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# Asymptotic safety vs. classicalization in Goldstone boson dynamics

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Two classes of nonperturbative phenomena could occur in the nonlinear sigma models at high energy. The first is the existence of a nontrivial fixed point for the renormalization group flow, leading to nonperturbative renormalizability (a.k.a. asymptotic safety). The second is the dominance of classical configurations in high energy scattering, known as classicalization. Both could lead to unitarization of the scattering amplitudes. We discuss both possibilities, emphasizing similarities and differences. The spacetime dimension is kept arbitrary most of the time.

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## 1. Introduction

The low energy dynamics of Goldstone bosons is described by a Nonlinear Sigma Model (NLSM), which is perturbatively nonrenormalizable in four dimensions. The theory is expected to break down at the characteristic energy scale $4\pi f_\phi$, where $f_\phi$ is a coupling which in four dimensions has dimension of mass (the so-called "pion decay constant"). At this scale the interactions become strong, and perturbative unitarity is violated. In spite of this, there have been suggestions that this theory may hold also beyond this limit and might perhaps be well behaved up to arbitrarily high energy due to nonperturbative effects. There are two main proposals. The first is nonperturbative renormalizability, a.k.a "asymptotic safety" [1]. It is based on the possibile existence of a nontrivial Fixed Point (FP) for the Renormalization Group (RG), which implies that trajectories ending (in the UV) at this FP do not exhibit unphysical divergences when energy goes to infinity. Such trajectories are called "renormalizable" or "asymptotically safe". If the UV basin of attraction of the FP is finite dimensional, the condition that the theory be described by a renormalizable trajectory leaves only a finite number of free parameters. All the others are fixed by the theory and thus provide predictions that could be experimentally verified. An asymptotically safe theory of this type is therefore as well-behaved and as predictive as QCD. This idea has been discussed especially in the context of gravity [2], but it could apply also to the case of Goldstone bosons [3].

The other possibility is that the scattering of Goldstone bosons is dominated by the formation of classical intermediate states called "classicalons", which would typically decay into a large number of low energy particles and would thus suppress the cross section for hard scattering with few highly energetic final states. This would lead to unitarization of Goldstone boson scattering. This scenario, which has been called "classicalization", has also originated in a gravitational context [4], and subsequently extended to models of Goldstone bosons [5]. Various aspects of classicalization have been also considered in [6, 7, 8, 9].

Both possibilities, being intrinsically nonperturbative, are hard to establish rigorously. So far several calculation in simplified settings have shown that these are at least plausible scenarios. A natural question, which has been put forward in [10], is whether they are really different or just two facets of the same physical phenomenon. At first this sounds quite implausible, due to the very different mechanisms invoked. Still, one may note that the final observable result may not be dissimilar; the details of what happens in the interaction region are not directly observable and thus perhaps not so important.

In order to analyze this question one would have to compare physical observables. This is hampered by the fact that so far calculations with asymptotic safety have been concerned mainly with the existence of a FP for the RG of coupling constants, viewed as parameters in the Lagrangian. It is generally believed that a FP for these couplings would correspond also to a FP for couplings defined more physically in terms of observables. A suggestive argument, based on the lowest order Lagrangian for the NLSM, goes as follows. The tree level scattering amplitude of Goldstone bosons behaves as $p^2/f_\phi^2$, where $p^2$ is some quadratic combination of the external momenta. The statement that this theory has a FP means that when the cutoff $\Lambda$ tends to infinity, the coupling scales as $f_\phi \sim \Lambda$. If we were to identify the cutoff with $p$, the amplitude would tend to a constant. Unfortunately so far there is no proper calculation of $n$-point functions that can be used to convalidate this argument. Thus, asymptotic safety and classicalization are, at the moment, quite complementary: the first deals with Lagrangian couplings, the second with scattering amplitudes. Eventually one will have to compare them directly at the level of observables.

We will review here the main aspects of asymptotic safety, classicalization and the possible connection between the two. Section 2 is a review of work on asymptotic safety in NLSMs done by the first author in collaboration with A. Codello, O. Zanusso, F. Bazzocchi, M. Fabbrichesi, A. Tonero, L. Vecchi. It covers Lagrangians that are quadratic in derivatives [11], quartic in derivatives [12], the coupling to gauge fields [13, 14] and to fermions [15]. In section 3 we examine classicalization in scattering processes, extending the results of [10] to arbitrary dimensions. Section 4 contains comparison of the results and discussion.



## 2. Asymptotic safety

### 2.1 Functional Renormalization

In order to study the asymptotic safety (or lack thereof) of the theory of Goldstone bosons, one needs a tool that can be reliably applied in a nonperturbative context. Our preferred tool is the exact, Functional Renormalization Group Equation (FRGE) which provides a convenient implementation of Wilson's idea of integrating out degrees of freedom one momentum shell at the time. We start from a bare Euclidean action $S[\phi]$, and we add to it a suppression term $\Delta S_k[\phi]$ that is quadratic in the field. In momentum space it is of the form

$$\Delta S_k[\phi] = \frac{1}{2}\int dq\, \phi(-q)\, R_k(q^2)\, \phi(q) \; . \tag{2.1}$$

The kernel $R_k(z)$ will be called "the cutoff". It is arbitrary, except for the general requirements of being a monotonically decreasing function both in $z$ and $k$, tending to zero for $z \gg k^2$ and to $k^2$ for $z \ll k^2$. These conditions ensure that the contribution to the functional integral of field modes with momenta $q^2 \ll k^2$ are suppressed, while the contribution of field modes with momenta $q^2 \gg k^2$ are unaffected. We define a $k$-dependent generating functional $W_k$

$$e^{-W_k[J]} = \int D\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] - \int dx J\phi\right\} \tag{2.2}$$

and a $k$-dependent "effective average action"

$$\Gamma_k[\phi] = W_k[J] - \int dx J\phi - \Delta S_k[\phi] \, , \tag{2.3}$$

by Legendre transform and subtraction of $\Delta S_k[\phi]$. This functional satisfies the Wetterich equation or FRGE [16, 17]

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right]^{-1} k\frac{dR_k}{dk} \; , \tag{2.4}$$

where the trace in the r.h.s. contains a volume and a momentum integration.

One can gain some feeling for this equation by considering the one loop effective average action. Given a bare action $S$, the one loop effective action is $\Gamma^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left[\frac{\delta^2 S}{\delta\phi\delta\phi}\right]$. Adding to

$S$ the cutoff term (2.1), we obtain "one loop effective average action" $\Gamma_k^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right]$ which satisfies the equation

$$k\frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right]^{-1} k\frac{dR_k}{dk}\ . \tag{2.5}$$

This is formally identical to (2.4) except that in the r.h.s. the bare action $S$ appears in place of $\Gamma_k$. Thus the FRGE is a "RG improved" one-loop equation, where the bare couplings have been replaced by the running couplings.

The r.h.s. of (2.4) can be regarded as the "beta functional" of the theory, giving the $k$–dependence of all the couplings of the theory. To see this let us assume that $\Gamma_k$ admits a derivative expansion of the form

$$\Gamma_k(\phi, g_i) = \sum_{n=0}^{\infty}\sum_i g_i^{(n)}(k)\mathscr{O}_i^{(n)}(\phi)\ , \tag{2.6}$$

where $g_i^{(n)}(k)$ are coupling constants and $\mathscr{O}_i^{(n)}$ are all possible operators constructed with the field $\phi$ and $n$ derivatives, which are compatible with the symmetries of the theory. We have

$$k\frac{d\Gamma_k}{dk} = \sum_{n=0}^{\infty}\sum_i \beta_i^{(n)}\mathscr{O}_i^{(n)}\ . \tag{2.7}$$

where $\beta_i^{(n)}(g_j, k) = k\frac{dg_i^{(n)}}{dk} = \frac{dg_i^{(n)}}{dt}$ are the beta functions of the couplings. Here we have introduced $t = \log(k/k_0)$, $k_0$ being an arbitrary initial value. If we expand the trace on the r.h.s. of (2.4) in operators $\mathscr{O}_i^{(n)}$ and compare with (2.7), we can read off the beta functions of the individual couplings.

The trace on the r.h.s. of the FRGE is free of UV and IR divergences, because the derivative of the cutoff kernel goes rapidly to zero for $q^2 > k^2$, and $k$ also acts effectively as a mass. So, given a "theory space" which consists of a class of functionals of the fields, one can define on it a flow without having to worry about regularizations. All the beta functions are finite. Then, one can pick an initial point and study the trajectory passing through it in either direction. The issue of the UV divergences is now related to the behavior of the trajectory for $k \to \infty$. If there are none, the chosen trajectory is a renormalizable one. In the other direction, studying the limit $k \to 0$ one obtains the usual effective action $\Gamma[\phi]$.

### 2.2 Two derivatives

One convenient way of approximating the FRGE is the derivative expansion. In the case of the NLSM, the lowest order term is quadratic in derivatives and has the general form

$$S(\varphi) = \frac{1}{2\gamma^2}\int dx\, \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta h_{\alpha\beta}(\varphi)\ . \tag{2.8}$$

Since $h$, the metric in the target space, is in general a nonpolynomial function of the target space coordinates, this action already contains, in general, infinitely many couplings.

Quantization of the NLSM requires splitting the field $\varphi^\alpha$ into a background and a quantum field $\delta\varphi^\alpha$. For notational simplicity in the following we will call the background $\varphi^\alpha$. Since the difference of two coordinates has no simple transformation property, it is convenient to use instead of $\delta\varphi^\alpha$ a variable $\xi^\alpha$ such that $(\varphi+\delta\varphi)(x) = Exp_{\varphi(x)}\xi(x)$, where $Exp$ is the exponential map [18]. A map such as $\xi$ that assigns to each point $x$ a vector tangent to the target space at $\varphi(x)$ is called a "vectorfield along $\varphi$". Let $\Gamma_\alpha{}^\beta{}_\gamma$ be the Christoffel symbols of $h$ and $R_{\alpha\beta}{}^\gamma{}_\delta$ its Riemann tensor. The covariant derivative of a vectorfield along $\varphi$ is $\nabla_\mu\xi^\alpha = \partial_\mu\xi^\alpha + \partial_\mu\varphi^\gamma\Gamma_\gamma{}^\alpha{}_\beta\xi^\beta$. The curvature of the pullback connection is the pullback of the curvature of the Levi-Civita connection: $[\nabla_\mu,\nabla_\nu]\xi^\gamma \equiv \Omega_{\mu\nu}{}^\gamma{}_\delta\xi^\delta = \partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta R_{\alpha\beta}{}^\gamma{}_\delta\xi^\delta$.

Further using an orthonormal frame field $e^a_\alpha$ the quadratic part of the action (1) is

$$\frac{1}{2\gamma^2}\int dx\, \xi^a\left(-\nabla^2\delta_{ab} - M_{ab}\right)\xi^b\ , \tag{2.9}$$

where $M_{ab} = e^\alpha_a e^\beta_b \partial^\mu\varphi^\gamma\partial_\mu\varphi^\delta R_{\alpha\gamma\beta\delta}$. It is convenient to choose a cutoff kernel of the form $\mathscr{R}_{k,ab} = \frac{1}{\gamma^2}\delta_{ab}R_k(-\nabla^2)$. In this way the modified inverse propagator is $\frac{1}{\gamma^2}(P_k(-\nabla^2)\delta_{ab} - M_{ab})$, where $P_k(z) = z + R_k(z)$. Introducing in (2.5) we have

$$k\frac{d\Gamma^{(1)}_k}{dk} = \frac{1}{2}\mathrm{Tr}\,\frac{\dot{R}_k\mathbf{1}}{P_k\mathbf{1}-M}\ . \tag{2.10}$$

Expanding in $M$, extracting the term with two derivatives of the background and performing the momentum integration, one can read off the beta function for the metric [11]

$$k\frac{d}{dk}\left(\frac{1}{\gamma^2}h_{\alpha\beta}(\varphi)\right) = 2c_d k^{d-2}R_{\alpha\beta}\ . \tag{2.11}$$

where $c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$. This agrees with old results when $d = 2+\varepsilon$ [19] or $d = 3$ [20].

Let us now suppose that the metric $h_{\alpha\beta}$ has some Killing vectors, generating a Lie group $G$. Since the cutoff is defined by means of the $G$–invariant Laplacian $-\nabla^2$, it preserves the $G$ invariance. Therefore if the initial point of the flow is an invariant metric, the flow takes place within the restricted class of invariant metrics. We will focus our attention to two classes of models: The $O(n+1)$ and the chiral $SU(N)$ models.

The $O(n+1)$ model has target space $S^n = O(n+1)/O(n)$. Invariance under $SO(n+1)$ completely fixes the metric up to a scale. Its Riemann and Ricci tensors are given by

$$R_{\alpha\beta\gamma\delta} = h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma}\ ;\quad R_{\alpha\beta} = (n-1)h_{\alpha\beta}\ ;\quad R = n(n-1)\ .$$

Introducing in (2.11) we find a flow equation for the single coupling $\gamma^2$, which we can write in the form:

$$k\frac{d}{dk}\frac{1}{\gamma^2} = 2c_d(n-1)k^{d-2}\ . \tag{2.12}$$

Passing to the dimensionless coupling $\tilde{\gamma}^2 = k^{d-2}\gamma^2$ we find the beta function

$$\frac{d\tilde{\gamma}^2}{dt} = (d-2)\tilde{\gamma}^2 - 2c_d(n-1)\tilde{\gamma}^4 \tag{2.13}$$

which, for $d > 2$, has a FP at $\tilde{\gamma}^2_* = \frac{1}{2c_d}\frac{d-2}{n-1}$.

The chiral $SU(N)$ model has the group $SU(N)$ as target space. Up to rescalings, there is a unique $Ad$-invariant inner product in the Lie algebra, which we choose as $h_{ab} = 2\mathrm{Tr}T_aT_b = \delta_{ab}$ [1]. Then the corresponding biinvariant metric is

$$h_{\alpha\beta} = L^a_\alpha L^b_\beta \delta_{ab} \,, \tag{2.14}$$

so that the left-invariant vectorfields $L_a$ can also be regarded as a vierbein. The Riemann and Ricci tensors and the Ricci scalar of $h$ are given by

$$R_{\alpha\beta\gamma\delta} = \frac{1}{4} L^a_\alpha L^b_\beta L^c_\gamma L^d_\delta f_{ab}{}^e f_{ecd} \,; \qquad R_{\alpha\beta} = \frac{1}{4} N h_{\alpha\beta} \,; \qquad R = \frac{1}{4} N(N^2-1) \,. \tag{2.15}$$

Repeating the previous steps we find the beta function

$$\frac{d\tilde{\gamma}^2}{dt} = (d-2)\tilde{\gamma}^2 - \frac{Nc_d}{2}\tilde{\gamma}^4 \,. \tag{2.16}$$

which, for $d > 2$, has a FP at $\tilde{\gamma}^2_* = \frac{2(d-2)}{Nc_d}$.

In both cases, in the present approximation the derivative of the beta function at the FP is $\beta'(\tilde{\gamma}_*) = 2-d < 0$, so this FP is UV attractive. We refer to [11] for further discussion of this model. Note for later reference that according to equation (2.11) a FP at positive coupling requires quite generally positive curvature.

### 2.3 Four derivatives

In this section we restrict ourselves to four dimensions. The most general Lorentz- and parity-invariant action containig up to four derivatives is:

$$\frac{1}{2}\int d^4x \Big[ \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta h^{(2)}_{\alpha\beta}(\varphi) + \Box\varphi^\alpha\Box\varphi^\beta h^{(4)}_{\alpha\beta}(\varphi) \\ +\nabla_\mu\partial_\nu\varphi^\alpha\partial^\mu\varphi^\beta\partial^\nu\varphi^\gamma A_{\alpha\beta\gamma}(\varphi) + \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta T_{\alpha\beta\gamma\delta}(\varphi) \Big] \,. \tag{2.17}$$

The tensors $h^{(2)}$, $h^{(4)}$ are assumed to be positive definite metrics. We will use $h^{(4)}$ to raise and lower indices, while $h^{(2)}$ is treated as any tensor. The tensor $A$ can be assumed to be totally symmetric without loss of generality. The tensor $T$ must have the symmetry properties $T_{\alpha\beta\gamma\delta} = T_{\beta\alpha\gamma\delta} = T_{\alpha\beta\delta\gamma} = T_{\gamma\delta\alpha\beta}$. One can derive general beta functions for these tensors. For example, the beta function of the metric $h^{(4)}$ is again a Ricci flow [12]

$$k\frac{d}{dk}h^{(4)}_{\alpha\beta} = \frac{1}{8\pi^2}R_{\alpha\beta} \,. \tag{2.18}$$

We will not discuss the problem at this level of generality but rather restrict ourselves to target spaces that are spheres or special unitary groups.

[1] Here the matrices are in the fundamental representation. The Cartan-Killing form just differs by a constant: $B_{ab} = \mathrm{Tr}(Ad(T_a)Ad(T_b)) = N\delta_{ab}$.

#### 2.3.1 The spherical models

There is only one $O(n+1)$-invariant metric on the sphere, there is no invariant rank three tensor and there are only two invariant rank four tensors with the desired index symmetries, up to overall constant factors. If we regard $S^n$ as embedded in $\mathbf{R}^{n+1}$, we call $h_{\alpha\beta}$ the metric of the sphere of unit radius. Both $h^{(2)}$ and $h^{(4)}$ must be proportional to $h$, and $T$ is a combination of $h$'s:

$$h^{(2)}_{\alpha\beta} = \frac{1}{\gamma^2} h_{\alpha\beta}\ ; \qquad h^{(4)}_{\alpha\beta} = \frac{1}{\lambda} h_{\alpha\beta}\ ; \qquad T_{\alpha\beta\gamma\delta} = \frac{\ell_1}{2}\left(h_{\alpha\gamma}h_{\beta\delta} + h_{\alpha\delta}h_{\beta\gamma}\right) + \ell_2 h_{\alpha\beta}h_{\gamma\delta}\ .$$

Here $\gamma^2$ has mass dimension $-2$, while $\lambda$, $\ell_1$, $\ell_2$ are dimensionless. It is convenient to regard $1/\lambda$ as the overall factor of the fourth order terms; then we define the ratios between the three coefficients of the four-derivative terms as $f_1 = \lambda\ell_1$ and $f_2 = \lambda\ell_2$. For the reader's convenience we rewrite the action of the $S^n$ models:

$$\int d^4x \left[\frac{1}{2\gamma^2} h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta + \frac{1}{2\lambda}\left(h_{\alpha\beta}\Box\varphi^\alpha\Box\varphi^\beta + \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta(f_1 h_{\alpha\gamma}h_{\beta\delta} + f_2 h_{\alpha\beta}h_{\gamma\delta})\right)\right] \tag{2.19}$$

One then finds the following beta functions:

$$\beta_\lambda = -\frac{n-1}{8\pi^2}\lambda^2 \tag{2.20}$$

$$\beta_{f_1} = \frac{\lambda}{48\pi^2}\left((n+21)f_1^2 + 20 f_2 f_1 + 4 f_2^2 + 6(n+3)f_1 + 24 f_2 + 8\right) \tag{2.21}$$

$$\beta_{f_2} = \frac{\lambda}{8\pi^2}\left(\frac{n+15}{12}f_1^2 + \frac{3n+17}{3}f_1 f_2 + \frac{6n+7}{3}f_2^2 - (n+3)f_1 - (3n+1)f_2 + n - \frac{7}{3}\right) \tag{2.22}$$

$$\beta_{\tilde\gamma^2} = 2\tilde\gamma^2 + \frac{\tilde\gamma^4}{16\pi^2}\left((5+n)f_1 + (2+4n)f_2 + 4(1-n)\right) - \frac{\lambda\tilde\gamma^2}{16\pi^2}\left((5+n)f_1 + (2+4n)f_2 + 2(1-n)\right) \tag{2.23}$$

The beta function of $\lambda$ depends only on $\lambda$ and the solution is

$$\lambda(t) = \frac{\lambda_0}{1 + \lambda_0\frac{n-1}{8\pi^2}(t - t_0)}\ , \tag{2.24}$$

where $\lambda_0 = \lambda(t_0)$. We assume $\lambda_0 > 0$, thus $\lambda$ is asymptotically free. The beta functions of $f_1$ and $f_2$ do not depend on $g$, so their flow can be studied independently. Here we do not discuss general solutions but merely look for FPs. The overall factor $\lambda$ in these beta functions can be eliminated by a simple redefinition $t = t(\tilde t)$ of the parameter along the RG trajectories: $\frac{d}{d\tilde t} = \frac{1}{\lambda}\frac{d}{dt}$. Since $\tilde t$ is a monotonic function of $t$, the FPs for $f_1$ and $f_2$ are the zeroes of the modified beta functions

$$\tilde\beta_{f_i} = \frac{df_i}{d\tilde t} = \frac{1}{\lambda}\beta_{f_i}\ .$$

They are just polynomials in $f_1$ and $f_2$. The model has no real FP for $n = 2$, but there are FPs for all $n > 2$. For $n = 3,4,5,6$ they are given in the fifth and sixth column in Table I. One can then insert the FP values of $f_1$ and $f_2$ in $\beta_{\tilde\gamma^2}$ and look for FP of $\tilde\gamma^2$. In each case there are two solutions, one at $\tilde\gamma^2 = 0$, the other at some nonzero value. These solutions are reported in the fourth column in Table I, for $n = 3,4,5,6$. The first solution describes the Gaussian FP (GFP), where all the couplings $\tilde\gamma^2$, $\lambda$, $1/\ell_1$, $1/\ell_2$ are zero, the others non Gaussian FP's (NGFP) where $\tilde\gamma^2$ has finite limits instead.

| $n$ | $\tilde{\gamma}_*^{(III)}$ | FP | $\tilde{\gamma}_*$ | $f_{1*}$ | $f_{2*}$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|
| 3 | 8.886 | NGFP1 | 6.626 | -0.693 | 0.453 | 0.094 | -0.0121 |
| 3 | | NGFP2 | 6.390 | -1.042 | 0.615 | 0.103 | 0.0119 |
| 3 | | GFP1 | 0 | -0.693 | 0.453 | 0.094 | -0.0121 |
| 3 | | GFP2 | 0 | -1.042 | 0.615 | 0.103 | 0.0119 |
| 4 | 7.255 | NGFP1 | 5.877 | -0.479 | 0.398 | 0.105 | -0.0412 |
| 4 | | NGFP2 | 5.442 | -1.555 | 0.852 | 0.132 | 0.0392 |
| 4 | | GFP1 | 0 | -0.479 | 0.398 | 0.105 | -0.0412 |
| 4 | | GFP2 | 0 | -1.555 | 0.852 | 0.132 | 0.0392 |
| 5 | 6.283 | NGFP1 | 5.310 | -0.400 | 0.400 | 0.118 | -0.0608 |
| 5 | | NGFP2 | 4.924 | -1.875 | 0.988 | 0.154 | 0.0567 |
| 5 | | GFP1 | 0 | -0.400 | 0.400 | 0.118 | -0.0608 |
| 5 | | GFP2 | 0 | -1.875 | 0.988 | 0.154 | 0.0567 |
| 6 | 5.620 | NGFP1 | 4.883 | -0.350 | 0.408 | 0.131 | -0.0780 |
| 6 | | NGFP2 | 4.577 | -2.131 | 1.091 | 0.171 | -0.0717 |
| 6 | | GFP1 | 0 | -0.350 | 0.408 | 0.131 | -0.0780 |
| 6 | | GFP2 | 0 | -2.131 | 1.091 | 0.171 | 0.0717 |
| 6 | | GFP3 | 0 | -0.814 | 1.369 | -0.161 | -0.0539 |
| 6 | | GFP4 | 0 | -2.363 | 2.091 | -0.164 | -0.0617 |

**Table 1:** Gaussian and non-Gaussian FPs of the $S^n$ model at one loop. The first column gives the dimension $n$. The second column gives the position of the NGFP in the two-derivative truncation, using a type III cutoff. The rest of the table refers to the four-derivative truncation, also using a type III cutoff. The third column gives the name of the FP. Columns 4,5,6 give the position of the NGFP, columns 7,8 the critical exponents, as defined in the text. The coupling $\lambda$, not listed, goes to zero and is marginal in this approximation.

Each FP can be approached only from specific directions in the space parametrized by $\lambda$, $\ell_1$, $\ell_2$, *i.e.* the ratios $f_1$ and $f_2$ take specific values. For each NGFP these values are unique, while for the GFP there may be several possible values: two if $n = 3,4,5$ and four if $n = 6$.

When one considers the linearized flow around any of the GFPs, one finds as expected that the critical exponents, defined as minus the eigenvalues of the matrix $\frac{\partial \beta_i}{\partial g_j}$, are (-2,0,0,0), corresponding to the canonical dimensions of the couplings. The critical exponents at the NGP are instead (2,0,0,0). Thus the dimensionless couplings are marginal, and of the two FPs, the trivial one is IR attractive and the nontrivial one UV attractive for $\tilde{g}$. For $\lambda$ it is clear that the FP is UV attractive (if we had chosen $\lambda < 0$ it would be IR attractive). In order to establish the attractive or repulsive character of $f_1$ and $f_2$, one can look at the linearized flow in the variable $\tilde{t}$, which is described by the $2\times 2$ matrix $\frac{\partial \tilde{\beta}_{f_i}}{\partial f_j}$. We define the "critical exponents" $\theta_{1,2}$ to be minus the eigenvalues of this matrix. They are reported in the last two columns of table I, for $n = 3,4,5,6$. It is important to realize that even for the GFP the eigenvectors of the stability matrix are not the operators that appear in the action but mixings thereof. We do not report the eigenvectors here.

The FP exists for all $n > 2$. For large $n$ one can study the FPs analytically, to some extent. There are four FPs for the system of the $f_i$'s, which are: $f_1 = 0$, $f_2 = 1$ with critical exponents

$\theta_1 = 6$, $\theta_2 = 12$; $f_1 = 0$, $f_2 = 1/2$ with critical exponents $\theta_1 = 6$, $\theta_2 = -12$; $f_1 = -6$, $f_2 = 5/2$ with critical exponents $\theta_1 = -6$, $\theta_2 = 12$; $f_1 = -6$, $f_2 = 2$ with critical exponents $\theta_1 = -6$, $\theta_2 = -12$. The numerical values at finite $n$ do indeed tend towards these limits for growing $n$.

### 2.3.2 The chiral models

Next we consider the case where the target space is the group $SU(N)$. In this case it is customary to denote $U(x)$ the matrix (in the fundamental representation) that corresponds to the coordinates $\varphi^\alpha$. The theory is invariant under left and right multiplications $U(x) \mapsto g_L^{-1} U(x) g_R$, forming the group $SU(N)_L \times SU(N)_R$ ("chiral symmetry"). Further we demand that the theory be invariant under the discrete symmetries $U(x) \mapsto U^T(x)$, which corresponds physically to charge conjugation, to the simple parity $x_1 \mapsto -x_1$, to the involutive isometry $\Phi_0 : U \to U^{-1}$ and hence to "Parity", defined as $U(x_1, x_2, x_3, x_4) \mapsto U^{-1}(-x_1, x_2, x_3, x_4)$.

Let $T_a$ be hermitian, traceless $N \times N$ matrices forming a basis of the algebra in the fundamental representation. We fix the normalization of the basis by the equation

$$T_a T_b = \frac{1}{2N}\delta_{ab} + \frac{1}{2}\left(d_{abc} + i f_{abc}\right) T_c \,. \tag{2.25}$$

The tensors $d_{abc}$ and $f_{abc}$ are a totally symmetric and a totally antisymmetric $Ad$-invariant three tensor in the algebra. (In the case of $SU(3)$ these matrices are one half the Gell-Mann $\lambda$ matrices.) There is a one to one correspondence between biinvariant tensors on $SU(N)$ and $Ad$-invariant tensors in the Lie algebra of $SU(N)$, where $Ad$ is the adjoint representation. Given an $Ad$-invariant tensor $t_{ab\ldots}{}^{cd\ldots}$ on the algebra, the corresponding biinvariant tensorfield on the group is

$$t_{\alpha\beta\ldots}{}^{\gamma\delta\ldots} = t_{ab\ldots}{}^{cd\ldots} L^a_\alpha L^b_\beta \ldots L^\gamma_c L^\delta_d \ldots$$

where $L^a_\alpha$ are the components of the left-invariant Maurer Cartan form $L = U^{-1} dU = L^a_\alpha dy^\alpha(-iT_a)$ and $L^\alpha_a$ are the components of the left-invariant vectorfields on $SU(N)$. The matrix $L^\alpha_a$ is the inverse of $L^a_\alpha$. (In this construction we could use equivalently right-invariant objects.)

As with the sphere, we define $h^{(2)}_{\alpha\beta} = \frac{1}{\gamma^2} h_{\alpha\beta}$, $h^{(4)}_{\alpha\beta} = \frac{1}{\lambda} h_{\alpha\beta}$. In principle chiral invariance would permit a term in the action with $A_{\alpha\beta\gamma} = L^a_\alpha L^b_\beta L^c_\gamma d_{abc}$; but this can be forbidden by requiring Parity invariance [12].

For $T$ there are several candidates:

$$\begin{aligned} T^{(1)}_{abcd} &= \frac{1}{2}\left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \; ; \qquad T^{(2)}_{abcd} = \delta_{ab}\delta_{cd} \; ; \qquad T^{(3)}_{abcd} = \frac{1}{2}\left(f_{ace} f_{bd}{}^e + f_{ade} f_{bc}{}^e\right) \; ; \\ T^{(4)}_{abcd} &= \frac{1}{2}\left(d_{ace} d_{bd}{}^e + d_{ade} d_{bc}{}^e\right) \; ; \qquad T^{(5)}_{abcd} = d_{abe} d_{cd}{}^e \; . \end{aligned} \tag{2.26}$$

They are not all independent, however. The identity (2.10) of [21] implies that

$$\frac{2}{N} T^{(1)} - \frac{2}{N} T^{(2)} + T^{(3)} + T^{(4)} - T^{(5)} = 0 \,, \tag{2.27}$$

so that $T^{(5)}$ can be eliminated. In the case $N = 3$ the identity (2.23) of [21], together with the preceding relation, further implies

$$T^{(2)} - T^{(3)} - 3T^{(4)} = 0 \,, \tag{2.28}$$

so that we can also eliminate $T^{(4)}$. Finally in the case $N = 2$ the tensor $d_{abc}$ is identically zero, so we can keep only $T^{(1)}$ and $T^{(2)}$ as independent combinations, and use $T^{(3)} = T^{(2)} - T^{(1)}$.

The action of the generic $SU(N)$ models can then be written in the form:

$$\int d^4x \left[ \frac{1}{2\gamma^2} h_{\alpha\beta} \partial_\mu \varphi^\alpha \partial^\mu \varphi^\beta + \frac{1}{2\lambda} h_{\alpha\beta} \Box\varphi^\alpha \Box\varphi^\beta + \frac{1}{2} \partial_\mu \varphi^\alpha \partial^\mu \varphi^\beta \partial_\nu \varphi^\gamma \partial^\nu \varphi^\delta \sum_{i=1}^{4} \ell_i T^{(i)}_{\alpha\beta\gamma\delta} \right] \tag{2.29}$$

and the sum stops at $i = 3$ for $N = 3$ and $i = 2$ for $N = 2$. As in (2.19), it will be convenient to use instead of the couplings $\ell_i$ the combinations $f_i = \lambda \ell_i$.

We do not give here the beta functions for general $N$, which are very long. Suffice it to say that one finds no nontrivial FP for $N > 3$ [22, 12]. The only nontrivial cases to discuss are $N = 3, 2$, where only three, respectively two, of the couplings $f_i$ are independent. In the case $N = 3$ one can eliminate $f_4$ in favor of the other three couplings. Then one can obtain

$$\begin{aligned}
\beta_{f_1} &= \frac{\lambda}{768\pi^2} \left[ 464 f_1^2 + 64 f_2^2 + 180 f_3^2 + 320 f_1 f_2 - 96 f_1 f_3 + 72 f_1 - 108 f_3 + 9 \right] , \\
\beta_{f_2} &= \frac{\lambda}{1536\pi^2} \times \\
&\quad \left[ 368 f_1^2 + 3520 f_2^2 + 180 f_3^2 + 2624 f_1 f_2 + 480 f_1 f_3 + 1728 f_2 f_3 - 144 f_1 - 432 f_2 - 108 f_3 + 9 \right] , \\
\beta_{f_3} &= \frac{\lambda}{32\pi^2} \left[ 2 f_3^2 + 16 f_1 f_3 + 8 f_2 f_3 - 4 f_1 - 4 f_2 - 3 f_3 \right] .
\end{aligned}$$

This system of the $f_i$'s has two FPs at

$$\begin{aligned}
&FP1: \quad f_{1*} = -0.154\,; \qquad f_{2*} = 0.050\,; \qquad f_{3*} = 0.085\,; \\
&FP2: \quad f_{1*} = -0.108\,; \qquad f_{2*} = 0.043\,; \qquad f_{3*} = 0.061\,.
\end{aligned}$$

The attractivity properties in the space spanned by the $f_i$'s is given, as in the spherical case, by studying the modified flow with parameter $\tilde{t}$. The critical exponents at FP1 are: 0.0303 with eigenvector (0.411, 0.630, 0.658); 0.0123 with eigenvector (0.515, -0.570, 0.640); 0.00289 with eigenvector (0.869, -0.148, -0.473), whereas at FP2 they are: 0.0280 with eigenvector (0.366, 0.618, 0.695); 0.0108 with eigenvector (0.513, -0.575, 0.638) and -0.00293 with eigenvector (0.887, -0.125, -0.445). Therefore FP1 is attractive in all three directions, while FP2 is attractive in two directions. For each of these two FP's, the beta function of $\tilde{g}$ has two FP's: the trivial FP, which has always critical exponents -2, and a nontrivial FP, which is located at $\tilde{g} = 11.17$ for NGFP1 or 11.50 for NGFP2, and having critical exponent 2 in both cases.

Finally, the case $N = 2$ is identical to the $S^3$ model that was already discussed above.

### 2.4 Coupling to gauge fields

We will consider now a chiral $SU(N)$ NLSM when only $SU(N)_L$ is gauged, and call $A_\mu$ the corresponding gauge field. The covariant derivative and the gauge field strength are defined to be:

$$D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + A^a_\mu R^\alpha_a(\varphi) \qquad\qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu\,. \tag{2.30}$$

Restricting our attention to terms containing two derivatives of the fields, the Euclidean action of this gauged NLSM, in $d$ dimensions, reads

$$S = \frac{1}{2\gamma^2} \int d^d x\, h_{\alpha\beta} D_\mu \varphi^\alpha D^\mu \varphi^\beta + \frac{1}{4g^2} \int d^d x\, F^a_{\mu\nu} F_a^{\mu\nu}, \tag{2.31}$$

where $\gamma$ and $g$ are couplings. We choose the background gauge fixing term:

$$S_{gf} = \frac{1}{2\alpha g^2}\int d^d x\, \delta_{ab}\chi^a\chi^b \qquad \text{with} \qquad \chi^a = D^\mu a^a_\mu + \beta\frac{g^2}{\gamma^2}R^a_\alpha \xi^\alpha\ , \tag{2.32}$$

where $D$ is the background covariant derivative, $\alpha$ and $\beta$ are parameters and $a^a_\mu$ is the quantum fluctuation of the gauge field around the background. The case $\alpha=\beta$ is a "background $R_\xi$-gauge". Moreover, for $\alpha=\beta=1$ we have the generalization of the 't Hooft-Feynman gauge fixing. The ghost action is $S_{gh}=S_{ghF}+S_{ghI}$, where

$$S_{ghF} = \int d^d x\, \bar{c}^a\left(-D^2+\beta\frac{g^2}{\gamma^2}\right)c_a \tag{2.33}$$

is the free ghost action and $S_{ghI}$ are interaction terms.

Let us now choose $\alpha=1$. The beta functions for $1/g^2$ and $1/\gamma^2$ are:

$$\frac{d}{dt}\frac{1}{g^2} = -\frac{1}{(4\pi)^{d/2}}\frac{N}{3}\frac{1}{\Gamma\left(\frac{d}{2}-1\right)}\frac{k^{d-4}}{1+\frac{\tilde{g}^2}{\tilde{\gamma}^2}}\left[d-\frac{7}{4}+\frac{\frac{\eta_\xi}{4}+d\eta_a}{d-2}-\frac{192}{d(d-2)}\frac{1+\frac{\eta_a}{d+2}}{(1+\frac{\tilde{g}^2}{\tilde{\gamma}^2})^2}\right]\ , \tag{2.34}$$

$$\frac{d}{dt}\frac{1}{\gamma^2} = \frac{1}{(4\pi)^{d/2}}\frac{N}{2}\frac{1}{\Gamma\left(\frac{d}{2}+1\right)}\frac{k^{d-2}}{(1+\frac{\tilde{g}^2}{\tilde{\gamma}^2})^2}\left[1+\frac{\eta_\xi}{d+2}+\frac{4\tilde{g}^2/\tilde{\gamma}^2}{1+\frac{\tilde{g}^2}{\tilde{\gamma}^2}}\left(2+\frac{\eta_\xi+\eta_a}{d+2}\right)\right]\ . \tag{2.35}$$

Here $\eta_\xi=-2\partial_t\log\gamma$ and $\eta_a=-2\partial_t\log g$. Omitting the terms containing them in the right hand sides one obtains the one loop beta functions. In general, expressing $\eta_\xi$ and $\eta_a$ one can solve these algebraic equations and obtain the beta functions proper of the (generally) dimensionful couplings. The corresponding beta functions of the dimensionless combinations $\tilde{\gamma}^2=\gamma^2 k^{d-2}$ and $\tilde{g}^2=g^2k^{d-4}$ can be obtained by simple algebra.

Note the appearance of the factors $1/(1+\tilde{g}^2/\tilde{\gamma}^2)$ which represent threshold effects: for $k^2\gg g^2/\gamma^2$ these factors tend to one, whereas for $k^2\ll g^2/\gamma^2$ the denominators become large and suppress the running, reflecting the decoupling of the corresponding massive field modes. The flow equations in general $R_\xi$-gauge have been given in [13].

Let us now restrict our attention to the case $d=4$. If one considers the regime $k^2\gg g^2/\gamma^2$, where thresholds can be neglected, the beta function of $g$ is given by:

$$\frac{dg}{dt} = -\frac{1}{2}A_2 g^3\ ; \tag{2.36}$$

with a universal coefficient $A_2=\frac{N}{(4\pi)^2}\frac{29}{4}$. Note that $29/4$ differs from the coefficient $22/3$ of the pure gauge theory by the Goldstone boson contribution $-1/12$. This contribution is quite small and does not spoil the asymptotic freedom of $g$. On the other hand, in the same limit the beta function of $\tilde{\gamma}$ becomes

$$\frac{d\tilde{\gamma}}{dt} = \tilde{\gamma}-\frac{1}{2}A_1\tilde{\gamma}^3 \tag{2.37}$$

with $A_1=\frac{1}{(4\pi)^2}\frac{N}{4}$. This beta function has a nontrivial FP at $\tilde{\gamma}_*=\sqrt{2/A_1}$.

In $d>4$, due to the nontrivial dimensionality of the gauge coupling, one finds also a nontrivial FP for the gauge coupling. One would expect it to be there also in the presence of the Goldstone bosons. At one loop and in the limit $k^2 \gg g^2/\gamma^2$, there is a FP at

$$\tilde{\gamma}_* = \sqrt{\frac{d-2}{A_1}} \; ; \qquad \tilde{g}_* = \sqrt{\frac{d-4}{A_2}} \; . \tag{2.38}$$

For a detailed discussion we refer the reader to [23]. We conclude by mentioning that gauging also a $U(1)$ subgroup of $SU(N)_R$ does not change the results in a very significant way. This more realistic model has been discussed in [14], where it has been shown to be compatible with electroweak precision data.

### 2.5 Coupling to fermions

We continue with the chiral $SU(N)$ model. It will be convenient to use as a variable the matrix $U = \exp(i\gamma\pi^a T_a)$ where $\pi^a$ are the pion fields. The Lagrangian (2.8) can be rewritten as

$$\mathscr{L}_\sigma = \frac{1}{\gamma^2} \mathrm{Tr}\left(U^\dagger \partial_\mu U U^\dagger \partial^\mu U\right) . \tag{2.39}$$

We couple the Goldstone bosons to left- and right-handed fermions $\psi_L^{ia}$ and $\psi_R^{ia}$ carrying the fundamental representation of $SU(N)_L$ and $SU(N)_R$ respectively (corresponding to the indices $i = 1, \ldots, N$), and also the fundamental representation of a color group $SU(N_c)$ (corresponding to the indices $a = 1, \ldots, N_c$). In the real world the latter group is gauged; here we merely retain it as a global symmetry to count fermionic states. We couple the fermions in a chiral invariant way to the $U$ field by adding to the NLSM lagrangian the fermion kinetic and the Yukawa terms:

$$\mathscr{L}_{\psi^2} = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R - \frac{2h}{f}\left(\bar{\psi}_L^{ia} U^{ij} \psi_R^{ja} + \mathrm{h.c.}\right) , \tag{2.40}$$

where we have explicitly written out the group indices in the interaction.

Using a sharp cutoff regularization, one gets the following one-loop RG equations [15]:

$$\frac{d\tilde{\gamma}}{dt} = \tilde{\gamma} - \frac{N}{64\pi^2}\tilde{\gamma}^3 + \frac{N_c}{4\pi^2} h^2 \tilde{\gamma} \, , \tag{2.41}$$

$$\frac{dh}{dt} = \frac{1}{16\pi^2}\left(4N_c - 2\frac{N^2-1}{N}\right) h^3 + \frac{1}{64\pi^2}\frac{N^2-2}{N} h \tilde{\gamma}^2 \; . \tag{2.42}$$

These $\beta$-functions have been obtained by assuming that the mass of the fermions is much smaller than the cut-off scale; this sets a condition $k > h/\gamma$ for their validity.

The system of equations (2.41)-(2.42) admits a number of possible UV FPs. There is a formal Gaussian FP $\tilde{\gamma} = 0$, $h = 0$ which is outside the domain of our approximation. In the following we study only RG trajectories for energy scales larger than $1/\gamma$, which corresponds to $\tilde{\gamma} > 1$. There is also a nontrivial FP at $h_* = 0$, $\tilde{\gamma}_* = 8\pi/\sqrt{N}$ for which $\tilde{\gamma}$ is a relevant (UV attractive) direction and $h$ is marginally irrelevant. Requiring that it is reached in the UV implies the triviality of the Yukawa coupling at all scales. We therefore reject this choice as uninteresting.

A physically interesting trajectory requires nonvanishing $h$ and $\tilde{\gamma}$. If $h$ is treated as a $t$-independent constant, the $\beta$-function for $\tilde{\gamma}$ has a zero at $\tilde{\gamma}_* = 4\sqrt{(4\pi^2 + N_c h^2)/N}$, which is a deformation of the one appearing in the pure bosonic model. The existence of a nontrivial FP for the coupled system thus hinges on the existence of a nontrivial zero in the $\beta$-function of $h$. This requires that the first term in the right hand side of eq. (2.42) be negative, which is true for $N > 2N_c$.

Unfortunately this condition is not satisfied for the phenomenologically most important case $N = 2$, $N_c = 3$. In this case the first term on the right hand side of eq. (2.41) is initially dominant, leading to linear growth of $\tilde{\gamma}$. The second term then grows in absolute value and at some point nearly balances the first one, leading to an approximate FP behavior in some range of energies. Eventually $h$, whose $\beta$-function is everywhere positive, becomes large and the third term dominates leading to a Landau pole. These trajectories are not asymptotically safe. The scale at which destabilization occurs is very sensitive to the initial conditions and for the Yukawa couplings corresponding to light fermions no destabilization takes place up to very large energies. We conclude that this model is not AS in the case $N = 2$, $N_c = 3$, in the one loop approximation. For it to be AS, either the one loop approximation must break down, or else new physical effects must enter in the fermion sector at some energy scale.

It is interesting to compare this behavior to similar models. In the linear sigma model and therefore also in the SM the quadratically divergent term proportional to $\tilde{\gamma}^2 h$ is absent: it is canceled by diagrams containing loops of the Higgs field. In this case the Yukawa coupling is perturbative up to very high scales [24]. A study of the linear version of the model in the context of functional renormalization has been presented in [25] for QCD. Another strictly related model is the linear sigma model coupled to one right-handed and $N_L$ left-handed fermions, studied in [26]. Our Goldstone modes are contained in their scalar sector, with the VEV $\upsilon = 2/\gamma$ corresponding to the minimum of the scalar potential. There it was found that the scalar potential and the Yukawa coupling admit a FP for $1 \leq N_L \leq 57$. The results quoted above differ due both to the different fermion content and to the non-linear boson-fermion coupling. For realistic matter content the situation could be improved by adding four-point fermion interactions [27, 28, 15]. For related results see also [29].

## 3. Classicalization

In this section we will consider again derivatively coupled theories which are employed to describe the low energy dynamics of Goldstone bosons and we examine the classical propagation of waves [5]. We will follow the arguments of [10] but generalize them to arbitrary dimensions. Since we want to disentangle classical from quantum effects, throughout this discussion we will not choose units such that $\hbar = 1$. Thus everything will have dimensions of powers of length and mass.

### 3.1 Goldstone boson scattering

We begin by considering a model of a single Goldstone boson with higher derivative interaction lagrangian of the form:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + \frac{g_4^{m-1}}{2m}\left((\partial\phi)^2\right)^m , \tag{3.1}$$

for some integer $m > 1$. The field has canonical dimension $M^{1/2}L^{(3-d)/2}$ and the coupling $g_4$ has dimension $L^{d-1}M^{-1}$. It defines a characteristic length scale $L_* = (g_4\hbar)^{1/d}$ and a characteristic energy scale $E_* = \hbar/L_* = (\hbar^{d-1}/g_4)^{1/d}$. The equation of motion coming from this lagrangian is

$$\Box\phi + g_4^{m-1}\partial^\mu\left[\partial_\mu\phi\left((\partial\phi)^2\right)^{m-1}\right] = 0 . \tag{3.2}$$

Assuming that free asymptotic states exist, the solution of the nonlinear equation (3.2) can be constructed perturbatively. We consider solutions with $(d-2)$-spherical symmetry. The initial ingoing unperturbed free wave, solving the equation $\Box\phi_0 = 0$, has the form

$$\phi_0(t,r) = \sqrt{\hbar}\psi(\omega(t+r))r_0^{(d-4)/2}/r^{d-3} , \tag{3.3}$$

where $\psi(z) = A\sin(z) + B\cos(z)$ is dimensionless. Notice that in dimensions $d \neq 4$ the canonical dimension of the field is not equal to the power of $r$ that is needed to solve the free Laplace equation, hence the appearance of the arbitrary radius $r_0$, which should be seen as a free parameter of the unperturbed solution. We will assume that the wavelength $\omega^{-1}$ is small compared to the radius $r$, so that we can think of the solution as a harmonic function with a slowly-varying $r$-dependent amplitude. The first order perturbation $\phi_1$ must satisfy $\Box\phi_1 = -g_4^{m-1}\partial^\mu\left[\partial_\mu\phi_0\left((\partial\phi_0)^2\right)^{m-1}\right]$. Making the ansatz

$$\phi_1(t,r) = \sqrt{\hbar}\eta(\omega(t+r))r_0^{(d-4)/2}f(r) \tag{3.4}$$

we find the following equation

$$-\frac{2\omega\sqrt{\hbar}r_0^{(d-4)/2}}{r^{(d-2)/2}}\eta'\left(fr^{(d-2)/2}\right)' = -2^{m-1}(m-1)g_4^{m-1}\omega^m\hbar^{(2m-1)/2}(d-3)^{m-1}\times \frac{r_0^{(d-4)(2m-1)/2}}{r^{(2d-5)m+3-d}}\psi^{m-1}\psi'^{m-2}\left[(3d-8)\psi'^2+(d-3)\psi\psi''\right] , \tag{3.5}$$

where a prime denotes derivative of a function with respect to its argument.

The solution of this equation for $d \neq 3$ can be expressed as

$$\phi_1 = \frac{2^{m-1}(m-1)(d-3)^{m-1}g_4^{m-1}E^{m-1}\sqrt{\hbar}}{2(5-2d)m+3d-6}\frac{r_0^{(d-4)(m-1/2)}}{r^{(2d-5)m+2-d}}\eta(\omega(t+r)) \tag{3.6}$$

where $E = \hbar\omega$ and $\eta(z) = \int^z \psi^{m-1}\psi'^{m-2}\left[(d-3)\psi\psi''+(3d-8)\psi'^2\right]dz'$. The function $\eta$ is again dimensionless and periodic with period $2\pi$, which means that the scattered wave has the same frequency as the incoming one. Since $\eta \sim \psi \sim 1$, the ratio of the amplitudes $|\phi_1/\phi_0|$ is

$$|f(r)r^{d-3}| \simeq \frac{2^{m-1}(m-1)(d-3)^{m-1}}{2(5-2d)m+3d-6}\frac{g_4^{m-1}E^{m-1}r_0^{(d-4)(m-1)}}{r^{(2d-5)(m-1)}} \simeq \left(\frac{r_*}{r}\right)^{(2d-5)(m-1)} , \tag{3.7}$$

where in the last step we defined the "classicalization radius"

$$r_* = \sqrt[2d-5]{2|d-3|g_4r_0^{d-4}E} . \tag{3.8}$$

Notice that it does not depend on $m$.

The meaning of this radius can be understood as follows. At low energy (i.e. $E \ll E_*$) the theory can be treated as an effective field theory. Due to the uncertainty relations, an incoming wave with energy $E$ can only probe distances of order $\hbar/E$. When one gets close to the characteristic energy scale $E_*$ one would normally expect the effective field theory to break down. What one sees here is that the scattered wave becomes significant at radius of order $r_*$, and therefore cannot resolve smaller distances. Since $r_*$ grows with energy, there is a turnover energy where this bound becomes stronger than the one set by the uncertainty principle. Let us call $E_{opt}$ this energy and $L_{opt} = \hbar/E_{opt}$

the corresponding resolving power. At $E > E_{opt}$ the resolving power *decreases* with energy. This is called "classicalization". In this regime the scattering is dominated by the production of classical states with high occupation number, which will typically decay into many low energy particles [9]. The hard scattering of few particles into few particles will be suppressed and unitarity will be restored [5]. In this way classicalization may provide a form of UV completion of an effective field theory that does not necessitate the introduction of new weakly coupled degrees of freedom.

Details of this process depend on the dimension. First we observe that it can only occur when $d > 5/2$. For lower dimensions, the first perturbation grows with radius faster than the unperturbed wave, in contrast to our initial assumptions. In fact, the initial free wave solution doesn't decrease with radius. In particular, classicalization cannot take place in $d = 2$.

From equation (3.3) one sees that the case $d = 3$ requires a separate treatment. This is because in three dimensions a free wave has a logarithmic dependence on radius $\phi_0 = \sqrt{\hbar}\psi(\omega(t + r))r_0^{-1/2}\log(r/r_1)$, where $r_1$ is another free parameter. Solution of the appropriate equations of motion in this case can be expressed as a digamma function, having $\log(r/r_1)$ as one of its arguments. The ratio of the first perturbation to the initial logarithmic solution increases when the center of the scattering is approached, and becomes large, actually blowing up at $r = r_1$. Note that this sets a bound on the resolving power which is independent of energy. One may call this a weak form of classicalization.

For dimensions $d \geq 4$ all the formulas written above hold. Note however that $d = 4$ is special in that the classicalization radius is independent of $r_0$. For $d > 4$ the scale $E_{opt}$ decreases when $r_0$ increases, and $L_{opt}$ increases when $r_0$ increases. If we set $r_0 = L_*$, the characteristic length scale of the theory, the classicalization radius is $r_*^{2d-5} = 2(d-3)L_*^{2d-5}L_*\omega$ and one finds $E_{opt} \approx E_*$ and $L_{opt} \approx L_*$. In four dimensions these relations are always true, because $r_0$ does not appear in the formula for the classicalization radius.

### 3.2 Nonlinear sigma model with 2 derivatives

Generic theories of Goldstone bosons have interactions that can be seen geometrically as arising from the curvature of the target space. In the preceding subsection we have considered for simplicity a single Goldstone boson. In this case the target space is one-dimensional, and since a one-dimensional space is flat the interactions we have considered were of a non-geometrical nature. They necessarily involved higher derivative terms. We are now going to consider more general theories where the target space is curved. There are then interaction terms containing just two derivatives. This is the type of terms we consider in this subsection. In the next we will consider terms with both two- and four-derivative interactions.

For definiteness we consider nonlinear sigma models with values in maximally symmetric spaces: for positive curvature the target space is a sphere $S^n$, for negative curvature a hyperboloid. We adopt a specific coordinate system in target space such that the lagrangian has the form

$$\mathscr{L} = \frac{1}{2}\left[(\partial\vec{\phi})^2 \pm \frac{(\vec{\phi}\cdot\partial\vec{\phi})^2}{f_\phi^2 \mp \vec{\phi}^2}\right] \tag{3.9}$$

Here $\vec{\phi} = (\phi^1, \ldots, \phi^n)$ are canonically normalized fields of dimension $(d-2)/2$ and $f_\phi$ is a coupling with the same dimension as the field. If we define the dimensionless fields $\varphi^a = \phi^a/f_\phi$, the action takes the form (2.8) with $\gamma = 1/f_\phi$. The two signs in (3.9) correspond to the sign of the curvature. The equations of motion are

$$\Box\phi^a \pm \frac{\phi^a\,\partial(\overrightarrow{\phi}\cdot\partial\overrightarrow{\phi})}{f_\phi^2 \mp \overrightarrow{\phi}^2} \pm \frac{\phi^a\,(\overrightarrow{\phi}\cdot\partial\overrightarrow{\phi})^2}{(f_\phi^2 \mp \overrightarrow{\phi}^2)^2} = 0 \tag{3.10}$$

As in the preceding section, we are going to look for perturbative solution in the form $\overrightarrow{\phi} = \overrightarrow{\phi}_0 + \overrightarrow{\phi}_1 + \ldots$. We will study the extent to which $\overrightarrow{\phi}_1$ can be treated as a small perturbation.

Without much loss of generality we will work with a general spherically symmetric incoming free wave

$$\phi_0^a(t,r) = \sqrt{\hbar}\psi_a(\omega(t+r))r_0^{(d-4)/2}/r^{d-3}\;, \tag{3.11}$$

where all components have the same frequency $\omega$, the same arbitrary parameter $r_0$ and we assume $\omega r \gg 1$, as before. The first order perturbation will be written in the form $\phi_1^a(r,t) = \sqrt{\hbar}\eta_a(\omega(t+r))r_0^{(d-4)/2}f(r)$. To leading order in $1/r\omega$ we find

$$-\frac{2\omega\sqrt{\hbar}r_0^{(d-4)/2}}{r^{(d-2)/2}}(fr^{(d-2)/2})'\eta_a' = \mp\frac{2\omega\hbar^{3/2}r_0^{3(d-4)/2}(d-3)}{f_\phi^2 r^{3d-8}}\frac{\psi_a\vec{\psi}\vec{\psi}'}{\left(1\mp\frac{\hbar\vec{\psi}^2 r_0^{d-4}}{f_\phi^2 r^{2(d-3)}}\right)^2}\;. \tag{3.12}$$

We note that in contrast to equation (3.5) the $\omega$-dependence cancels out. Instead, the behavior of the solution is governed by the new dimensionless parameter $\kappa = f_\phi r^{d-3}/(\sqrt{\hbar}r_0^{(d-4)/2})$. As long as $\kappa \gg 1$, (which is naturally expected for the perturbations far away from the center in $d > 3$) the denominator in the r.h.s. can be approximated by one and the equation can be solved by separation of variables. Now we can notice that after the separation the radial equation for $f$ is the same for all components of $\phi_1^a$, therefore the choice $f^a(r) = f(r)$ is justified. The solution can be written in the form

$$\phi_1^a = \mp\sqrt{\hbar}\frac{2(d-3)\hbar r_0^{3(d-4)/2}}{(5d-16)f_\phi^2 r^{3d-9}}\eta_a(\omega(t+r))\;, \tag{3.13}$$

where $\eta^a(z) = \int^z \psi^a\overrightarrow{\psi}\,\overrightarrow{\psi}'dz'$.

In contrast to the case of the preceding section, the amplitudes of these oscillations of the scattered wave are independent of $\omega$. For $d > 3$ the ratio between the amplitude of the first perturbation and the incoming wave is

$$\left|f(r)r^{d-3}\right| = \frac{2(d-3)\hbar r_0^{d-4}}{(5d-16)f_\phi^2 r^{2(d-3)}} = \left(\frac{r_*}{r}\right)^{2(d-3)}\;, \tag{3.14}$$

with the classicalization radius

$$r_* \simeq \left(\frac{\sqrt{\hbar}r_0^{(d-4)/2}}{f_\phi}\right)^{\frac{1}{d-3}} \tag{3.15}$$

independent on the frequency or energy of the incoming wave packet. Again, incoming waves with arbitrarily high frequency do not probe distances shorter than $r_*$, but in contrast to the preceding case $r_*$ does not increase with frequency, so we have a weaker form of classicalization.

Let us now consider the effect of curvature, which (aside from the immaterial overall sign) is contained in the denominator of the r.h.s. of (3.12). We observe that since $0 \le \vec{\psi}^2$ is of order one,



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the effect of the denominator is to enhance the amplitude of the scattered wave for positive curvature (upper sign) and to decrease it for negative curvature (lower sign). In fact, with the positive curvature the amplitude reaches a pole for some $r \approx r_*$, strengthening the case for classicalization of the preceding analysis. In the case of negative curvature, the arguments for classicalization are considerably weaker.

Again the case $d \leq 3$ is special. Far from the origin we expect that $\kappa \ll 1$ and in equation (3.12) we can neglect the unity in the denominator in the r.h.s. and the equations can be solved by separation of variables. The ratio $|\phi_1/\phi_0|$ is

$$\left|f(r)r^{d-3}\right| = 2\frac{3-d}{4-d} = \text{const} \tag{3.16}$$

for $d < 3$ and

$$\left|\frac{f(r)}{\log(r/r_1)}\right| = \frac{2}{\log(r/r_1)} \tag{3.17}$$

for $d = 3$. We see that for $d < 3$ classicalization doesn't occur in these NLSM, because the ratio of the first perturbation to initial free wave solution is in the first approximation constant. This reflects the fact, that the free solution does not vanish at spatial infinity. In the special case of three spacetime dimensions this ratio acquires a value of order one for $r \approx r_1$ and this could be taken as a classicalization radius $r_*$. However here the $r-$dependence is not a power law, like in the all previously considered examples, but logarithmic.

### 3.3 Nonlinear sigma model with 2 and 4 derivatives

In a maximally symmetric NLSM a general four derivative interaction has the form

$$\mathscr{L}^{(4)}_{\text{int}} = g_4(\ell_1 h_{ab}h_{cd} + \ell_2 h_{ac}h_{bd})\partial_\mu\phi^a\partial^\mu\phi^b\partial_\nu\phi^c\partial^\nu\phi^d \ , \tag{3.18}$$

where $\ell_1$ and $\ell_2$ are dimensionless couplings of order one and $g_4$ has dimension $L^{d-1}M^{-1}$. We will not be interested in the detailed dependence of results on $\ell_1$ and $\ell_2$. The overall coupling $g_4$ can be treated as an independent coupling. However, in effective field theory one expects the coefficients of operators with different number of derivatives to be all proportional to powers of the same scale $f_\phi$ in natural units. Then one may identify $g_4 = \hbar^{2/(d-2)} f_\phi^{-(2d)/(d-2)}$, where $f_\phi$ is as in the previous section. We will discuss both points of view below.

When this interaction is added to the two-derivative Lagrangian (3.9), the background solution will still have the form (3.11) and we are led to the following equation for the first perturbation:

$$\begin{aligned}\Box\phi_1^a = &\mp\frac{2\omega\hbar^{3/2}r_0^{3(d-4)/2}(d-3)}{f_\phi^2 r^{3d-8}}\frac{\psi_a\vec{\psi}\vec{\psi}'}{\left(1\mp\frac{\hbar\vec{\psi}^2 r_0^{d-4}}{f_\phi^2 r^{2(d-3)}}\right)^2} \\ &-\frac{4\omega^2\hbar^{3/2}g_4 r_0^{3(d-4)/2}(d-3)}{r^{3d-7}}\left[(\ell_1+3\ell_2)\psi_a\vec{\psi}'^2+(3\ell_1+5\ell_2)\psi_a'\vec{\psi}'\vec{\psi}+(\ell_1+\ell_2)\psi_a\vec{\psi}''\vec{\psi}+\ell_2\psi_a''\vec{\psi}^2\right]\end{aligned} \tag{3.19}$$

One cannot solve this equation by separation of variables unless one of the two terms can be neglected. However, we can get a reasonable estimate of the size of the terms involved by simply setting to one all the factors $\eta$ in the l.h.s. and the terms involving $\psi$ in the r.h.s.. The resulting equation for $f(r)$ can then be easily integrated and we find

$$|f(r)r^{d-3}| = \mp\frac{\hbar r_0^{d-4}}{f_\phi^2 r^{2(d-3)}} - \frac{E g_4 r_0^{d-4}}{r^{2d-5}} - \frac{\hbar^2 r_0^{2(d-4)}}{f_\phi^4 r^{4(d-3)}} + \dots \tag{3.20}$$

$$= \mp\left(\frac{r_{2*}}{r}\right)^{2(d-3)} - \left(\frac{r_{4*}}{r}\right)^{2d-5} - \left(\frac{r_{2*}}{r}\right)^{4(d-3)} + \dots \tag{3.21}$$

where the first and third term come from the expansion of the two-derivative term and the second comes from the four-derivative term. We have defined the classicalization radii

$$r_{2*} = \left(\frac{\hbar}{f_\phi^2}\right)^{\frac{1}{d-2}} \quad ; \qquad r_{4*} = \left(E g_4 r_0^{d-4}\right)^{\frac{1}{2d-5}} = \left(\frac{E\hbar^{\frac{2}{d-2}} r_0^{d-4}}{f_\phi^{\frac{2d}{d-2}}}\right)^{\frac{1}{2d-5}} . \tag{3.22}$$

The radius $r_{4*}$ has been written in two possible equivalent ways, the first in terms of the generic parameter $g_4$ defined in (3.18), the second when $g_4$ is expressed in terms of $f_\phi$. All the terms in the expansion of the two-derivative term correspond to the same classicalization radius $r_{2*}$. These terms are dominant for $E < \sqrt[d-2]{\hbar^{d-3} f_\phi^2}$. For higher energy the four-derivative terms dominate and the system behaves like several copies of the single Goldstone boson model of section 3.1, in the special case with $2m = 4$ derivatives. In fact, the formula for $r_{4*}$ agrees with (3.8). Strong classicalization occurs for $\omega > r_{4*}^{-1}$ regardless of the sign of the curvature. We see here that adding higher derivatives for $d \le 3$ brings us back to the case analyzed in section 3.1 with the same conclusions, namely that when $d = 2$ classicalization doesn't occur and for $d = 3$ we have its weak form with $r_* = r_1$.

## 4. Discussion

Goldstone bosons are ubiquitous in low energy effective field theories and in particular the chiral models have important phenomenological applications both to strong interactions ("chiral perturbation theory" [30]). and electroweak interactions ("electroweak chiral perturbation theory" [31]). The two implementation differ mainly in their characteristic energy scale, which is $f_\phi = 93$ MeV for the low energy QCD and $f_\phi = 246$ GeV for the electroweak case. These theories are generally assumed to break down at momenta of order $4\pi f_\phi$, which is where perturbative unitarity fails. Beyond this energy "new physics", generally in the form of new weakly coupled degrees of freedom, is expected to manifest itself. Interestingly, already in perturbation theory there are signs that in some cases the effective field theory can self-unitarize even in the absence of new weakly coupled degrees of freedom [32]. This would extend the applicability of the effective field theory to somewhat higher energies than generally believed. If either asymptotic safety or classicalization or both occur in the chiral NLSM, the domain of applicability of these theories could extend much further, possibly up to arbitrarily high energy. In practice the first and most important sign would be the unitarization of the scattering amplitudes. A plausibility argument for this has already been mentioned in the introduction for the two-boson to two-boson scattering. A proper discussion of this issue requires a detailed calculation of the scattering amplitude as a function of the external momenta, which is not yet available.

In the meantime, we can compare and contrast the results of section 2, for the existence of a nontrivial FP, to those of section 3, for classicalization, with the aim of understanding whether there could be some relation between these apparently very different phenomena. We begin by the observation that in a general NLSM there can be several classes of interactions: those that have more fields than derivatives, those that have equal number of fields and derivatives and those that have more derivatives than fields. Interactions that are due to the curvature of the target space metric, such as arise from the expansion of the metric in (2.8), belong to the first class. We have also considered here interaction monomials where all fields appear under exactly one derivative; they belong to the second class. For short we will refer to the former as curvature interactions and the latter as higher derivative interactions. We have not considered in this paper interactions with more derivatives than fields, nor more generally interactions with two or more derivatives acting on a single field.

The first case to discuss is that of a single Goldstone boson. A one dimensional space cannot have intrinsic curvature, so any interaction must necessarily contain higher derivatives. In dimension $d \geq 4$ this model exhibits a strong form of classicalization. A detailed analysis of the RG flow of this theory, containing arbitrarily high powers of derivatives, is not yet available, but it is not expected to possess a nontrivial FP. For example, neglecting possible effects due to the topology of the target space, we can specialize the results of section 2.3.1 to the case $n = 1$; then we find no FP. Classicalization of this model seems unlikely to have to do with asymptotic safety.

Consider now general models with several Goldstone bosons parametrizing a curved target space. In two spacetime dimensions the NLSM is asymptotically free and a nontrivial FP appears as soon as one goes to $2+\varepsilon$ dimensions [19]. Its existence in three dimensions is well known [20], and it is widely believed to be in the same universality class of the Wilson-Fisher FP. One can thus study its properties also in the linear realization of the theory. In four dimension the existence of a FP is much less certain. The calculations presented in section 2 show that if one restricts oneself to truncations with two derivatives, positive scalar curvature is sufficient to give a FP, whereas addition of four derivative terms imposes tighter restrictions.

Let us now see, how this compares to classicalization. In two dimensions classicalization does not occur in any of the models considered. This seems to agree with the absence of a nontrivial FP there. In three dimensions a weak (logarithmic) form of classicalization takes place. However, the classicalization radius cannot be expressed as a function of the couplings: it is instead a free parameter of the unperturbed solution of the wave equations. The general formulas that we have given in section 3 hold in four and higher dimensions. We find that curvature interactions, as already present in the two-derivative NLSM, give rise to a weak form of classicalization, whose classicalization radius contains $\hbar$. One may therefore reasonably argue that this particular form of classicalization is really a quantum phenomenon and it may be an alternative way of looking at asymptotic safety. Another piece of evidence in favor of this interpretation is that positive curvature strengthens the case for classicalization, while negative curvature weakens it. This is reminiscent of the fact that a nontrivial FP requires positive curvature.

Interactions due to higher derivative terms, whether including the effect of curvature or not, give instead rise to a strong form of classicalization, in the sense that the classicalization radius grows with some power of the energy. If one uses the parameterization of the action in term of the coupling $g_4$, the classicalization radius is independent of $\hbar$, justifying its name. The available evidence therefore suggests that curvature interactions give rise to a weak form of classicalization that may be a manifestation of a nontrivial FP, whereas higher derivative interactions give a strong form of classicalization, which seems to be a different phenomenon.

This interpretation, though reasonable, is not the only possible one. Different interpretations could arise from different ways of defining the classcial limit of the theory. As recently discussed in [33], this limit is ambiguous even in the familiar case of QED. For example, one may argue whether it is $E$ or $\omega$ that is being kept fixed in the limit $\hbar \to 0$. In the models considered here, another choice is whether $g_4$ or $f_\phi$ is kept fixed. In the preceding discussion we have implicitly assumed that $E$ and $g_4$ were kept constant and this implied that the classicalization radius $r_{4*}$ stays constant in the limit. However, we may define the limit assuming that either $\omega$ or $f_\phi$ or both are held fixed; then, additional factors of $\hbar$ appear in such a way that $r_{4*} \to 0$ in the limit $\hbar \to 0$ (see equation (3.22)). For example, in chiral perturbation theory in four dimensions, if one starts from the two-derivative action (2.8) the four-derivative interactions are induced by quantum loops with a logarithmically divergent coefficient proportional to $\hbar\gamma^4 = \hbar f_\phi^{-4}$ [31]. Then, the interactions (3.18) would not be seen as part of the classical (bare) action of the theory but rather of its quantum effective action. The analysis of section 3 would still apply but it would not be appropriate to view it as a purely classical phenomenon. The plan to return to this point in the future.

In concluding we must stress once more that, as is usually the case with nonperturbative problems, our whole discussion is based on partial results and that all its conclusions are tentative. Especially if one has in mind realistic applications a purely bosonic model may be misleading. For example, Yukawa interactions with too many fermions seem to destroy the FP; although four-fermion interactions improve the picture [27, 28, 15] it is not entirely clear that this is a viable scenario. As a further caveat, even assuming that a FP with the correct properties exists, the effective field theory may just happen not to lie on a renormalizable trajectory, in which case it will break down at some scale. In these cases one may still hope that the RG trajectory passes near a FP, or more generally in a region where the beta functions are small. This would slow down the flow in such a way that the breakdown of the theory occurs at a scale that is higher than one would normally guess [34].

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