,3} H_i^a \delta_a^b H_{ib}\,. \end{aligned} \tag{4.16}$$

Finally the soft breaking terms are

$$\begin{aligned} V_{\rm soft} &= \left(\frac{4R_1^2}{R_2^2 R_3^2} - \frac{8}{R_1^2}\right)\langle H_1|H_1\rangle + \left(\frac{4R_2^2}{R_1^2 R_3^2} - \frac{8}{R_2^2}\right)\langle H_2|H_2\rangle \\ &+ \left(\frac{4R_3^2}{R_1^2 R_2^2} - \frac{8}{R_3^2}\right)\left(\langle H_3|H_3\rangle + |\theta|^2\right) \\ &+ 80\sqrt{2}\left(\frac{R_1}{R_2 R_3} + \frac{R_2}{R_1 R_3} + \frac{R_3}{R_1 R_2}\right)\left(d_{abc} H_1^a H_2^b H_3^c + \text{h.c.}\right). \end{aligned} \tag{4.17}$$

The $(G^A)_a^b$ are structure constants, thus antisymmetric in a and b. The vector $|\phi\rangle$ and its hermitian conjugate $\langle\phi|$ represent the 9-dimensional bi-fundamental fields shown above.

The potential can be written in a more convenient form, as suggested in [41]. It amounts to writing the vectors in complex 3×3 matrix notation. The various terms in the scalar potential can be then interpreted as invariant Lie algebra polynomials. We identify

$$H_1 \sim (\bar{3},1,3) \longrightarrow (q^c)^{\alpha}_{p} \quad H_2 \sim (3,\bar{3},1) \longrightarrow Q^{a}_{\alpha} \quad H_3 \sim (1,3,\bar{3}) \longrightarrow L^{p}_{a} \tag{4.18}$$

where

$$q^c = \begin{pmatrix} d^1_R & u^1_R & D^1_R \\ d^2_R & u^2_R & D^2_R \\ d^3_R & u^3_R & D^3_R \end{pmatrix}, \; Q = \begin{pmatrix} d^1_L & d^2_L & d^3_L \\ u^1_L & u^2_L & u^3_L \\ D^1_L & D^2_L & D^3_L \end{pmatrix}, \; L = \begin{pmatrix} H^0_d & H^+_u & \nu_L \\ H^-_d & H^0_u & e_L \\ \nu_R & e_R & S \end{pmatrix}. \tag{4.19}$$

Evidently $d_{L,R}, u_{L,R}, D_{L,R}$ tranforms as $3,\bar{3}$ under colour. Then we introduce

$$\hat{q^c}^{p}_{\alpha} = \frac{1}{3}\frac{\partial I_3}{\partial {q^c}^{\alpha}_{p}} \qquad \hat{L}^{a}_{p} = \frac{1}{3}\frac{\partial I_3}{\partial L^{p}_{a}} \qquad \hat{Q}^{\alpha}_{a} = \frac{1}{3}\frac{\partial I_3}{\partial Q^{a}_{\alpha}}, \tag{4.20}$$

where

$$I_3 = \det[\mathrm{Q}] + \det[\mathrm{q^c}] + \det[\mathrm{L}] - \mathrm{tr}(q^c\cdot L\cdot Q)\,. \tag{4.21}$$

In terms of these matrices, we have $\langle H_1|H_1\rangle = \mathrm{tr}({q^c}^{\dagger} q^c)$, $\langle H_2|H_2\rangle = \mathrm{tr}(Q^{\dagger}Q)$, $\langle H_3|H_3\rangle = \mathrm{tr}(L^{\dagger}L)$ and

$$d_{abc}H_1^a H_2^b H_3^c = \mathrm{det{q^c}^{\dagger}} + \mathrm{detM^{\dagger}} + \mathrm{detL^{\dagger}} - \mathrm{tr}(N^{\dagger}M^{\dagger}L^{\dagger})\,. \tag{4.22}$$

The F-terms which explicitly read

$$V_F \;=\; 40 d_{abc}d^{cde}({H_1}^a {H_2}^b H_{1d}H_{2e} + {H_2}^a {H_3}^b H_{2d}H_{3e} + {H_1}^a {H_3}^b H_{1d}H_{3e}). \tag{4.23}$$

can be now written as

$$V_F = 40\mathrm{tr}(\hat{q^c}^{\dagger}\hat{q^c} + \hat{Q}^{\dagger}\hat{Q} + \hat{L}^{\dagger}\hat{L}). \tag{4.24}$$

5. Gauge symmetry breaking

Consider the following vevs:

$$L^{(1)}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V \end{pmatrix}, \qquad L^{(2)}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V & 0 & 0 \end{pmatrix} \tag{5.1}$$

for $H_3^{(1)}$ and $H_3^{(2)}$ respectively. These vevs leave the $SU(3)_c$ part of the gauge group unbroken but trigger the spontaneous breaking of the rest. More precisely, $L^{(1)}_0$ breaks the gauge group according to

$$SU(3)_c\times SU(3)_L\times SU(3)_R\times U(1)_A\times U(1)_B \longrightarrow SU(3)_c\times SU(2)_L\times SU(2)_R\times U(1) \tag{5.2}$$

while $L^{(2)}_0$ according to

$$SU(3)_c\times SU(3)_L\times SU(3)_R\times U(1)_A\times U(1)_B \longrightarrow SU(3)_c\times SU(2)_L\times SU(2)'_R\times U(1)'\,. \tag{5.3}$$

The combination of the two gives [44]

$$SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \longrightarrow SU(3)_c \times SU(2)_L \times U(1)_Y\,. \tag{5.4}$$

Electroweak (EW) breaking then proceeds by a second vev v, for example by [45] $L_0^{(1)} = \mathrm{diag}(v,v,V)$. We first look at $V^{(1)}$ in the presence of the vevs. Using the fact that the coefficients $(G^A)_a^{\,b}$ are antisymmetric in a and b, it is easy to see that for these vevs, the quadratic form $\langle\phi|G^A|\phi\rangle$ vanishes identically in the vacuum, and so do the corresponding $SU(3)$ D-terms D^A. The other terms give in the vacuum

$$\begin{aligned}
V_{D_1} &= 15(V^2+2v^2)^2 \\
V_{D_2} &= \frac{5}{9}(V^2+2v^2-\theta_0^2)^2 \\
V_F &= \frac{40}{9}v^2(2V^2+v^2) \\
V_{\mathrm{soft}} &= \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)(V^2+2v^2)+\left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)(\theta_0^{(1)})^2 \\
&+\ 160\sqrt{3}\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_1R_2}\right)Vv^2\,.
\end{aligned} \tag{5.5}$$

As expected, already in the vacuum where EW symmetry is unbroken, supersymmetry is broken by both D and F-terms, in addition to its breaking by the soft terms. The potential is positive definite so we are looking for a vacuum solution with $V_0^{(1)} = 0$. For simplicity we choose $R_1 = R_2 = R_3 = R$ (strictly speaking in this case the manifold becomes nearly-Kähler). Then, if the vevs satisfy the relation

$$\begin{aligned}
(\theta_0^{(1)})^2 &= \frac{1}{10R^2}\Big[5R^2V^2+10R^2v^2+9 \\
&+\ \big(-675V^4R^4-3100V^2v^2R^4+270V^2R^2-2900v^4R^4 \\
&+\ 540v^2R^2+27-21600\sqrt{3}Vv^2R^3\big)^{1/2}\Big]\,,
\end{aligned} \tag{5.6}$$

the potential is zero at the minimum. We stress that in contrast to exactly supersymmetric theories, the zero of the potential at the minimum does not imply unbroken supersymmetry. This is because the potential is a perfect square (which is a consequence of its higher dimensional origin from $F_{MN}F^{MN}$) with the soft breaking terms included.

It is interesting to notice that generically the solution makes sense when $V < 1/R$ and then if we set, with no loss of generality $V = 1$, we find that the quantity under the square root is positive if $v \sim \mathcal{O}(0.1)$ for $R \sim \mathcal{O}(1/2)$. It is interesting that the desired hierarchy of scales is naturally generated by the structure of the scalar potential. Therefore a consistent picture emerges assuming that the compactification and the supersymmetry breaking scales are both in the few TeV regime [43].

The analysis of $V^{(2)}$ in the presence of the second of the vevs in eq. (5.1) is similar. The potential is zero at the minimum if the vev of $\theta^{(2)}$ satisfies

$$(\theta_0^{(2)})^2 = \frac{1}{10R^2}\left(5V^2R^2+9+3\sqrt{-75V^4R^4+30V^2R^2+3}\right). \tag{5.7}$$

The vevs $\theta_0^{(1)}$ and $\theta_0^{(2)}$ need not be equal and $\theta_0^{(3)}$ can, but does not need to be zero.

5.1 $U(1)$ structure and Yukawa couplings

The breaking pattern of the bifundamental representations that V induces is

$$(\overline{3},1,3)_{(3,1/2)} \longrightarrow (\overline{3},1,1+1+1)_{(3,1/2)} \tag{5.8}$$
$$(3,\overline{3},1)_{(-3,1/2)} \longrightarrow (3,2+1,1)_{(-3,1/2)} \tag{5.9}$$
$$(1,3,\overline{3})_{(0,-1)} \longrightarrow (1,2+1,1+1+1)_{(0,-1)} \tag{5.10}$$

from which we can read off the representations under the SM gauge group and the extra $U(1)$'s. From (5.8) we obtain $\overline{\mathbf{u}}$, $\overline{\mathbf{d}}$ and $\overline{\mathbf{D}}$, that is the two right handed quarks and an extra quark type state. From (5.9) we obtain the quark doublet $\mathbf{Q}$ and the vector-like partner $\mathbf{D}$ of the extra quark. Notice however that the extra quarks are not completely vector-like, since they have the same $U(1)_B$ charge. From (5.10) we obtain the lepton doublet L, the right handed lepton singlet $\overline{e}$, two right handed neutrinos and two electroweak doublets. We will denote the latter doublets as H_u and H_d like in the minimal supersymmetric SM (MSSM). Notice that the scalar components of these doublets are the components of the H_3 Higgs field that takes the vev v. We will denote the former singlets as $\overline{N}_{1,2}$ while the singlet chiral superfields whose lowest component are the $\theta^{(l)}$ we will call $\Theta^{(l)}$. In the table we summarize the states contained in one family, with their $U(1)$ charges. We have separated the MSSM spectrum from new states by a double line.

$SU(3)_c \times SU(2)_L$	$U(1)_Y$	$U(1)_A$	$U(1)_B$
$\mathbf{Q} \sim (3,2)$	$1/6$	-3	$1/2$
$\overline{\mathbf{u}} \sim (\overline{3},1)$	$-2/3$	3	$1/2$
$\overline{\mathbf{d}} \sim (\overline{3},1)$	$1/3$	3	$1/2$
$L \sim (1,2)$	$-1/2$	0	-1
$\overline{e} \sim (1,1)$	1	0	-1
$H_u \sim (1,2)$	$1/2$	0	-1
$H_d \sim (1,2)$	$-1/2$	0	-1
$\mathbf{D} \sim (3,1)$	$-1/3$	-3	$1/2$
$\overline{\mathbf{D}} \sim (\overline{3},1)$	$1/3$	3	$1/2$
$\overline{N}_1 \sim (1,1)$	0	0	-1
$\overline{N}_2 \sim (1,1)$	0	0	-1
$\Theta^{(1)} \sim (1,1)$	0	0	-1

We immediately recognize that

$$U(1)_A = -9B\,, \tag{5.11}$$

where B is baryon number and $U(1)_B$ as a Peccei-Quinn type of symmetry. Lepton number on the other hand does not appear to be a conserved symmetry (e.g. $L\mathbf{Q}\overline{\mathbf{d}}$ is allowed). The presence of a conserved global baryon number is clearly a welcome feature from the point of view of the stability of the proton. The two extra $U(1)$'s at this stage, are both anomalous and at least one of them will remain anomalous after charge redefinitions. They both break by the vev V, however their respective global subgroups remain at low energies and constrain the allowed (non-renormalizable) operators in the superpotential. Gauge invariance in the presence of the anomalous symmetries can be maintained by the addition of a specific combination of terms to the low energy effective Lagrangian, including a Stückelberg coupling and an axion-like interaction. These interactions introduce a new, phenomenologically interesting sector in the effective action [46]. Let A_μ be the anomalous $U(1)$ gauge field and F_A its field strength. Then the terms that render the action gauge invariant are

$$\mathscr{L}_{\rm St-WZ} = \frac{1}{2}(\partial_\mu a + MA_\mu)^2 + c\frac{a}{M}F_A \wedge F_A + \mathscr{L}_{\rm an}\,. \tag{5.12}$$

The axion a shifts under the anomalous symmetry so that the kinetic term is invariant and coefficient c is such that the Wess-Zumino term cancels the 1-loop anomaly $\mathscr{L}_{\rm an}$. The scale M is related to the vev V. These couplings are added by hand because they are not part of the the interactions of the original ten-dimensional gauge multiplet, neither can be generated by its dimensional reduction. In fact, the axion field is the four-dimensional remnant of the two-form B_{MN}. This is the (minimum) price to pay for neglecting the gravitational and two-form sectors (and actually also the second E_8 factor) along with the ten-dimensional anomaly cancellation mechanism, for which their presence is essential [47].

A few comments regarding the Yukawa sector are in order. Every operator originating from the superpotential $d_{abc}H_1^a H_2^b H_3^c$ will appear at tree level. At the quantum level operators that break the CSDR constraints and the supersymmetric structure will eventually develop, as long as they are gauge invariant. As an example of the former case, the extra vector-like pair of quarks will develop a mass term in the V-vacuum

$$\Theta^{(1)}\overline{\mathbf{D}}\mathbf{D} \tag{5.13}$$

which is a singlet. As an example of the latter, notice that in the quark sector the standard Yukawa terms in the superpotential appear at tree level. In the lepton sector however the term $L\overline{e}H_d$ is not invariant under $U(1)_B$. An effective Yukawa coupling can come though from the higher-dimensional operator

$$L\overline{e}H_d\left(\frac{\theta^{(1)*}}{M}\right)^3 \tag{5.14}$$

in the V-vacuum, with M a high scale such as the string scale and $\theta^{(1)*}$ the complex conjugate of $\theta^{(1)}$. Similar arguments apply to the entire lepton sector: effective Yukawa couplings appear via higher dimensional operators

$$LH_u\overline{N}\left(\frac{\theta^{(1)*}}{M}\right)^3 \qquad M\overline{N}\overline{N}\left(\frac{\theta^{(1)*}}{M}\right)^2 . \tag{5.15}$$

Similar terms are generated for the second and third families. Evidently, after electroweak symmetry breaking, fermion mass hierarchies and mixings can be generated [48] not because the $U(1)$'s have flavour dependent charges, but from the different values that the vevs $\theta^{(l)}$ can have. A term that mixes flavours is, for example,

$$L^{(1)}\overline{e}^{(2)}H_d^{(2)}\left(\frac{\theta^{(1)*}}{M}\right)\left(\frac{\theta^{(2)*}}{M}\right)^2 , \tag{5.16}$$

where we have made the flavour superscripts explicit on all fields.

Clearly the general flavour problem is a crucial question to address in this model. The symmetries of the coset space are very constraining on the four-dimensional effective action and so will be the quark and lepton mass hierarchies. It will be therefore interesting to see if the observed pattern in the mass hierarchies and mixings is possible to be accommodated. In a future work, we plan to investigate this important issue.

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Appendices

A. Reducing the 10-dimensional 32-spinor to 8-spinor by Majorana-Weyl conditions

The case we are going to examine is $\mathscr{N}=1$ SYM theory in D=10. In particular we would like to demonstrate how the Dirac spinor with $2^{[D/2]}=32$ components is reduced to a Weyl-Majorana spinor with 8 components in order to have the same degrees of freedom as the gauge fields. We choose the following representation of the Γ-matrices

$$\Gamma^{\mu}=\gamma^{\mu}\otimes I_8\ ,\ \mu=0,1,2,3\ . \tag{A.1}$$

The Dirac spinor can be written as

$$\psi=(\psi_1\ \dots\ \psi_4\ \chi_1\ \dots\chi_4\ )^T\ , \tag{A.2}$$

where all $\psi_i,\ \chi_i,\ i=1,\dots,4$ tranform as $SO(1,3)$ Dirac spinors. Let us present the rest Γ-matrices

$$\begin{aligned}\Gamma^4=\gamma^5\otimes\sigma^1\otimes\sigma^2\otimes\sigma^2\ ,\ \Gamma^5=\gamma^5\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2,\\ \Gamma^6=\gamma^5\otimes I_2\otimes\sigma^3\otimes\sigma^2\ ,\ \Gamma^7=\gamma^5\otimes I_4\otimes\sigma^1,\\ \Gamma^8=\gamma^5\otimes\sigma^3\otimes\sigma^2\otimes\sigma^2\ ,\ \Gamma^9=\gamma^5\otimes I_2\otimes\sigma^1\otimes\sigma^2\end{aligned} \tag{A.3}$$

and hence

$$\Gamma^{11}=\Gamma^0\dots\Gamma^9=-\gamma^5\otimes I_2\otimes I_2\otimes\sigma^3=\gamma^5\otimes\begin{pmatrix}-I_4 & 0\\ 0 & I_4\end{pmatrix} \tag{A.4}$$

The spinor ψ is reducible, $\Gamma^{11}\psi_{\pm}=\pm\psi_{\pm}$, where $\psi_{\pm}=\frac{1}{2}\left(1\pm\Gamma^{11}\right)\psi$. Then the Weyl condition,

$$\Gamma^{11}\psi=\psi \tag{A.5}$$

selects the ψ_+ , where

$$\psi_+=(L\psi_1\ \dots\ L\psi_4\ R\chi_1\ \dots R\chi_4\ )^T \tag{A.6}$$

where $L=\frac{1}{2}\left(1-\gamma^5\right)$ (left handed) and $R=\frac{1}{2}\left(1+\gamma^5\right)$ (right handed). The ψ_i form the 4 and the χ_i the $\bar{4}$ representations of $SO(6)$. Imposing further the Majorana condition on the 10-dimensional spinor,

$$\psi=C_{10}\Gamma^0\psi^* \tag{A.7}$$

where $C_{10}=C_4\otimes\sigma_2\otimes\sigma_2\otimes I_2$ we are led to the relations $\chi_{1,3}=C\gamma_0\psi^*_{2,4}$ and $\chi_{2,4}=-C\gamma_0\psi^*_{1,3}$. Therefore by imposing both Weyl and Majorana conditions in 10 dimensions we obtain a Weyl spinor in 4 dimensions transforming as 4 of $SO(6)$ i.e.

$$\psi=(L\psi_1\ L\psi_2\ L\psi_3\ L\psi_4\ R\tilde{\psi}_2\ R\tilde{\psi}_1\ R\tilde{\psi}_4\ R\tilde{\psi}_3\ )^T\ ,\ \tilde{\psi}_i=(-1)^iC\gamma_0\psi_i^*\ . \tag{A.8}$$

In addition we need the gamma matrices in the coset space $SU(3)/U(1)\times U(1)$. The metric is given $g_{ab}=diag(R_1^2,R_1^2,R_2^2,R_2^2,R_3^2,R_3^2)$, $g^{ab}=diag(\frac{1}{R_1^2},\frac{1}{R_1^2},\frac{1}{R_2^2},\frac{1}{R_2^2},\frac{1}{R_3^2},\frac{1}{R_3^2})$ and hence the Γ-matrices are given by

$$\begin{aligned}\Gamma^4=\frac{1}{R_1}\gamma^5\otimes\sigma^1\otimes\sigma^2\otimes\sigma^2\ ,\ \Gamma^5=\frac{1}{R_1}\gamma^5\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2,\\ \Gamma^6=\frac{1}{R_2}\gamma^5\otimes I_2\otimes\sigma^3\otimes\sigma^2\ ,\ \Gamma^7=\frac{1}{R_2}\gamma^5\otimes I_4\otimes\sigma^1,\\ \Gamma^8=\frac{1}{R_3}\gamma^5\otimes\sigma^3\otimes\sigma^2\otimes\sigma^2\ ,\ \Gamma^9=\frac{1}{R_3}\gamma^5\otimes I_2\otimes\sigma^1\otimes\sigma^2\end{aligned} \tag{A.9}$$

B. Commutation Relations

Table 1:

Six-dimensional non-symmetric cosets with $rankS = rankR$		
S/R	$SO(6)$ vector	$SO(6)$ spinor
$G_2/SU(3)$	$3+\overline{3}$	$1+3$
$Sp(4)/(SU(2)\times U(1))_{non-max}$	$1_2+1_{-2}+2_1+2_{-1}$	$1_0+1_2+2_{-1}$
$SU(3)/U(1)\times U(1)$	$(a,c)+(b,d)+(a+b,c+d)$ $+(-a,-c)+(-b,-d)$ $+(-a-b,-c-d)$	$(0,0)+(a,c)+(b,d)$ $+(-a-b,-c-d)$

Table 2:

Non-trivial commutation relations of $SU(3)$ according to the decomposition given in (3.4)		
$[Q_1,Q_0]=\sqrt{3}Q_1$	$[Q_1,Q'_0]=Q_1$	$[Q_2,Q_0]=-\sqrt{3}Q_2$
$[Q_2,Q'_0]=Q_2$	$[Q_3,Q_0]=0$	$[Q_3,Q'_0]=-2Q_3$
$\left[Q_1,Q^1\right]=-\sqrt{3}Q_0-Q'_0$	$\left[Q_2,Q^2\right]=\sqrt{3}Q_0-Q'_0$	$\left[Q_3,Q^3\right]=2Q'_0$
$[Q_1,Q_2]=\sqrt{2}Q^3$	$[Q_2,Q_3]=\sqrt{2}Q^1$	$[Q_3,Q_1]=\sqrt{2}Q^2$

The normalization in the above table is

$$Tr(Q_0Q_0)=Tr(Q'_0Q'_0)=Tr(Q_1Q^1)=Tr(Q_2Q^2)=Tr(Q_3Q^3)=2$$

Table 3:

Non-trivial commutation relations of E_8 according to the decomposition given by eq.(3.8)		
$[Q_1,Q_0]=\sqrt{30}Q_1$	$[Q_1,Q'_0]=\sqrt{10}Q_1$	$[Q_2,Q_0]=-\sqrt{30}Q_2$
$[Q_2,Q'_0]=\sqrt{10}Q_2$	$[Q_3,Q_0]=0$	$[Q_3,Q'_0]=-2\sqrt{10}Q_3$
$\left[Q_1,Q^1\right]=-\sqrt{30}Q_0-\sqrt{10}Q'_0$	$\left[Q_2,Q^2\right]=\sqrt{30}Q_0-\sqrt{10}Q'_0$	$\left[Q_3,Q^3\right]=2\sqrt{10}Q'_0$
$[Q_1,Q_2]=\sqrt{20}Q^3$	$[Q_2,Q_3]=\sqrt{20}Q^1$	$[Q_3,Q_1]=\sqrt{20}Q^2$
$[Q_{1i},Q_0]=\sqrt{30}Q_{1i}$	$[Q_{1i},Q'_0]=\sqrt{10}Q_{1i}$	$[Q_{2i},Q_0]=-\sqrt{30}Q_{2i}$
$[Q_{2i},Q'_0]=\sqrt{10}Q_{2i}$	$[Q_{3i},Q_0]=0$	$[Q_{3i},Q'_0]=-2\sqrt{10}Q_{3i}$
$[Q_{1i},Q_{2j}]=\sqrt{20}d_{ijk}Q^{3k}$	$[Q_{2i},Q_{3j}]=\sqrt{20}d_{ijk}Q^{1k}$	$[Q_{3i},Q_{1j}]=\sqrt{20}d_{ijk}Q^{2k}$
$\left[Q^\alpha,Q^\beta\right]=2ig^{\alpha\beta\gamma}Q^\gamma$	$[Q^\alpha,Q_0]=0$	$[Q^\alpha,Q'_0]=0$
$[Q^\alpha,Q_{1i}]=-(G^\alpha)_i^jQ_{1j}$	$[Q^\alpha,Q_{2i}]=-(G^\alpha)_i^jQ_{2j}$	$[Q^\alpha,Q_{3i}]=-(G^\alpha)_i^jQ_{3j}$
$\left[Q_{1i},Q^{1j}\right]=-\frac{1}{6}(G^\alpha)_i^jQ^\alpha-\sqrt{30}\delta_i^jQ_0-\sqrt{10}\delta_i^jQ'_0$		
$\left[Q_{2i},Q^{2j}\right]=-\frac{1}{6}(G^\alpha)_i^jQ^\alpha+\sqrt{30}\delta_i^jQ_0-\sqrt{10}\delta_i^jQ'_0$		
$\left[Q_{3i},Q^{3j}\right]=-\frac{1}{6}(G^\alpha)_i^jQ^\alpha+2\sqrt{10}\delta_i^jQ'_0$		

The normalization in the table 3 is

$$\begin{aligned} Tr(Q_0Q_0) &= Tr(Q'_0Q'_0)=Tr(Q_1Q^1)=Tr(Q_2Q^2)=Tr(Q_3Q^3)=2\,, \\ Tr(Q_{1i}Q^{1j}) &= Tr(Q_{2i}Q^{2j})=Tr(Q_{3i}Q^{3j})=2\delta_i^j,\ \ Tr(Q^\alpha Q^\beta)=12\delta^{\alpha\beta}. \end{aligned}$$

C. Useful relations to calculate the gaugino mass.

We use the real metric of the coset, $g_{ab} = diag(a,a,b,b,c,c)$ with $a = R_1^2, b = R_2^2, c = R_3^2$. Using the structure constants of $SU(3)$, ${f_{12}}^3 = 2$, ${f_{45}}^8 = {f_{67}}^8 = \sqrt{3}$, ${f_{24}}^6 = {f_{14}}^7 = {f_{25}}^7 = -{f_{36}}^7 = -{f_{15}}^6 = -{f_{34}}^5 = 1$, (where the indices 3 and 8 correspond to the $U(1) \times U(1)$ and the rest are the coset indices) we calculate the components of the D_{abc}:
$D_{523} = D_{613} = D_{624} = D_{541} = -D_{514} = -D_{532} = -D_{631} = -D_{624} = \frac{1}{2}(c-a-b)$.
$D_{235} = D_{136} = D_{624} = D_{154} = -D_{145} = -D_{253} = -D_{163} = -D_{264} = \frac{1}{2}(a-b-c)$.
$D_{352} = D_{361} = D_{462} = D_{415} = -D_{451} = -D_{325} = -D_{316} = -D_{426} = \frac{1}{2}(b-c-a)$.
From the D's we calculate the contorsion tensor

$$\Sigma_{abc} = 2\tau(D_{abc} + D_{bca} - D_{cba}),$$

and then the tensor

$$G_{abc} = D_{abc} + \frac{1}{2}\Sigma_{abc}$$

which is
$G_{523} = G_{613} = G_{642} = G_{541} = -G_{514} = -G_{532} = -G_{631} = -G_{642} = \frac{1}{2}[(1-\tau)c - (1+\tau)a - (1+\tau)b]$.
$G_{235} = G_{136} = G_{246} = G_{154} = -G_{145} = -G_{253} = -G_{163} = -G_{264} = \frac{1}{2}[-(1-\tau)a + (1+\tau)b + (1+\tau)c]$.
$G_{352} = G_{361} = G_{462} = G_{415} = -G_{451} = -G_{325} = -G_{316} = -G_{426} = \frac{1}{2}[-(1+\tau)a + (1-\tau)b - (1+\tau)c]$.

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