PROCEEDINGS OF SCIENCE

T-DUALITY OF MASSIVE STRINGY STATES

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The evidence for the target space duality symmetry associated with massive excited states of closed bosonic string is presented. The vertex operators for excited masive levels of closed bosonic string are constructed for the case where d-dimensions are compactified on a torus. The existence of T-duality is verified for a few massive levels. A systematic procedure is presented to study T-duality symmetry of vertex operators of massive levels of closed bosonic string. It is argued that all vertex operators corresponding to excited massive states can be cast in an $O(d,d)$ invariant form, d being the number of compact dimensions .

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1. Introduction

The string theory holds the promise of unifying the fundamental forces of Nature. There have been several important steps to achieve this goal [1]. String theory has also resolved many important issues in quantum gravity with considerable success. The computation of Bekenstein-Hawking entropy and insight into the nature of Hawking radiation in black hole physics from a microscopic theory are considered as important achievements. Moreover, outcome of important investigations of string theory have influenced research in the frontiers of cosmology. It is well known that string theory is very rich in symmetries. These can be identified as local and global symmetries. The former is generally associated with the massless states of the strings such as graviton and antisymmetric tensor field (for closed string). Moreover, the string theory is endowed with symmetries on the worldsheet in the first quantized formulation as well as through its description from the perspective of target space effective action. The duality symmetries play an important role to understand string dynamics [2]. The web of dualities unravel the intimate relationships between the five superstring theories in various dimensions [3, 4] although they are perturbatively distinct in the critical dimension, $\hat{D} = 10$. The target space duality, T-duality, is a special attribute of the theory which owes its origin to the one dimensional nature of the string. In its simplest form, we encounter T-duality in the worldsheet description of string's evolution. If σ denotes its coordinate, in the temporal evolution (τ-evolution), a string sweeps a surface and therefore, its coordinates, $X^{\hat{\mu}}(\sigma,\tau)$ are parametrized by them. Thus $\tau \leftrightarrow \sigma$ interchange describes the same physical evolution process. When particles, which belong to the string spectrum, scatter they form strings as intermediate states corresponding to exchange of towers of particles. Interchange of τ and σ is to be interpreted as direct channel and crossed channel processes from the quantum field theory perspective.

I shall present recent investigations [5] on duality symmetry associated with massive excited stated of closed bosonic string in the Hamiltonian formalism from the worldsheet view point. The evolution of the string in the background of its massless excitation corresponds to a 2-dimensional σ-model where the backgrounds are identified as coupling constants of the theory. These are constrained if we demand that the theory respects conformal invariance. We intend to follow a similar approach where the string evolves in the background of higher massive levels in order to study the duality symmetry associated with the excited states.

Let us recall that all the massive states of closed string belong to irreducible representations of rotation group, $SO(\hat{D}-1)$, $\hat{D}$ being the number of spacetime dimensions. Moreover, at each level of the spectrum, the states are degenerate with different spins. The importance of massive string states is recognized when one computes the β-functions associated with the massless states, in the σ-model approach, beyond the leading order. In particular, when one computes the second order corrections to the β-function in the graviton background, it was observed that [7, 8] it is necessary to introduce counter terms which correspond to infinite number of massive modes to cancel the loop diverges that appear in the bosonic theory. One could couple the string to excited massive states analogously, generalizing the procedures of coupling to the massless excitations; however, such terms will be suppressed by appropriate factors α' on dimensional considerations. Therefore, the resulting effective actions (obtained for such massive states) will play a subdominant role in the low energy regime. Let us recall that compactification of string effective action, derived for a string in critical dimensions(26 and 10 for bosonic and superstrings respectively), leads to appearance of noncompact duality group (see [6] and references therein). Thus it is worth while to examine whether excited states are endowed with any duality-like symmetry. We conjecture that there are evidences for dualities which can be verified, for toroidally compactified closed bosonic string, in the case of a few excited levels. Moreover, we present a systematic procedure to verify the validity of our proposal that such dualities persist for all massive levels. It is well known that the excited states are endowed with several interesting attributes. The degeneracy of higher states, for a given mass, grows exponentially which is the *raison de etre* for limiting (Hagedorn) temperature. Moreover, it has been argued that in the Planckian energy scattering regime, such stringy states play an important role [9, 10]. There are hints that these states might be endowed with higher gauge symmetries [11, 12, 13, 14, 15, 16]. The recent interests to study high spin massless field theories have utilized properties of excited stringy states in certain limits [17, 18]. We recall some of the useful results in order to formulate our problem. In this optics, it is worth while to investigate duality symmetry associated with excited massive levels of closed string where d of its spatial coordinates are compactified on T^d.

2. T-DUALITY SYMMETRY FOR EXCITED LEVELS

In this section we discuss T-duality properties of massive excited states of closed bosonic string. We consider a scenario where some of the spatial coordinates of the string are compactified on T^d. We designate noncompact string coordinates as spacetime coordinates (which includes time coordinate). We choose the massless background along spacetime directions to be trivial i.e. the metric is Minkowskian and antisymmetric tensor, $B_{\mu\nu} = 0$. Thus the free Hamiltonian density is sum of two pieces; one expressed in terms of spacetime coordinates and their conjugate momenta and the other piece contains compact coordinates and their conjugate momenta. Then we supplent it with terms coming from various vertex operators which are treated in the weak field approximation. If we require the vertex operators to be $(1,1)$ primaries [12, 13, 14] then they satisfy equations of motion as well as some transversality conditions. We shall not provide details of these calculations which are available in the literature [12, 13], although we shall utilize them when the need arises. The target space duality has been examined from several perspectives [19]. Now we recall some salient results of T-duality in the frame work of the worldsheet theory [20, 21, 6, 22]. In particular we focus on toroidal compactification for massless states in the worldsheet approach [20, 21, 6]. In this context it is assumed that string coordinates $Y^\alpha(\sigma,\tau), \alpha, \beta = 1,2,..d$ are compactified on torus T^d. The noncompact coordinates are $X^\mu(\sigma,\tau), \mu,\nu = 0,1,2..D-1$ with $D+d = \hat{D}$. The corresponding backgrounds after dimensional reduction[6], for the metric, are $G_{\mu\nu}(X), A^{(1)}_{\mu\alpha}(X)$and $G_{\alpha\beta}(X)$ and from the 2-form we get $B_{\mu\nu}(X), B_{\mu\alpha}$ and $B_{\alpha\beta}(X)$. It is assumed that all the backgrounds depend only on the spacetime string coordinates X^μ. The gauge fields $A^{(1)}_{\mu\alpha}$ are associated with the isometries and $B_{\mu\alpha}$ are another set of gauge fields coming from dimensional reductions of the 2-form. It was shown that, after introducing a set of dual coordinates $\tilde{Y}^\alpha$ the combined worldsheet equations of motion (of Y and $\tilde{Y}$) can be cast in a duality covariant form. Note that if one resorts to the Hamiltonian formulation for a slightly simplified version of above compactification [23], the resulting Hamiltonian is expressed in duality invariant form [24]. Our strategy will be to utilize the results of Hamiltonian formulation and adopt a simple compactification procedure for the higher levels and unveil the duality symmetry for these states. Let us consider toroidal compactification where we set $G_{\alpha\beta} = \delta_{\alpha\beta}$ and $B_{\alpha\beta} = 0$; in other words the radii of T^d are set to unity. The stress energy momentum tensors used to compute the conformal weights are

$$T_{++} = \frac{1}{2}(G^{(0)}_{\mu\nu}\partial X^\mu \partial X^\nu + \delta_{\alpha\beta}\partial Y^\alpha \partial Y^\beta) \tag{2.1}$$

$$T_{--} = \frac{1}{2}(G^{(0)}_{\mu\nu}\bar\partial X^\mu \bar\partial X^\nu + \delta_{\alpha\beta}\bar\partial Y^\alpha \bar\partial Y^\beta) \tag{2.2}$$

where $G^{(0)}_{\mu\nu} = \mathrm{diag}(1,-1,-1..)$ is the flat D-dimensional metric, $\partial X^\mu = \dot X^\mu + X'^\mu$, $\partial Y^\alpha = \dot Y^\alpha + Y'^\alpha$, $\bar\partial X^\mu = \dot X^\mu - X'^\mu$ and $\bar\partial Y^\alpha = \dot Y^\alpha - Y'^\alpha$; 'overdot' and 'prime' stand for derivatives with respect to τ and σ here and everywhere. Note that the string coordinates, X^μ and Y^α will be decomposed as right and left movers, $X^\mu = X^\mu_R + X^\mu_L$, and $Y^\alpha = Y^\alpha_R + Y^\alpha_L$ and therefore, $\partial\bar\partial X^\mu = \partial\bar\partial Y^\alpha = 0$. Therefore, we do not include such terms in construction of excited level vertex operators. The vertex operator for first excited massive state [12, 13] for the uncompactified closed string is

$$\hat\phi_1 = \hat V_1^{(1)} + \hat V_1^{(2)} + \hat V_1^{(3)} + \hat V_1^{(4)} \tag{2.3}$$

where

$$\hat V_1^{(1)} = A^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'}(X)\partial X^{\hat\mu}\partial X^{\hat\nu}\bar\partial X^{\hat\mu'}\bar\partial X^{\hat\nu'} \tag{2.4}$$

$$\hat V_1^{(2)} = A^{(2)}_{\hat\mu\hat\nu,\hat\mu'}(X)\partial X^{\hat\mu}\partial X^{\hat\nu}\bar\partial^2 X^{\hat\mu'}, \quad \hat V_1^{(3)} = A^{(3)}_{\hat\mu,\hat\mu'\hat\nu'}(X)\partial^2 X^{\hat\mu}\bar\partial X^{\hat\mu'}\bar\partial X^{\hat\nu'} \tag{2.5}$$

$$\hat V_1^{(4)} = A^{(4)}_{\hat\mu,\hat\mu'}(X)\partial^2 X^{\hat\mu}\bar\partial^2 X^{\hat\mu'} \tag{2.6}$$

The subscript '1' appearing in $\hat V_1^{(i)}, i = 1-4$ is indicative of the fact that they correspond to ones for the first excited massive level. Notices that the tensor indices are labeled with unprimed and primed indices. This convention is adopted to keep track of the operators (or oscillators in mode expansions of $X^{\hat\mu}$) coming from the right moving sector such as $\partial X^{\hat\mu}$ and from the left moving sector, $\bar\partial X^{\hat\mu'}$, or powers of ∂, $\bar\partial$ acting on $X^{\hat\mu}$. It facilitates our future computation and will be useful notation when we dwell on duality symmetry in sequel. If we demand $\hat\phi_1$ to be a $(1,1)$ primary, with respect to $T_{\pm\pm}$, then $V_1^{(i)}$ are are constrained (actually the $X^{\hat\mu}$-dependent tensors,$A^{(i)}$ are restricted). It is a straight forward calculation to obtain these conditions. We follow the methods of [12, 13] and summarize the relevant results below. These will be utilized when we explore the associated T-duality properties of these vertex operators for the compactified scenario. Note that each one of the functions, $(V_1^{(2)} - V_1^{(4)})$, is not $(1,1)$ on its own; however, $V^{(1)}$ is $(1,1)$ as is easily verified. Second point, we mention in passing, is that conformal invariance imposes two types of constraints on these vertex functions. We designate $A_1^{(i)}$ or $V_1^{(i)}$ as vertex functions to distinguish them from full vertex operator for a given level, like $\hat\phi_1$ for the first excited sate, which is expressed as sum of vertex functions : each one satisfies a mass-shell condition (recall that same is true for tachyon and all massless vertex operators) and gauge (or transversality) conditions which is also known for all the massless sectors. These are listed below

$$(\hat{\nabla}^2-2)A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'}(X)=0,\ \ (\hat{\nabla}^2-2)A^{(2)}_{\hat{\mu}\hat{\nu},\hat{\mu}'}(X)=0, \tag{2.7}$$

and

$$(\hat{\nabla}^2-2)A^{(3)}_{\hat{\mu},\hat{\mu}'\hat{\nu}'}(X)=0,\ \ (\hat{\nabla}^2-2)A^{(4)}_{\hat{\mu},\hat{\mu}'}(X)=0 \tag{2.8}$$

The $\hat{D}$-dimensional laplacian, $\hat{\nabla}^2$, is defined in term of the flat spacetime metric. The mass levels are in in units of the string scale which has been set to *one* in eqs.(2.7) and (2.8). The four vertex functions are also related through following equations

$$A^{(2)}_{\hat{\mu}\hat{\nu},\hat{\mu}'}=\partial^{\hat{\nu}'}A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'},\ \ A^{(3)}_{\hat{\mu},\hat{\mu}'\hat{\nu}'}=\partial^{\hat{\nu}}A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'},\ \ A^{(4)}_{\hat{\mu},\hat{\mu}'}=\partial^{\hat{\nu}'}\partial^{\hat{\nu}}A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'} \tag{2.9}$$

Here $\partial^{\hat{\mu}}$ etc. stand for partial derivatives with respect to spacetime coordinates. Furthermore, besides eqs. (2.7),(2.8) and eq. (2.9) there are further constraints (like gauge conditions) which also follow from the requirements of that the vertex functions be $(1,1)$ primaries [12, 13]

$$A^{(1)\hat{\mu}}_{\hat{\mu}}{}_{,\hat{\mu}'\hat{\nu}'}+2\partial^{\hat{\mu}}\partial^{\hat{\nu}}A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'}=0,\ \ \text{and}\ \ A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'}{}^{\hat{\mu}'}+2\partial^{\hat{\mu}'}\partial^{\hat{\nu}'}A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'}=0 \tag{2.10}$$

The above relations, eq.(2.9) and eq.(2.10), will be useful for our investigation of the duality in what follows.

Let us very briefly recapitulate how the T-duality group $O(d,d)$ plays an important role in the worldsheet Hamiltonian description of a closed string compactified on T^d. We shall proceed in two steps. First consider a simple compactification scheme [23]where the metric and and the 2-form decompose as follows

$$G_{\hat{\mu}\hat{\nu}}(X)=\begin{pmatrix} g_{\mu\nu}(X) & 0 \\ 0 & G_{\alpha\beta}(X)\end{pmatrix},\ \ B_{\hat{\mu}\hat{\nu}}=\begin{pmatrix} b_{\mu\nu}(X) & 0 \\ 0 & B_{\alpha\beta}(X)\end{pmatrix} \tag{2.11}$$

We shall consider the following action. We intend to go over to canonical Hamiltonian description; note that γ^{ab} is already chosen to be ON gauge metric).

$$S=\frac{1}{2}\int d\sigma d\tau\Big(\gamma^{ab}\sqrt{-\gamma}G_{\hat{\mu}\hat{\nu}}(X)\partial_a X^{\hat{\mu}}\partial X^{\hat{\nu}}+\varepsilon^{ab}B_{\hat{\mu}\hat{\nu}}(X)\partial_a X^{\hat{\mu}}\partial_b X^{\hat{\nu}}\Big) \tag{2.12}$$

We introduce a pair of vectors $\mathscr{V}$ and $\mathscr{W}$ of dimensions $2D$ and $2d$ respectively as defined below

$$\mathscr{V}=\begin{pmatrix}\tilde{P}_\mu \\ X'^\mu\end{pmatrix},\ \ \mathscr{W}=\begin{pmatrix}P_\alpha \\ Y'^\alpha\end{pmatrix} \tag{2.13}$$

where $\tilde{P}_\mu$ and P_α are conjugate momenta of string coordinates X^μ and Y^α respectively. The canonical Hamiltonian is expressed as sum of two terms

$$\mathscr{H}_c=\frac{1}{2}\Big(\mathscr{V}^T\tilde{M}\mathscr{V}+\mathscr{W}^TM\mathscr{W}\Big) \tag{2.14}$$

whereas $\tilde{M}$ is a $2D\times 2D$ matrix and M is another $2d\times 2d$ matrix, given by

$$\tilde{M}=\begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}b_{\rho\nu} \\ b_{\mu\rho}g^{\rho\nu} & g_{\mu\nu}-b_{\mu\rho}g^{\rho\lambda}b_{\lambda\nu}\end{pmatrix},\quad M=\begin{pmatrix} G^{\alpha\beta} & -G^{\alpha\gamma}B_{\gamma\beta} \\ B_{\alpha\gamma}G^{\gamma\beta} & G_{\alpha\beta}-B_{\alpha\gamma}G^{\gamma\delta}B_{\delta\beta}\end{pmatrix} \tag{2.15}$$

Let us focus on the second term of (2.14), define it to be $H_2=\frac{1}{2}\mathscr{W}^TM\mathscr{W}$, which is of importance to us. Under the global $O(d,d)$ transformations

$$M\rightarrow \Omega M\Omega^T,\ \mathscr{W}\rightarrow\Omega\mathscr{W},\ \Omega^T\eta\Omega=\eta,\ \Omega\in O(d,d),\ \eta=\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix} \tag{2.16}$$

η is the $O(d,d)$ metric and $\mathbf{1}$ is $d\times d$ unit matrix and $\mathscr{W}$ is the $O(d,d)$ vector and $M\in O(d,d)$. Since, $\tilde{M}$ and $\mathscr{V}$ are *inert* under this duality transformation; as a consequence, $\mathscr{H}_c$ is indeed T-duality invariant. The moduli, $\tilde{M}$ and M are classical backgrounds. However, we shall work in the weak field approximation: $G_{\alpha\beta}=\delta_{\alpha\beta}+h_{\alpha\beta}$. Let us focus on the $O(d,d)$ invariance of graviton vertex operator along compact directions

$$V_h=h_{\alpha\beta'}\partial Y^\alpha\bar{\partial}Y^{\beta'} \tag{2.17}$$

$h_{\alpha\beta'}$ is a symmetric tensor. Noting that $P_\alpha=\delta_{\alpha\beta}\dot{Y}^\alpha$, we rewrite (2.17) as

$$V_h=h^{\alpha\beta'}P_\alpha P_{\beta'}-h_{\alpha\beta'}Y'^\alpha Y'^{\beta'} \tag{2.18}$$

This is expressed in $O(d,d)$ invariant form

$$V_h=\mathscr{W}^T\mathbf{H}\mathscr{W},\ \ \mathbf{H}=\begin{pmatrix}h & 0\\ 0 & -h\end{pmatrix} \tag{2.19}$$

$\mathbf{H}\in O(d,d)$ and transforms according to (2.16); the appropriate assignment of indices can be read off from (2.18). We can repeat the same procedure for the combined vertex of graviton and antisymmetric tensor, $b_{\alpha\beta'}$.

Let us examine T-duality properties of the first excited massive level where we adopt a simple compactification scheme. We focus the attention on $V_1^{(1)}$ as an example. Note that if we follow the toroidal comactification scheme adopted in [6] in the context of worldsheet duality for the case at hand the vertex function $A^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'}(X)\partial X^{\hat\mu}\partial X^{\hat\nu}\bar\partial X^{\hat\mu'}\bar\partial X^{\hat\nu'}$ will decompose into following forms: (i) A tensor $A^{(1)}_{\mu\nu,\mu'\nu'}$, one which has all Lorentz indices (ii) another which has three Lorentz indices and one index corresponding to compact directions, (iii) a tensor with two Lorentz indices and two indices in compact directions, (iv) another, which has a single Lorentz index and three indices in in internal directions and (v) a tensor with all indices corresponding to compact directions i.e. $A^{(1)}_{\alpha\beta,\alpha'\beta'}$. It is obvious these tensors will be suitably contracted with $\partial X^\mu,\bar\partial X^\mu,\partial Y^\alpha,\bar\partial Y^\alpha$ with all allowed combinations. We adopt, to start with, a compactification scheme where only $A^{(1)}_{\alpha\beta,\alpha'\beta'}$ is present and the tensors with mixed indices are absent. We shall return to more general case later. We may allow the presence of $A^{(1)}_{\mu\nu,\mu'\nu'}$; note however, that its presence is not very essential for the discuss of T-duality symmetry since the spacetime tensors and coordinates are assumed be to inert under the T-duality transformations, as a consequence this term will be duality invariant on its own right. Therefore, we shall deal with a single vertex function to discuss T-duality symmetry as a prelude

$$V_1^{(1)}=A^{(1)}_{\alpha\beta,\alpha'\beta'}(X)\partial Y^\alpha\partial Y^\beta\bar\partial Y^{\alpha'}\bar\partial Y^{\beta'} \tag{2.20}$$

As argued earlier, if we expand out the expression for $V_1^{(1)}$, eq.(2.20), in terms of P_α and Y'^α we get terms of the following type contacted with the tensor $A^{(1)}_{\alpha\beta,\alpha'\beta'}(X)$; note that we do not use any symmetry(antisymmetry) properties of this tensor under $\alpha \leftrightarrow \beta$ and $\alpha' \leftrightarrow \beta'$. Moreover, although we express the vertex in terms of Y' and P, we still like to retain the memory whether these terms came from left movers or right movers. The full expression for the vertex function is classified into five types. These are listed below: (I) All are P^α's (index raised by $\delta^{\alpha\beta}$): $A^{(1)}_{\alpha\beta,\alpha'\beta'}(X)P^\alpha P^\beta P^{\alpha'} P^{\beta'}$.
(II) All are Y'^α's: $A^{(1)}_{\alpha\beta,\alpha'\beta'}(X)Y'^\alpha Y'^\beta Y'^{\alpha'} Y'^{\beta'}$.
(III) The four terms with three P^α's;e.g - $A^{(1)}_{\alpha\beta,\alpha'\beta'}(X)P^\alpha P^\beta P^{\alpha'} Y'^{\beta'}$.
(IV) There are also four terms with three Y'^α's which combine with the terms in (III) to give a T-duality invariant term; one such term is: -$A^{(1)}_{\alpha\beta,\alpha'\beta'}(X)Y'^\alpha Y'^\beta P^{\alpha'} Y'^{\beta'}$.
(V) There are six terms, each of is a product of a pair of momenta (P^α and a pair Y'^α. A generic term is: $A^{(1)}_{\alpha\beta,\alpha\beta'}P^\alpha P^\beta Y'^{\alpha'} Y'^{\beta'}$.
We conclude from careful inspections of altogether 16 terms that, we can combine terms in (I) and (II), those in (III) and (IV) and those in (V) to compose $O(d,d)$ invariant functions. However, this is not an efficient method.
Let us consider the following three vertex functions in the present compactification scheme

$$V_1^{(2)} = A^{(2)}_{\alpha\beta,\alpha'}(X)\partial Y^\alpha \partial Y^\beta \bar\partial^2 Y^{\alpha'}, \quad V_1^{(3)} = A^{(3)}_{\alpha,\alpha'\beta'}(X)\partial^2 Y^\alpha \partial Y^{\alpha'} \partial Y^{\beta'} \tag{2.21}$$

and

$$V_1^{(4)} = A^{(4)}_{\alpha,\alpha'}(X)\partial^2 Y^\alpha \bar\partial^2 Y^{\alpha'} \tag{2.22}$$

It follows from (2.9) that these vertex functions are related to derivatives of $A^{(1)}$ as a consequence of the constraints that they be $(1,1)$ primaries. For the case in hand, when these carry all internal indices and our focus is on (2.20), they will vanish. Moreover, in order to study T-duality properties, of (2.21) and (2.22) we encounter another problem since higher order derivatives, $\partial, \bar\partial$ act on Y; consequently, the classification scheme adopted to group terms as in (I)-(V), discussed above are not quite suitable.
Furthermore, we must recognize that, verifying T-duality symmetry for higher excited states will provide obstacles since we have to deal with product of a string of $\partial Y, \bar\partial Y$ and more and more worldsheet derivatives acting on Y^α when we consider higher mass levels. One of the vertex functions for second massive level is

$$V_2^{(1)} = C^{(1)}_{\alpha\beta\gamma,\alpha'\beta'\gamma'}\partial Y^\alpha \partial Y^\beta \partial Y^\gamma \bar\partial Y^{\alpha'} \bar\partial Y^{\beta'} \bar\partial Y^{\gamma'} \tag{2.23}$$

expressed as products of P^α, Y'^α, $P^{\beta'}$ and $Y^{\beta'}$, all the terms can be reorganized and reexpressed in such a way that $V_2^{(1)}$ is T-duality invariant. However, second level has lot more terms and it is not easy to verify whether these vertex functions are T-duality invariant following the method alluded to above.
If one considers vertex operators for higher massive levels, the vertex operator for each level is composed of large number of vertex functions. We propose the following procedure to systematically organize various vertex functions at a given level. (i) The first observation is that the basic building blocks of vertex functions are $\partial Y^\alpha = P^\alpha + Y'^\alpha$ and $\bar\partial Y^{\alpha'} = P^{\alpha'} - Y'^{\alpha'}$. (ii) Each vertex

function at a given level is either string of products of these basic blocks or these blocks are operated by ∂ and $\bar{\partial}$ respectively so that each vertex operator at a given mass level has the desired dimensions. Thus it is not convenient to deal with P^α and Y'^α separately in order to study the T-duality properties and the same is true for the combinations $P \pm Y'$. However, P^α and Y'^α can be projected out from the $O(d,d)$ vector, $\mathscr{W}$.

Let us first introduce following projection operators for later conveniences

$$\mathbf{P}_\pm = \frac{1}{2}(\mathbf{1} \pm \tilde{\sigma_3}), \quad \tilde{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.24}$$

where $\mathbf{1}$ is $2d \times 2d$ unit matrix and and the diagonal entries $(1,-1)$ stand for $d \times d$ unit matrices. It is easy to check that the projection operators are $O(d,d)$ matrices since each one of them is. We project out two $O(d,d)$ vectors as follows

$$P = \mathbf{P}_+ \mathscr{W}, \quad Y' = \mathbf{P}_- \mathscr{W} \tag{2.25}$$

Therefore,

$$P + Y' = \frac{1}{2}\Big(\mathbf{P}_+ \mathscr{W} + \eta \mathbf{P}_- \mathscr{W}\Big), \quad P - Y' = \frac{1}{2}\Big(\mathbf{P}_+ \mathscr{W} - \eta \mathbf{P}_- \mathscr{W}\Big) \tag{2.26}$$

notice that η flips lower component Y' vector to an upper component one. Thus when we have only products of $P+Y'$ and $P-Y'$, we can express them first as products of $O(d,d)$ vector and subsequently contract their indices with appropriate tensors endowed with $O(d,d)$ indices. Next we deal with worldsheet partial derivatives ∂ and $\bar{\partial}$ operating on basic building blocks. Let us define

$$\Delta_\tau = \mathbf{P}_+ \partial_\tau, \quad \Delta_\sigma = \mathbf{P}_+ \partial_\sigma \quad \text{and} \quad \Delta_\pm(\tau,\sigma) = \frac{1}{2}(\Delta_\tau \pm \Delta_\sigma) \tag{2.27}$$

Therefore,

$$\partial(P+Y') = \Delta_+(\tau,\sigma)\Big(\mathbf{P}_+ \mathscr{W} + \eta \mathbf{P}_- \mathscr{W}\Big) \tag{2.28}$$

Thus the above expression is an $O(d,d)$ vector. Similarly, when $\bar{\partial}$ operates on $P-Y'$, we can express it as

$$\bar{\partial}(P-Y') = \Delta_-(\tau,\sigma)\Big(\mathbf{P}_+ \mathscr{W} - \eta \mathbf{P}_- \mathscr{W}\Big) \tag{2.29}$$

The vertex operators we have considered in eqs.(2.20) and (2.23) which are expressed as only string of products of ∂Y^α and $\bar{\partial} Y^{\alpha'}$ can be rewritten in terms of the $O(d,d)$ vectors $\mathscr{W}$ and subsequently contracted with suitable $O(d,d)$ tensors. We remind the reader that, now familiar, M-matrix which expresses the Hamiltonian in $O(d,d)$ invariant form is also parametrized in terms of backgrounds $G_{\alpha\beta}$ and $B_{\alpha\beta}$. Let us turn our attentions to the other three vertex operators appearing in (2.21) and (2.22). The procedure outlined above can be adopted to cast $V_1^{(2)}, V_1^{(3)}$ and $V_1^{(4)}$ in a straight forward manner using the relations (2.28) and (2.29).

In order to illustrate the variety of vertex functions that can arise as we go to higher levels; let us consider the second massive state as an example. We list below the vetex function which is assumed to be sum of all vertex functions with *only* internal indices, momentarily assume all other vertex functions are set to zero and this is the vertex operator. Thus vertex operator assumes the form [13]

$$\phi_2 = V_2^{(1)} + V_2^{(2)} + V_2^{(3)} + V_2^{(4)} + V_2^{(5)} + V_2^{(6)} \tag{2.30}$$

The vertex functions $V_2^{(i)}, i = 1,6$ are given below (expression for $V_2^{(1)}$ is given by (2.23)).

$$V_2^{(2)} = C^{(2)}_{\alpha\beta,\alpha'\beta'\gamma'} \partial^2 Y^\alpha \partial Y^\beta \bar\partial Y^{\alpha'} \bar\partial Y^{\beta'} \bar\partial Y^{\gamma'} + C^{(3)}_{\alpha\beta\gamma,\alpha'\beta'} \partial Y^\alpha \partial Y^\beta \partial Y^\gamma \bar\partial^2 Y^{\alpha'} \bar\partial Y^{\beta'} \tag{2.31}$$

These vertex functions have a term of the form $\partial^2 Y^\alpha$ or $\bar\partial^2 Y^{\alpha'}$ and rest of the structure is decided by dimensional considerations. Some of the other vertex functions are

$$\begin{aligned} V_2^{(3)} &= C^{(4)}_{\alpha,\alpha'\beta'\gamma'} \partial^3 Y^\alpha Y^{\alpha'} \bar\partial Y^{\beta'} \bar\partial Y^{\gamma'} + C^{(5)}_{\alpha\beta\gamma,\alpha'} \partial Y^\alpha \partial Y^\beta \partial Y^\gamma \bar\partial^3 Y^{\alpha'}, \\ V_2^{(4)} &= C^{(6)}_{\alpha\beta,\alpha'\beta'} \partial^2 Y^\alpha \partial Y^\beta \bar\partial^2 Y^{\alpha'} \bar\partial Y^{\beta'} \end{aligned} \tag{2.32}$$

and

$$V_2^{(5)} = C^{(7)}_{\alpha\alpha'\beta'} \partial^3 Y^\alpha \bar\partial^2 Y^{\alpha'} \bar\partial Y^{\beta'} + C^{(8)}_{\alpha\beta,\alpha'} \partial^2 Y^\alpha \partial Y^\beta \bar\partial^3 Y^{\alpha'},\ V_2^{(6)} = C^{(9)}_{\alpha,\alpha'} \partial^3 Y^\alpha \bar\partial^3 Y^{\alpha'} \tag{2.33}$$

The tensors $C^{(2)} - C^{(9)}$ appearing in eqs.(2.31-2.33) are all functions of X^μ, independent of compact coordinates Y^α, and constrained by requirements of conformal invariance (not necessarily nonvanishing in the compactification scheme we envisage). We observe from the structures of vertex functions $V_2^{(1)} - V_2^{(6)}$ that, each one with the combinations of the terms will be $O(d,d)$ invariant when we follow the prescriptions of introducing projection operators, rewrite the combinations $P+Y'$ and $P-Y'$ as $O(d,d)$ vectors and convert ∂ and $\bar\partial$ to $\Delta_\pm(\tau,\sigma)$ to operate on $P \pm Y'$ (reexpressed in terms of the projected $\mathcal{W}$'s) respectively. Let us consider n^{th} excited massive level as an example. The the dimension of all right movers obtained from products of ∂Y higher powers of ∂ acting on ∂Y should be $(n+1)$ and same hold good for the left moving sector as well. Consider the right moving sector of the type $\partial Y^{\alpha_1} \partial Y^{\alpha_2} .. \partial Y^{\alpha_{n+1}}$ and the left moving sector $\bar\partial Y^{\alpha'_1} \bar\partial Y^{\alpha'_2} .... \bar\partial Y^{\alpha'_{n+1}}$ The vertex function is

$$V_{\alpha_1,\alpha_2...\alpha_{n+1},\alpha'_1\alpha'_2...\alpha'_{n+1}}(X) \Pi_1^{n+1} \partial Y^{\alpha_i} \Pi_1^{n+1} \bar\partial Y^{\alpha'_i} \tag{2.34}$$

and these products of ∂Y^{α_i} and $\bar\partial Y^{\alpha'_i}$ can be converted to products of $(n+1)$ projected $\mathcal{W}$ for right movers and $(n+1)$ projected $\mathcal{W}$ from left movers. Let us consider for a vertex function for such a high level state. A generic vertex will have a structure

$$\partial^p Y^{\alpha_i} \partial^q Y^{\alpha_j} \partial^r Y^{\alpha_k} ... \bar\partial^{p'} Y^{\alpha'_i} \bar\partial^{q'} Y^{\alpha'_j} \bar\partial^{r'} Y^{\alpha'_k} ...,\ \ p+q+r = n+1,\ p'+q'+r' = n+1 \tag{2.35}$$

The product is an $O(d,d)$ tensor whose rank is decided by the constraints on sum of p,q and r and p',q' and r' since number of Y^{α_i}'s and $Y^{\alpha'_i}$'s appearing in (2.35) is determined from those conditions. Thus this tensor will be contracted with an appropriate tensor $T_{\alpha_i\alpha_j\alpha_k..,\alpha'_i\alpha'_j\alpha'_k..}(X)$ which will give us to a vertex function. Let us discuss how to express eq.(2.35) as a product of $O(d,d)$ vectors using the projection operators introduced earlier.

(i) The first step is to rewrite
$\partial^p Y = \partial^{p-1}(P+Y'), \quad \bar\partial^{p'}(P-Y') = \bar\partial^{p'-1}(P-Y')$
(ii) We arrive at
$\partial^{p-1}(P+Y') = {\Delta_+}^{p-1}(P+Y'),\ \bar\partial^{p'-1}(P-Y') = {\Delta_-}^{p'-1}(P-Y')$
from (2.28) and (2.29)
(iii) Finally, using the projection operators (2.26) we get
${\Delta_+}^{p-1}(P+Y') = {\Delta_+}^{p-1}\Big(\mathbf{P}_+\mathscr{W} + \eta\mathbf{P}_-\mathscr{W}\Big),\ \ \Delta^{p'-1}(P-Y') = {\Delta_-}^{p'-1}\Big((\mathbf{P}_+\mathscr{W} - \eta\mathbf{P}_-\mathscr{W}\Big)$
Thus the products in (2.35) can be expressed as products of $O(d,d)$ vectors. We need to contract these indices with suitable $O(d,d)$ tensors which have the following form:

$$V_{n+1} = \mathscr{A}_{klm..,k'l'm'..}{\Delta_+}^{p-1}\mathscr{W}_+^k\Delta^{q-1}\mathscr{W}_+^l{\Delta_+}^{r-1}\mathscr{W}_+^m..{\Delta_-}^{p'-1}\mathscr{W}_-^{k'}{\Delta_-}^{p'-1}\mathscr{W}_-^{l'}{\Delta_-}^{p'-1}\mathscr{W}_-^{m'} \tag{2.36}$$

where $\mathscr{W}_\pm = (\mathbf{P}_+\mathscr{W} \pm \eta\mathbf{P}_-\mathscr{W})$ with $p+q+r = n+1$ and $p'+q'+r' = n+1$. Note that superscripts $\{k,l,m;k',l',m'\}$ appearing on $\mathscr{W}_\pm$ in eq. (2.36) are the indices of the components of the projected $O(d,d)$ vectors. Moreover, $\mathscr{A}_{klm..,k'l'm'..}$ is X-dependent $O(d,d)$ tensor. Note that (2.36) will be $O(d,d)$ invariant if coefficients transform as follows

$$\mathscr{A}_{klm..,k'l'm'..} \to \Omega_k^p\Omega_l^q\Omega_m^r...\Omega_{k'}^{p'}\Omega_{l'}^{q'}\Omega_{m'}^{r'}\mathscr{A}_{pqr..,p'q'r'..} \tag{2.37}$$

since each term in the product $\Delta^{p-1}\mathscr{W}_+^k....\Delta_-^{p'-1}\mathscr{W}^{k'}$, above, transforms like an $O(d,d)$ vector. Now we turn our attention in another direction. Note that ϕ_2 was expressed as sum of vertex functions where tensors with only internal indices were contracted with various types of derivatives of Y^α. All these levels are scalars under $SO(D-1)$. However, once we allow tensors appearing in vertex functions to carry Lorentz indices, these tensors will be contracted with derivatives of X^μ and internal indices will contract with derivatives of compact coordinates. We claim that those vertex functions which have expressions with contraction of Lorentz indices with X^μ's will be $O(d,d)$ invariant with respect to rest of the tensor indices contracted with indices of Y^α's. Let us consider the first excited massive level to illustrate our strategy which can be generalized to any level. We claim that full vertex operator, for this level, are $O(d,d)$ invariant. We recall X^μ and tensors with only spacetime indices (i.e. $\mu,\nu,..$etc.) a tensor transform trivially under the T-duality for these set of indices. Thus

$$\tilde V_1^{(1)} = \tilde A^{(1)}_{\mu\nu,\mu'\nu'}\partial X^\mu\partial X^\nu\bar\partial X^{\mu'}\bar\partial X^{\nu'} \tag{2.38}$$

is $O(d,d)$ invariant as per above prescription. Similarly, vertex function constructed out of :
$\tilde A^{(2)}_{\mu,\mu'\nu'}\partial^2X^\mu\bar\partial X^{\mu'}\bar\partial X^{\nu'}$, $\tilde A^{(3)}_{\mu\nu,\mu'}\partial X^\mu\partial X^\nu\bar\partial^2X^{\mu'}$ and $\tilde A^{(3)}_{\mu,\mu'}\partial^2X^\mu\bar\partial^2X^{\mu'}$ are also $O(d,d)$ invariant. Let us classify the vertex functions according to the spacetime and 'internal' indices they carry (with appropriate contractions of course).
(A) Vertex functions which have one Lorentz index and three internal indices:
$\tilde B^{(1)}_{\mu\alpha,\alpha'\beta'}\partial X^\mu\partial Y^\alpha\bar\partial Y^{\alpha'}\bar\partial Y^{\beta'} +$ other terms by permuting the indices.
(B) Vertex functions which have two Lorentz indices and two internal indices:
$\tilde B^{(2)}_{\mu\beta,\mu'\beta'}\partial X^\mu\partial Y^\beta\bar\partial X^{\mu'}\bar\partial Y^{\beta'} +$ other similar terms.
(C) Vertex functions with three Lorentz indices and one internal index:

$\tilde{B}^{(3)}_{\mu\nu,\mu'\beta'}\partial X^\mu \partial X^\nu \bar{\partial} X^{\mu'} \bar{\partial}^2 Y^{\beta'} +$ other similar terms.
(D) Vertex functions of the type:
(i)$\tilde{B}^{(4)}_{\mu\nu,\alpha'}\partial X^\mu \partial X^\nu \bar{\partial} Y^{\beta'} +$ other similar terms.
(ii) $\tilde{B}^{(5)}_{\mu\beta,\alpha'}\partial X^\mu \partial Y^\alpha \bar{\partial}^2 Y^{\beta'} +$ other similar terms.
(iii) Vertex functions with second derivatives:
$\tilde{B}^{(6)}_{\alpha,\mu'}\partial^2 Y^\alpha \bar{\partial}^2 X^{\mu'}$ and $\tilde{B}^{(7)}_{\mu,\alpha'}\partial^2 X^\mu \bar{\partial}^2 Y^{\alpha'}$
The vertex functions whose Lorentz index/indices are contracted with $\partial X^\mu, \partial^2 X^\mu$, $\bar{\partial} X^{\mu'}, \partial^2 X^\mu$ will be inert under $O(d,d)$ rotations; however, rest of the indices correspond to internal indices and those are contracted with $\partial Y^\alpha, \bar{\partial} Y^{\alpha'}, \partial^2 Y^\alpha, \bar{\partial}^2 Y^{\alpha'}$ and so on. Moreover, the vertex functions considered above, (A)-(D), do not necessarily vanish unlike the cases when some vertex function, carrying only internal indices ($V_1^{(2)}$ - $V_1^{(4)}$), vanished as the consequences of conformal invariance i.e. that these are $(1,1)$ primaries. This conclusion can be easily verified from relations eqs. (2.9) and (2.10). We conclude that only the worldsheet variables with internal indices, such as $P \pm Y'$ are relevant to construct $O(d,d)$ vectors which contract with corresponding indices of the relevant tensors. We have laid down a procedures to construct $O(d,d)$ vectors from $\partial Y^\alpha, \bar{\partial} Y^{\alpha'}, \partial^2 Y^\alpha, \bar{\partial}^2 Y^{\alpha'}$ and other higher derivative objects. For example, $\tilde{B}^{(1)}_{\mu\alpha,\alpha'\beta'}\partial X^\mu \partial Y^\alpha \bar{\partial} Y^{\alpha'} \bar{\partial} Y^{\beta'}$ has three 'internal' indices of $\tilde{B}^{(1)}$ contracted with $\partial Y^\alpha \bar{\partial} Y^{\alpha'} \bar{\partial} Y^{\beta'}$ and therefore, this vertex function will be converted to an $O(d,d)$ invariant vertex function which has a generic form

$$\tilde{T}^{(1)}_{k,k'l'}\mathscr{W}^k \mathscr{W}^{k'} \mathscr{W}^{l'} \tag{2.39}$$

This argument can be carried forward for all vertex operators at any massive level of the closed bosonic string. Moreover, the type of vertex functions discussed in (A)-(D) correspond to massive particles of various spins which fall into the representations of $SO(D-1)$. Therefore, we are able to conclude that vertex functions for massive levels of a closed bosonic string can be cast in an $O(d,d)$ invariant form for every level following the procedure presented here.

3. SUMMARY AND DISCUSSION

We have proposed a systematic procedure to obtain T-duality invariant vertex operators for massive levels of a closed bosonic string when it is compactified on T^d. It is assumed that the tensor fields associated with these vertex operators depend only on the spacetime coordinates, $X^\mu(\sigma,\tau)$ and are independent of the compact coordinates, $Y^\alpha(\sigma,\tau)$. The duality invariance is manifest for vertex operators of each level aided by the projection technique to convert $\{P, Y'\}$ to $O(d,d)$ vectors and/or their $\Delta_\pm$ derivatives.
The T-duality symmetry plays an important role in string theory. We expect that these symmetry properties will have important applications. The T-duality symmetry has been widely applied to obtain new solutions to the background configurations through judicious implementations of the solution generating techniques. Thus given a configuration of massive level background field it will be possible, in principle, to generate another background within the same massive level. Furthermore, there are evidences that massive excited states are endowed with local symmetries. It is worth while to examine the implications of T-duality for those local symmetries.
Another point which deserves attention is to study the zero-norm states in this formulation. It is

well known that the existence of zero-norm states is quite essential in order that the bosonic string respects Lorentz invariance in critical dimensions i.e. $\hat{D} = 26$. This issue has been carefully analyzed in [13, 25]. We expect that these results will continue to hold good for toroidally compactified closed bosonic string.

It is well known that very massive stringy states possess exponential degeneracy which has played crucial role in deriving Bekenstein-Hawking entropy relation for stringy back holes from the counting of microscopic states. This high degree of degeneracy is also instrumental in deducing the thermal nature of emission spectrum of a stringy black hole. We expect that some of supermassive states which also belong to the spectrum of the compactfied string might exhibit symmetry properties which are yet to be discovered.

In summary, we have investigated T-duality properties of the vertex operators of excited massive closed string. We have proposed a prescription to show that the vertex operators at every level can be expressed in manifestly duality invariant form. These results might have important consequences to discover new stringy symmetries.

It will be very interesting to implement the toroidal compactification procedure adopted in [6] for the vertex operators of excited states extending the method presented here. There are need to improve this prescription considerably in order to overcome some difficulties. However, the toroidal compactification in its totality applied to vertex operators through dimensional reduction is expected to unravel more interesting features of T-duality in string theory.

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