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Linear approximation in Møller gravity

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Linear approximation in the Møller gravity theory is presented. It is shown that in the case of linear approximation we can treat the frame field over the background as the usual metric field (gravity) and an antisymmetric second rank tensor field with spin 1 (matter). Thus, we have some bosonic matter obtained from geometrical considerations. This matter "feels" the reference frame structure by means of interacting with a background frame, and in the general case has an effective mass, depending on time and position. Thus, the properties of that matter depend in an unusual way upon the background. We review briefly the theory for an arbitrary choice of the background and pay a special attention to some interesting cases, such as flat the Minkowski space or Schwarzschild backgrounds.

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1. Introduction

In this paper we present a linear approximation in an interesting General Relativity generalization suggested by C. Møller [1]. With the exception of this little introduction and a brief review of the Møller gravity (section 2), we are going to discuss the properties of an antisymmetric second rank tensor field with spin 1.

In the case of linear approximation such field corresponds to the antisymmetric part of the small frame fluctuation above an arbitrary background solution. It has a simple geometrical interpretation. The action of the Møller gravity theory is not invariant under the rotations of the frame [1, 2]. Thus, the reference frame is a dynamical variable in this theory. In this paper we demonstrate that in the case of linear approximation the convenient dynamical variables to describe the small frame fluctuation above an arbitrary background are infinitesimal rotations of the reference frame. There is a hope that we can treat this “field of frame rotations” as a “Dark Matter”, obtained from geometrical considerations only.

As was shown in [2] the characteristic constants of the Møller gravity Lagrangian do not need to be necessarily small in comparison with the General Relativity constant to obtain the same physically interesting background solutions, which appear in General Relativity. (For example, the spherically symmetric Schwarzschild solution or the Kerr solution). These solutions appear in the case of arbitrary constants too, where some relations for these constants are valid [2, 3]. If the couplings of the theory are not small, in the case when background solutions exist, we can also hope that even small fluctuations above a background give an effect, which can be comparable with the effects of General Relativity. And, as is shown below, in the case of linear approximation such an effect can be treated as the presence of a bosonic matter, which is described by the antisymmetric second rank tensor field.

2. Møller gravity theory

It was C. Møller, who first put forward this theory in 1978 [1]. The Møller gravity theory is a metric theory, in which the metric tensor

$$g_{\mu\nu} \equiv \sum_{\alpha=0}^{3} \delta(\alpha\alpha) g(\alpha)_\mu g(\alpha)_\nu \equiv g(\alpha)_\mu g(\alpha)_\nu\,, \quad \delta(\alpha\beta) \equiv diag\{-1,1,1,1\} \tag{2.1}$$

is constructed from an orthonormal tetrad:

$$g(\alpha)_\mu g(\beta)^\mu \equiv \delta(\alpha\beta) \tag{2.2}$$

Here indices in the brackets are frame (vielbein) indices, which run from 1 to 4, and the summation over the repeated indices is implied. The stress tensor for this vector fields (denoted by $f(\alpha)_{\mu\nu}$) is, as usually:

$$f(\alpha)_{\mu\nu} \equiv \partial_{[\mu} g(\alpha)_{\nu]} \tag{2.3}$$

Where the antisymmetrization over the indices in the square brackets is implied. The coordinate indices can be turned into vielbein indices as it is presented below:

$$C_{\mu\nu}(\alpha)g(\alpha)_{\lambda} \equiv C_{\mu\nu\lambda} \tag{2.4}$$

We can obtain the Riche tensor from the stress tensor:

$$R(\alpha\beta)=\frac{1}{2}\partial(\alpha)f(\mu\mu\beta)+\frac{1}{2}\partial(\beta)f(\mu\mu\alpha)+\frac{1}{2}\partial(\mu)f(\beta\alpha\mu)$$
$$+\frac{1}{2}\partial(\mu)f(\alpha\beta\mu)-\frac{1}{2}f(\alpha\beta\mu)f(\nu\nu\mu)-\frac{1}{2}f(\beta\alpha\mu)f(\nu\nu\mu)+$$
$$-\frac{1}{2}f(\mu\nu\alpha)f(\nu\mu\beta)-\frac{1}{2}f(\mu\nu\alpha)f(\mu\nu\beta)+\frac{1}{4}f(\alpha\mu\nu)f(\beta\mu\nu)$$

(2.5)

Where $\partial(\alpha)\equiv g(\alpha)^{\mu}\partial_{\mu}$ (2.6)

Contacting indices in different ways, we can build from the stress tensor three different scalars:

$$L_1 \equiv f_{\alpha\beta\gamma}f^{\alpha\beta\gamma}\ \ L_2 \equiv f_{\alpha\beta\gamma}f^{\beta\alpha\gamma}\ \ L_3 \equiv f^{\alpha}{}_{\alpha\gamma}f_{\beta}{}^{\beta\gamma} \tag{2.7}$$

Then, in general case, the simplest action quadratic in the partial derivatives is:

$$S=\int_X\left(k_0+k_1L_1+k_2L_2+k_3L_3\right)\sqrt{g}\,dx \tag{2.8}$$

where k_0, k_1, k_2, k_3 are arbitrary dimensional constants.

Using (2.5) we can obtain more usual expression for Einstein-Møller action (up to a complete divergence):

$$S=\int_X\left(k_0+k_3R+k_1'L_1+k_2'L_2\right)\sqrt{g}\,dx \tag{2.9}$$

We could not write the action with the terms L_1 and L_2, if we used just the metric tensor without the vielbein formalism. Thus, we have a generalization of metric gravity.

As we can see, Møller gravity theory coincides with General Relativity, if k_1', k_2' is equal to 0.

Denoting the variation of the action with respect to $g(\mu)_{\alpha}$ as

$$X(\mu)^{\alpha}\equiv\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g(\mu)_{\alpha}}, \tag{2.10}$$

we can write the symmetric and the antisymmetric parts of the equations of motion $X(\mu)^{\alpha}\equiv\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g(\mu)_{\alpha}}=0$ separately. The symmetric part gives:

$$X^{(\alpha\beta)}=-4k_3\left(R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}\left(R+\Lambda\right)\right)+2\left(2k_1'+k_2'\right)\nabla_{\mu}f^{(\alpha\beta)\mu}-\left(2k_1'-k_2'\right)f^{[\mu\nu]\alpha}f_{[\mu\nu]}{}^{\beta}+$$
$$+\left(2k_1'+k_2'\right)f^{\mu\nu(\alpha}f^{\beta)}{}_{\mu\nu}-2k_2'f^{\alpha}{}_{\mu\nu}f^{\beta\mu\nu}+2g^{\alpha\beta}\left(k_1'f^{\mu\nu\lambda}f_{\mu\nu\lambda}+k_2'f^{\mu\nu\lambda}f_{\nu\mu\lambda}\right)=0 \tag{2.11}$$

The antisymmetric part of the equations of motion is

$$X^{[\alpha\beta]}=2\left(2k_1'-k_2'\right)\nabla_{\mu}f^{[\alpha\beta]\mu}-4k_2'\nabla_{\mu}f^{\mu\alpha\beta}-\left(2k_1'-3k_2'\right)f^{\mu\nu[\alpha}f^{\beta]}{}_{\mu\nu}=0 \tag{2.12}$$

If k_1', k_2' are equal to 0, the antisymmetric part vanishes, and the symmetric part gives us General Relativity. As we can see, in the Møller gravity theory we have more restrictions on frame vectors, than in General Relativity, because of the additional antisymmetric part of the equations of motion. And now the action of the theory is not invariant under the rotations of the frame, unlike in General Relativity.

3.Linear approximation

3.1 Small rotations as convenient dynamical variables

Let us divide the variation $\delta g(\alpha)_\beta$ in two parts:

$$\delta g(\alpha)_\beta \equiv \frac{1}{2} g(\alpha)^\mu \delta g_{\mu\beta} + \omega(\alpha)_\beta \tag{3.1}$$

where $\delta g_{\mu\beta}$ is a variation of the metric tensor and $\omega(\alpha)_\beta$ is a remnant. Then $\omega_{\alpha\beta}$ is an antisymmetric second rank tensor, which can be defined by means of frame vectors:

$$\omega_{\alpha\beta} = \frac{1}{2} g(\lambda)_{[\alpha} \delta g(\lambda)_{\beta]} \tag{3.2}$$

$$\omega(\beta\alpha) = \frac{1}{2}\left[g(\alpha)^\mu \delta g(\beta)_\mu - g(\beta)^\mu \delta g(\alpha)_\mu \right] \tag{3.3}$$

Let us consider the variations with respect to $\delta g_{\mu\beta}$ and $\omega(\beta\alpha)$ independently. Then in (3.1) we can do the integration over $\omega(\beta\alpha)$ and, finally, obtain for *finitesimal* $\Omega(\beta\alpha)$:

$$g'(\alpha)_\beta = \left\{ \delta^\nu_\beta e^{\Omega(\alpha\beta)} + \frac{1}{2}\delta(\alpha\beta) g^{\nu\mu} \delta g_{\mu\beta} \right\} g(\beta)_\nu \tag{3.4}$$

Here $\Omega(\beta\alpha)$ are the parameters of the rotations, because of $e^{\Omega(\alpha\beta)} = e^{-\frac{i}{2}\Omega(\mu\nu)T(\mu\nu\alpha\beta)}$, where $T(\mu\nu\alpha\beta) \equiv i\left[\delta(\mu\alpha)\delta(\nu\beta) - \delta(\mu\beta)\delta(\nu\alpha)\right]$, as we can see, are generators of the rotation group.

Thus, in Møller gravity we can consider the parameters of the frame rotations as reasonable dynamical variables. But unfortunately, the Lagrangian of the theory is non-polynomial for these variables. On the contrary, the Lagrangian is polynomial in the case of linear approximation, when the frame fluctuation above the background are small, so that we can consider the infinitesimal parameters of the frame rotations as the antisymmetric second rank tensor field.

3.2 Equations for the small frame fluctuation above an arbitrary background

If $g(\mu)_\alpha$ and $g'(\mu)_\alpha = g(\mu)_\alpha + \delta g(\mu)_\alpha$ are two close solutions of the motion equations, then $g(\mu)_\alpha$ is a background and $\delta g(\mu)_\alpha$ are small fluctuations above this background. If we consider the pure rotations $\delta g(\alpha)_\beta = g(\alpha)^\mu \omega_{\mu\nu}$ only, then from the antisymmetric part of the equations of motion we obtain the equation for the small fluctuations:

$$\begin{aligned} &-\left(2k_1 + k_2 + k_3\right)\nabla_\mu \nabla^\mu \omega^{\alpha\beta} + \frac{1}{2}\left(k_2 + k_3\right)\nabla_\mu \nabla^{[\alpha}\omega^{\beta\mu]} + \\ &-2k_1 C^{\mu[\alpha}{}_\lambda \nabla_\mu \omega^{\beta]\lambda} - 2k_1 \omega^{\lambda[\alpha}\nabla_\mu C^{|\mu|\beta]}{}_\lambda + \left(k_2 + k_3\right) C_{\mu\nu}{}^{[\alpha}\nabla^{\beta]}\omega^{\mu\nu} - \left(k_2 + k_3\right) C^{[\alpha\beta]}{}_\mu \nabla_\nu \omega^{\mu\nu} + \\ &+\left(k_2 + k_3\right)\left\{\nabla^\mu C^{[\beta}{}_{\mu\lambda} + C^{\mu\nu}{}_\lambda C^{[\beta}{}_{\mu\nu}\right\}\omega^{\alpha]\lambda} + \left(k_2 + k_3\right)\left\{\nabla^\nu C^{[\alpha\beta]\lambda} - C^{\nu[\alpha}{}_\mu C^{\beta]\lambda\mu}\right\}\omega_{\nu\lambda} = 0 \end{aligned} \tag{3.5}$$

Where $C_{\alpha\beta}(\gamma) \equiv \nabla_\alpha g(\gamma)_\beta$

In this paper we will investigate this equation for some special backgrounds:

3.2.1 Flat Minkowski space as a background

For the Minkowski background, the equation for the small fluctuations (3.5) takes a simple form:

$$-\left(2k_1+k_2+k_3\right)\nabla_\mu\nabla^\mu\omega^{\alpha\beta}+\frac{1}{2}\left(k_2+k_3\right)\nabla_\mu\nabla^{[\alpha}\omega^{\beta\mu]}=0 \tag{3.6}$$

Since for Minkowski space we can take the frames with $C_{\alpha\beta\gamma}=0$

If $2k_1+k_2+k_3=0$ and $k_1\neq 0$, we have the equation for the massless antisymmetric second rank tensor field with spin 1,

$$-k_1\nabla_\mu\nabla^{[\alpha}\omega^{\beta\mu]}=0 \tag{3.7}$$

This field has, as usually, two transverse polarizations. In the general case we have longitudinal polarizations also, but later we will consider the last case only, because physically interesting background solutions (like the spherically symmetric Schwarzschild solution or the Kerr solution) appear only if $2k_1+k_2+k_3=0$ [2, 3].

3.2.2 Schwarzschild solution as a background

An expression for the reference frame in the case of Schwarzschild solution is suggested in [2]. It is easy to show that for the Schwarzschild reference frame we have two non-trivial expressions for the components of $C_{\alpha\beta\gamma}$:

$$C_{001}=-e^{2\gamma}\gamma_{,r}\,,\ C_{ab1}=g_{ab}\alpha_{,r} \tag{3.8}$$

where $\gamma=\ln\dfrac{r-r_0}{r+r_0}$, $\alpha=\alpha_0+2\ln\left(1+r_0/r\right)$ and r_0 is the Schwarzschild radius.

Thus, we have from (3.5) the equation for the small fluctuations above the Schwarzschild background:

$$\nabla^\mu\nabla_{[\alpha}\omega_{\beta\mu]}+A^1\nabla_{[\alpha}\omega_{\beta]1}+B\omega_{\alpha\beta}=0 \tag{3.9}$$

where $A^1=-\dfrac{4e^{-2\gamma}r_0^2}{r(r^2-r_0^2)}$, $B=-4e^{-2\gamma}r_0^2\left\{\dfrac{1}{r^2\left(r^2-r_0^2\right)}+\dfrac{2}{r\left(r+r_0\right)\left(r^2-r_0{}^2\right)}+\dfrac{2}{r^2\left(r+r_0\right)^2}\right\}$

As we can see, this equation describes the waves, which have an effective mass, which depends on the radial component.

3.2.3 General remarks about an arbitrary homogeneous and isotropic background

If we try to investigate an arbitrary homogeneous and isotropic background (like de Sitter space, anti de Sitter space or Friedman solutions) we can use the following expression for the reference frame:

$$g(0)_0=i\,,\ g(a)_q=e^{\alpha(t)}\overline{g}(a)_q \tag{3.10}$$

where $\alpha(t)$ - is a function of time and $\bar{g}(a)_q$ - a frame on the three-dimensional sphere or hyperboloid. Then, for such a reference frame we have non-trivial expressions for the components of $C_{\alpha\beta\gamma}$:

$$C_{ab0} = g_{ab}\partial_0\alpha \, , \; C_{abc} = \varsigma e^{-\alpha}\varepsilon_{abc} \qquad (3.11)$$

where ς is 1, 0, *i*, for spaces with the positive, zero and negative curvature correspondingly.

Thus, we have from (3.5) the equation for the small fluctuation above an arbitrary homogeneous and isotropic background:

$$\nabla^{\mu}\nabla_{[\alpha}\omega_{\beta\mu]} + A^0\nabla_{[\alpha}\omega_{\beta]0} + B\omega_{\alpha\beta} = 0 \qquad (3.12)$$

where A^0 and B are some functions of time and $\omega_{\alpha\beta}$ can be interpreted as waves with some effective mass, which depends on time.

3.3 An interaction between spinors and small frame fluctuations above an arbitrary background

It is obvious that our antisymmetric second rank tensor field can interact with spinor matter, because it comes from small frame fluctuations above a background, because spinors “feel” the frame structure.

Let us write the Dirac equation in curved space-time:

$$\left(\gamma^{\nu}\nabla_{\nu} + m\right)\psi = 0 \, , \text{ where } \nabla_{\nu}\psi \equiv \left(\partial_{\nu} + \frac{1}{4}C_{\nu\lambda\tau}\gamma^{\lambda}\gamma^{\tau}\right)\psi \qquad (3.13)$$

is a covariant derivative for the spinor ψ and $\gamma^{\mu} = g(\alpha)^{\mu}\gamma(\alpha)$, where $\gamma(\alpha)$ are gamma-matrices for the flat space. Then from (3.13), $\delta g(\alpha)_{\beta} = g(\alpha)^{\mu}\omega_{\mu\nu}$ and $C_{\alpha\beta}(\gamma) \equiv \nabla_{\alpha}g(\gamma)_{\beta}$ we obtain an equation with an interaction between the spinors and the antisymmetric second rank tensor field $\omega_{\mu\nu}$:

$$\left\{\gamma^{\mu}\nabla_{\mu} + m + \gamma^{\mu}\left(\omega_{\mu}{}^{\tau}\nabla_{\tau} + \frac{1}{4}\gamma^{\alpha}\gamma^{\beta}\nabla_{\mu}\omega_{\alpha\beta}\right)\right\}\psi = 0 \qquad (3.14)$$

4. Conclusions

In this paper we presented a linear approximation in the Møller gravity theory. It is shown that in the case of linear approximation, we can treat the reference frame field above the background as the usual metric field (gravity) and some antisymmetric second rank tensor field with spin 1 (matter).

This second rank tensor field has a simple geometrical interpretation. We can consider it as a field of the small rotations of the frame. We can do that, because frame is a dynamical variable in this theory, unlike in General Relativity. We suppose that in such case, we have some variant of “Dark Matter” obtained from geometrical considerations only.

The effects from such matter can be comparable with the effects of General Relativity, because the characteristic constants of the Møller gravity Lagrangian do not necessarily need to be small in comparison with the General Relativity constant in the case when the physically interesting background solutions exist.

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