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The Integrand Reduction of One- and Two-Loop Scattering Amplitudes

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The integrand-level methods for the reduction of scattering amplitudes are well-established techniques, which have already proven their effectiveness in several applications at one-loop. In addition to the automation and refinement of tools for one-loop calculations, during the past year we observed very interesting progress in developing new techniques for amplitudes at two- and higher-loops, based on similar principles. In this presentation, we review the main features of integrand-level approaches with a particular focus on algebraic techniques, such as Laurent series expansion which we used to improve the one-loop reduction, and multivariate polynomial division which unveils the structure of multi-loop amplitudes.

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1. Overview

Scattering amplitudes are analytic functions of the momenta of the particles involved and they can be studied by exploring their singularity structure [1, 2]. The investigation of the residues at the poles, which correspond to particles going on their mass-shell, led to the discovery of new important relations. The BCFW recurrence relation [3], its link to the leading singularity of one-loop amplitudes [4], and the OPP integrand-decomposition formula for one-loop integrals [5] have shown the underlying simplicity beneath the rich mathematical structure of quantum field theory. Moreover they provided the theoretical framework to develop efficient computational techniques at the next-to-leading order in perturbation theory [6–13].

The integrand reduction methods, originally developed for one-loop diagrams [5], use the singularity structure of the integrands to decompose the (integrated) amplitudes in terms of Master Integrals (MI's). The multi-particle pole expansion of the integrand is equivalent to the decomposition of the numerator in terms of products of denominators, multiplied by polynomials which correspond to the residues at the multiple-cuts.

The parametric form of the polynomial residues is *process-independent* and can be determined once and for all from the structure of the corresponding multiple cut. The actual value of the coefficients which appear in the residues is instead *process-dependent* and, in the framework of the integrand-reduction their determination is achieved by *polynomial fitting*, through the evaluation of the (known) integrand at values of the loop-momenta fulfilling the cut conditions [14].

Extensions of the integrand reduction method beyond one-loop, first proposed in [14, 15], have been systematized within the mathematical framework of multivariate polynomial division and basic principles of algebraic geometry [16, 17]. Recently they become the target of several new developments [18–20], thus giving birth to a new direction in the study of multi-loop amplitudes. An alternative approach to two-loop calculations based on maximal unitarity has been pursued in [21–24].

2. Automation of One-Loop Calculations

The continuous improvement of new techniques for one-loop computations led to tremendous progress in the field of NLO QCD corrections [25]. Calculations of increasing complexity have been performed with improved algebraic reduction methods based on Feynman-diagrammatic algorithms, as well as with new numerical techniques based on the idea of reconstructing one-loop amplitudes from their unitarity cuts. These theoretical developments found an ideal counterpart in the integrand-level reduction algorithm.

The GoSam framework [10,26] combines the automated algebraic generation of d-dimensional unintegrated amplitudes obtained via Feynman diagrams, with the numerical integrand-level reduction. Concerning the reduction, GoSam allows to choose at run-time (namely without regenerating the code) among integrand-level d-dimensional reduction [5, 27], as implemented in SAMURAI [28], or traditional tensor reduction interfaced through tensorial reconstruction [29, 30]. The coefficients determined by the reduction are then multiplied by the corresponding scalar (master) integrals [31]. Recent examples of full NLO QCD calculations performed within this framework include $pp \to b\bar{b}b\bar{b}$ [32] and $pp \to W^+W^- + 2\,\text{jet}$ [33] at the LHC.

3. Integrand-Reduction via Laurent Expansion

An improved version of integrand-reduction method for one-loop amplitudes was presented in [34], elaborating on the the techniques proposed in [35, 36]. This method allows, whenever the analytic form of the numerator is known, to extract the unknown coefficients of the integrand decomposition by performing a Laurent expansion.

In general, the multiple-cut conditions constrain the loop momentum. Therefore, the on-shell solutions are parametrized by those components which are not completely determined in terms of the external kinematics.

The original reduction algorithm [5] requires, in these cases: to sample the numerator on a finite subset solutions; to subtract from the integrand all the non-vanishing contributions coming from higher-point residues; and finally to solve a linear system of equations in order to find the value of the unknown coefficients parametrizing the residue of the cut.

This algorithm can be simplified by exploiting the knowledge of the analytic expression of the integrand. Indeed, by performing a *Laurent expansion* with respect to one of the free parameters which appear in the solutions of the cut, both the integrand and the subtraction terms exhibit the same polynomial behavior of the residue. Moreover, the contributions coming from the subtraction terms can be implemented as *corrections at the coefficient level*, hence replacing the subtractions at the integrand level of the original algorithm. The parametric form of this corrections can be computed once and for all, in terms of a subset of the higher-point coefficients. With this method the number of coefficients entering in each subtraction term is significantly reduced. For instance, box and pentagons do not affect at all the computation of lower-points coefficients.

In summary, this method identifies the coefficients of a residue with the ones of the Laurent expansion of the numerator (with respect to one of the free components of the loop momentum which are not fixed by the cut conditions). The result must be corrected by a subtraction term which is a known function of the higher point coefficients. If either the analytic expression of the integrand or the tensor structure of the numerator is known, this procedure can also be implemented in a semi-numerical algorithm. Indeed, the coefficients of the Laurent expansion of a rational function can be computed, either analytically or numerically, by performing a polynomial division between the numerator and the denominator. This method has been implemented in a C++ library, and preliminary tests show an improvement in the computational performance with respect to the standard algorithm.

4. Higher-rank integrands

The integrand decomposition was originally developed for renormalizable gauge theories, where, at one-loop, the rank r of the numerator cannot be greater than the number of external legs n. In [34], we extended the decomposition to the case where the rank becomes larger than n. This extension is required, for instance, for applying the integrand reduction to the production of Higgs in combination with jets, in the gluon-fusion channel via effective-gluon vertex, generated by the large top-mass limit. As a first step along this direction, we implemented within SAMURAI the extension of the polynomial residues and the corresponding additional sampling required to fit their coefficients.

5. Integrand-Reduction for Two-Loop Scattering Amplitudes and beyond

The first extension of the *integrand reduction method* beyond one-loop was proposed in [14]. A key point of the higher-loop extension is the proper parametrization of the residues at the multi-particle poles. We define *irreducible scalar products* (ISP's) the set of scalar products, among the loop momenta and either external momenta or polarization vectors constructed out of them, which cannot be expressed in terms of denominators. Residues at the multi-particle poles can be written as a multivariate polynomial in the ISP's. Hence, a systematic classification of the polynomial structures of the residues is mandatory.

This task has been successfully achieved in [17], where we have shown that the shape of the residues is uniquely determined by the on-shell conditions alone, without any additional constraint. We have derived a simple *integrand recurrence relation* that generates the required multi-particle pole decomposition for arbitrary amplitudes, independently of the number of loops. The algorithm presented in [17] relies on general properties of the loop integrands

$$\mathscr{I}_{i_1\cdots i_n} = \frac{\mathscr{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} \,. \tag{5.1}$$

- When the number n of denominators D_i is larger than the total number of the components of the loop momenta, the *weak Nullstellensatz theorem* yields the trivial reduction of an n-denominator integrand in terms integrands with $(n-1)$ denominators.

- When n is equal or less than the total number of components of the loop momenta, we divide the numerator $\mathscr{N}_{i_1\cdots i_n}$ modulo the Gröbner basis of the n-ple cut, namely modulo a set of polynomials vanishing on the same on-shell solutions as the cut denominators. The *remainder* of the division is the *residue* $\Delta_{i_1\cdots i_n}$ of the n-ple cut. The *quotients* generate integrands with $(n-1)$ denominators which should undergo the same decomposition.

  This allows us to cast the each numerator $\mathscr{N}_{i_1\cdots i_n}$, sitting on a set of denominators D_i, in the form

  $$\mathscr{N}_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathscr{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n} D_{i_\kappa} + \Delta_{i_1\cdots i_n} \,, \tag{5.2}$$

  which inserted in the expression for the generic the n-denominator integrand, provides the aforementioned recurrence relation

  $$\mathscr{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{k} \mathscr{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} \,. \tag{5.3}$$

  We remark that the procedure, together with Eqs.(5.2) and (5.3), hold for any number of loops and in all dimensions.

- By iteration, we extract the polynomial forms of *all* residues. The algorithm will stop when all cuts are exhausted, and no denominator is left, leaving us with the complete integrand reduction formula.

In [17], we have also proved a theorem on the *maximum-cuts*, i.e. the cuts defined by the maximum number of on-shell conditions which can be simultaneously satisfied by the loop momenta.

The on-shell conditions of a maximum cut lead to a zero-dimensional system. The *Finiteness Theorem* and the *Shape Lemma* ensure that the residue at the maximum-cut is parametrized by n_s coefficients, where n_s is the number of solutions of the multiple cut-conditions. This guarantees that the corresponding residue can *always be reconstructed by evaluating the numerator at the solutions of the cut*. The maximum cut theorem generalizes at any loop the simplicity of the one-loop quadruple-cut [4,5], where the only two solutions of the cut univocally determine the two coefficients needed to parametrize the residue.

As a first application, we applied the algorithm to a generic one-loop integrand, reproducing the well-known d-dimensional integrand decomposition formula [5, 27]. Very recently [37], we have applied our algorithm to the two-loop five-point planar and non-planar amplitudes in $\mathscr{N}=4$ Super Yang-Mills (SYM) and $\mathscr{N}=8$ Supergravity (SUGRA). The numerators of the integrals have at most rank two in the integration momenta. In particular we perform the integrand reduction both semi-numerically, by polynomial fitting, and analytically. The latter computation has been performed generalizing the method of integrand reduction through Laurent expansion [34] discussed above.

6. Conclusions

The integrand reduction methods, which use the singularity structure of the integrands to decompose the (integrated) amplitudes in terms of Master Integrals, already proved their effectiveness in applications at the one-loop level. Several efforts are under way in order to extend this formalism to higher orders, starting with two-loop amplitudes.

We recently proposed a new approach for the reduction of scattering amplitudes, based on multivariate polynomial division. This technique yields the complete integrand decomposition for arbitrary amplitudes, regardless of the number of loops. We have shown that the shape of the residues is uniquely determined by the on-shell conditions, and we have derived a simple integrand recurrence relation that generates the multi-particle pole decomposition for arbitrary multi-loop amplitudes. We have successfully applied the new reduction algorithm to one-loop and two-loop examples.

The method is well suited for a numerical implementation, but it also allows for the extraction of full analytic results, thus providing a universal and powerful tool for the generalization of integrand-level reduction techniques to all orders in perturbation theory.

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