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Massive Yang-Mills: Divergences Removal, Unitarity and Lattice Simulation

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A consistent subtraction procedure of the infinities in the massive Yang-Mills theory (a nonrenormalizable model), recently proposed, appears to be doable only via dimensional regularization and moreover to face problems with unitarity (a remnant of nonrenormalizability).
Here we show how the lattice regularization turns out to be a promising tool for shedding some light on both problems. We consider a lattice model for the massive YM theory with a Wilsonian action plus a term, which yields the Stückelberg mass invariant in the naïve continuum limit.
After a birds-eyed view of the parameters space (coupling constant and mass) we study the property of the phases. Suitable gauge invariant fields are introduced and their correlators are calculated. Some informations can be obtained from the energy gaps.
The results of this investigation are positive both for the lattice regularization of the massive YM akin to the dimensional subtraction and for the question of unitarity.

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PoS(QCD-TNT-II)054

1. Introduction

A common structure is present in the nonlinear sigma model (NLSM), in the massive Yang-Mills (YM) and in the Higgless Electro-Weak model (EW). For $SU(2)$ one has the action structures: NLSM [1]

$$S_{NLSM} = \Lambda^{D-4}\frac{M^2}{4}\int d^Dx\, Tr\Big\{\partial^\mu\Omega^\dagger\partial_\mu\Omega\Big\} \tag{1.1}$$

the Stückelberg mass for YM [2]

$$S_{YM} \sim \Lambda^{D-4}M^2\int d^Dx\, Tr\left\{\Big[A_\mu - i\Omega\partial_\mu\Omega^\dagger\Big]^2\right\} \tag{1.2}$$

and EW mass terms [3],[4]

$$\begin{aligned} S_{EW} \sim \Lambda^{D-4}M^2\int d^Dx\Bigg(& Tr\left\{\Big(gA_\mu - \frac{g'}{2}\Omega\tau_3B_\mu\Omega^\dagger - i\Omega\partial_\mu\Omega^\dagger\Big)^2\right\} \\ & +\frac{\kappa}{2}\Big[Tr\Big\{\Big(gA_\mu - \frac{g'}{2}\Omega\tau_3B_\mu\Omega^\dagger - i\Omega\partial_\mu\Omega^\dagger\Big)\tau_3\Big\}\Big]^2\Bigg). \end{aligned} \tag{1.3}$$

The $2\times 2 \in SU(2)$ matrix may be parameterized by the real fields

$$\Omega = \phi_0 + i\tau_i\phi_i, \qquad \phi_0 = \sqrt{1-\vec{\phi}^2}. \tag{1.4}$$

The constraint is implemented in the path integral measure

$$\prod_x \mathscr{D}^4\phi(x)\theta(\phi_0)\delta(\vec{\phi}(x)^2+\phi_0^2(x)-1) = \prod_x \mathscr{D}^3\phi(x)\frac{2}{\sqrt{1-\vec{\phi}^2}}. \tag{1.5}$$

The non trivial measure in the path integral is the source of very interesting facts. This has led us [1]-[4] to propose a subtraction procedure which preserves locality and perturbative unitarity for nonrenormalizable theories as nonlinear sigma model, massive Yang-Mills and the electroweak model with no Higgs boson.

2. Subtraction and Unitarity

The theory is expected to be fundamental (not an effective one), where the number of parameters is fixed. Two questions are discussed here.

A) The suggested subtraction procedure is based on **dimensional regularization**. The question is whether it works with other regularizations.

B) Although perturbative unitarity is valid, the behavior of some cross sections at high energy, evaluated at fixed order, is untenable (e.g. the celebrated case of W_LW_L elastic scattering [5]).

3. Decoupling of the Longitudinal Modes

In a recent work [6] we showed that this problem (Unitarity) is tightly connected to the coupling/decoupling of the longitudinal mode at very high energy. Roughly speaking, by means of the Equivalence Theorem [7] one can show that in the Higgs mechanism the longitudinal mode does not decouple, while in the nonlinear case a phase transition separates the low energy regime from the high energy's where the longitudinal mode decouples.

In the present work we investigate these problems in lattice gauge theory. Our preliminary results support this scenario.

4. The Lattice Action ($SU(2)$Yang-Mills)

We use a naïve continuous limit to write the lattice (L^4) action ($D = 4$ cubic lattice with periodic boundary conditions): the Wilson plaquette action [8] with a mass term.

$$\begin{aligned} S_E = & -\frac{\beta}{2}\,\Re e \sum_{\Box} Tr(U_{\Box}) \\ & -\frac{\beta}{2} M^2 a^2 \Re e \sum_{x\mu} Tr\Big\{\Omega(x)^{\dagger} U(x,\mu)\Omega(x+\mu)\Big\}, \end{aligned} \tag{4.1}$$

where $\beta = \frac{4}{g^2}$. We use the dimensionless parameter $m^2 \equiv M^2 a^2$.
The limit for the classical action yields

$$\begin{aligned} & -\lim_{a=0}\frac{\beta}{2} M^2 a^2 \Re e \sum_{x\mu} Tr\Big\{\Omega(x)^{\dagger}U(x,\mu)\Omega(x+\mu)-1\Big\} = \frac{M^2}{g^2}\int d^4x Tr\,\Big\{(A_\mu - i\Omega\partial_\mu\Omega^{\dagger})^2\Big\} \\ & = \frac{M^2}{g^2}\int d^4x Tr\,\Big\{[(i\partial_\mu + A_\mu)\Omega]^{\dagger}(i\partial_\mu + A_\mu)\Omega\Big\}. \end{aligned} \tag{4.2}$$

5. Simulation (Heat Bath)

The partition function is obtained by summing over all configurations given by the link variables $U(x,\mu)$ and the gauge field $\Omega(x)$ both $\in SU(2)$

$$Z[\beta, m^2, N] = \sum_{\{U,\Omega\}} e^{-S_E}, \tag{5.1}$$

where N is the number of sites.

This model has been studied for a long time [9] to describe the Higgs field with frozen length. In our approach the integration over $\Omega(x)$ is redundant, since by a change of variables ($U'(x) := \Omega(x)^{\dagger}U(x,\mu)\Omega(x+\mu)$) we can factor out the volume of the group

$$Z = \Big[\sum_{\{\Omega\}}\Big]\sum_{\{U\}} e^{\beta\left(\frac{1}{2}\Re e\sum_{\Box} Tr\left\{U_{\Box}\right\} + \frac{1}{2}m^2 \Re e \sum_{x\mu} Tr\left\{U(x,\mu)\right\}\right)}. \tag{5.2}$$

In eq. (5.2) the integration over Ω has disappeared; consequently Ω in eq. (5.1) does not describe any degree of freedom. We force the integration over the gauge orbit U_Ω by means of the explicit sum over Ω. In doing this we gain an interesting theoretical setup of the model and its relation with the continuum limit as in eq. (4.2). Moreover we get results which are less noisy.

PoS(QCD-TNT-II)054

6. Order Parameter and Functionals

The energy-per-site functional $E = \frac{1}{N}\frac{\partial}{\partial\beta}\ln Z$

$$E = \frac{1}{2N}\left\langle \Re e\sum_{\Box} Tr\{U_{\Box}\} + m^2\sum_{x\mu} Tr\{\Omega^{\dagger}(x)U(x,\mu)\Omega(x+\mu)\}\right\rangle. \tag{6.1}$$

Moreover we introduce the order parameter

$$\mathfrak{C} = \frac{1}{DN\beta}\frac{\partial}{\partial m^2}\ln Z = \frac{1}{2ND}\left\langle \Re e\sum_{x\mu} Tr\{\Omega^{\dagger}(x)U(x,\mu)\Omega(x+\mu)\}\right\rangle. \tag{6.2}$$

Then we have the plaquette energy

$$E_P = \frac{2}{D(D-1)N}\left\langle \frac{1}{2}\Re e\sum_{\Box} Tr\{U_{\Box}\}\right\rangle = \frac{2}{D(D-1)}\left[E - Dm^2\mathfrak{C}\right]. \tag{6.3}$$

There are some simple properties that will be of some help in the sequel. Under the mapping

$$U(x,\mu) \to -\,U(x,\mu) \tag{6.4}$$

the Wilson action is invariant while the mass part changes sign.
The measure of the group integration is invariant, then we have from eqs. (5.1), (6.1) and (6.2)

$$\begin{aligned}
Z[\beta,-m^2,N] &= Z[\beta,m^2,N]\\
E[\beta,-m^2,N] &= E[\beta,m^2,N]\\
\mathfrak{C}[\beta,-m^2,N] &= -\mathfrak{C}[\beta,m^2,N].
\end{aligned} \tag{6.5}$$

7. Gauge Invariance

The action is invariant under the local gauge transformations (left)

$$\begin{aligned}
&U(x,\mu) \to g_L(x)U(x,\mu)g_L(x+\mu)^{\dagger}\\
&\Omega(x) \to g_L(x)\Omega(x), \qquad \forall g_L(x) \in SU_L(2)
\end{aligned} \tag{7.1}$$

and under the global gauge transformations (right)

$$\begin{aligned}
&U(x,\mu) \to U(x,\mu)\\
&\Omega(x) \to \Omega(x)g_R^{\dagger}, \qquad \forall g_R \in SU_R(2).
\end{aligned} \tag{7.2}$$

8. The gauge invariant Fields

The presence Ω allows the introduction of a set of *gauge invariant* fields.

$$\begin{aligned} C(x,\mu) &\equiv \Omega^\dagger(x)U(x,\mu)\Omega(x+\mu) = C_0(x,\mu) + i\tau_a C_a(x,\mu) \\ i\tau_a C_a(x,\mu) &= -ia\Omega^\dagger\left(A_\mu(x) - i\Omega\partial_\mu\Omega^\dagger\right)\Omega + \mathscr{O}(a^2) \\ C_0(x,\mu) &= 1 - \frac{a^2}{4}Tr\left\{(A_\mu - i\Omega\partial_\mu\Omega^\dagger)^2\right\} + \mathscr{O}(a^4). \end{aligned} \tag{8.1}$$

$C_0(x,\mu)$ is the mass term density in the action (4.1) and it is a $SU(2)_R$-scalar, while $C_a(x,\mu)$ are vectors under the same group of transformations. Since $C(x,\mu) \in SU(2)$, we get that all fields are real and moreover

$$C_0(x,\mu)^2 + \sum_a C_a(x,\mu)^2 = 1. \tag{8.2}$$

Moreover we expect the vacuum to be invariant under $SU_R(2)$ global transformations and therefore

$$\begin{aligned} &\langle C_a(x,\mu)\rangle = 0 \\ &\langle C_a(x,\mu)C_b(y,\nu)\rangle_C = 0, \quad a \neq b, \quad a,b = 1,2,3. \end{aligned} \tag{8.3}$$

In order to investigate the transition between phases we consider also

$$\begin{aligned} \frac{\partial}{\partial m^2}\mathfrak{C} &= \frac{\beta}{DN}\sum_{x\mu}\sum_{y\nu}\Big\langle C_0(x,\mu)C_0(y,\nu)\Big\rangle_C \\ &= \frac{\beta}{DN}\sum_{x\mu}\sum_{y\nu}\Big(\Big\langle C_0(x,\mu)C_0(y,\nu)\Big\rangle - \Big\langle C_0(x,\mu)\Big\rangle\Big\langle C_0(y,\nu)\Big\rangle\Big). \end{aligned} \tag{8.4}$$

It should be noticed that the mean square error of $\mathfrak{C}$ is related to its derivative

$$\frac{\partial}{\partial m^2}\mathfrak{C} = \beta DN\Big\langle\Big(\mathfrak{C} - \langle\mathfrak{C}\rangle\Big)^2\Big\rangle. \tag{8.5}$$

This relation is very important for numerical simulations. If the derivative of $\mathfrak{C}$ increases by increasing the lattice size (as it is expected in a inflection point becoming more and more steep), we expect some anomalous behavior of the mean square error. I.e. the system becomes noisy. If the derivative of $\mathfrak{C}$ has a finite limit for $N \to \infty$, the standard deviation has the normal $1/\sqrt{N}$ behavior. If instead the derivative diverges then the standard error might be anomalous in the limit. If this is the case, then the calculation of the derivative by using the incremental ratio yields a noisy signal. The noise might not decrease for $N \to \infty$.

9. Numerical results

We plot in Fig. 1 the points β, m^2 where the order parameter $\mathfrak{C}$ has a marked inflection as function of m^2, as shown in Figs. 2, 3 and 4. On these points all the derivatives $(\partial E/\partial m^2, \partial E/\partial\beta, \partial\mathfrak{C}/\partial\beta)$ have some peak. The peak heights increase with the size of the lattice for $\beta > 2.2$. In the literature

PoS(QCD-TNT-II)054

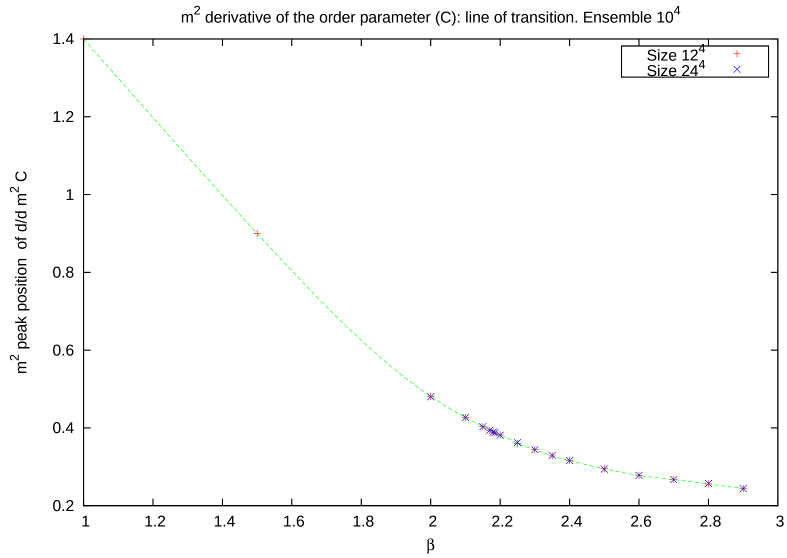

Figure 1: Phase diagram

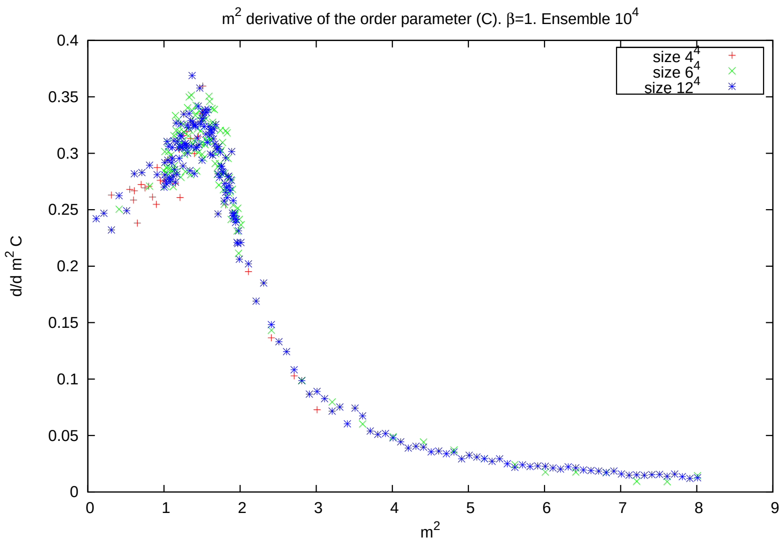

Figure 2: m^2 derivative of the order parameter. $\beta = 1.0$

[10] the exact position of the end point is discussed together with the nature of the phases above and below the transition line. It is commonly accepted [10] that one has confinement below the PT line for $\beta > 2.2$ and a cross-over line from $\beta \simeq 2.2, m^2 = 0$ to $\beta \simeq 2.2, m^2 \simeq 0.381$ (end point). The sequence of Figures 5, 6, 7, 8 and 9 show the β-dependence of the energy E along the straight lines at constant m^2. One observes the presence of the crossover region and of the phase transition line while they merge.

We do not discuss these items, since we are primarily interested in the features of the phases.

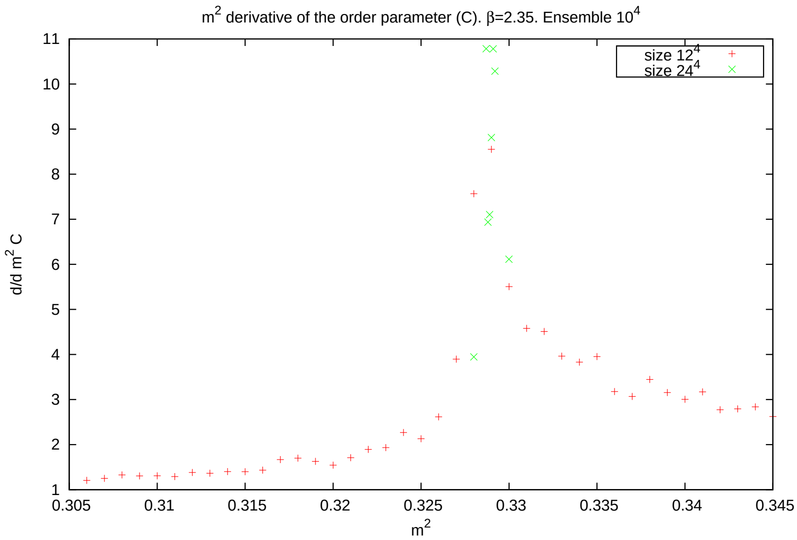

Figure 3: m^2 derivative of the order parameter. $\beta = 2.35$

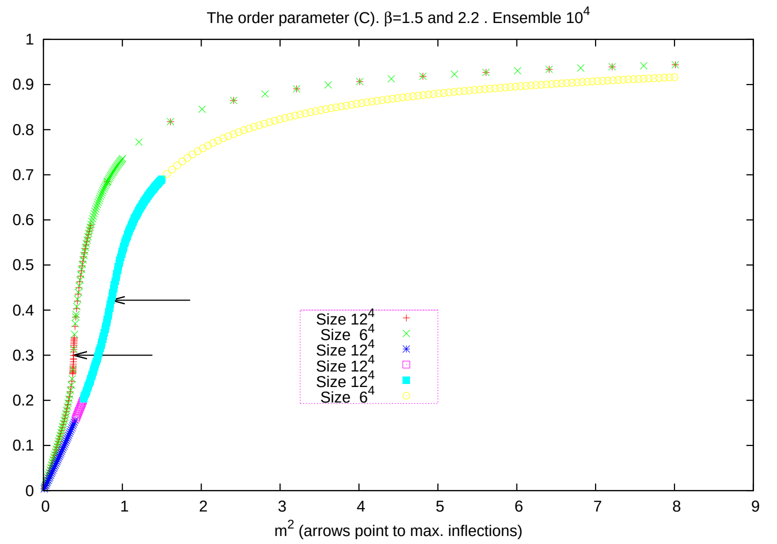

Figure 4: Order parameter at $\beta = 1.5$ and $\beta = 2.2$.

We have done some systematic search on the operators

$$C_{a,\mu}(t) := \frac{1}{\sqrt{N^{\frac{3}{4}}}} \sum_{\vec{x}} C_a(\vec{x}, x_4, \mu)\Big|_{x_4=t}, \quad a = 0,1,2,3, \;\; \mu, \nu = 1,2,3,4 \tag{9.1}$$

by evaluating the connected two-point correlators as in Fig. 10

$$C_{ab,\mu\nu}(t) := \Big\langle C_{a,\mu}(t+t_0) C_{b,\nu}(t_0) \Big\rangle_C . \tag{9.2}$$

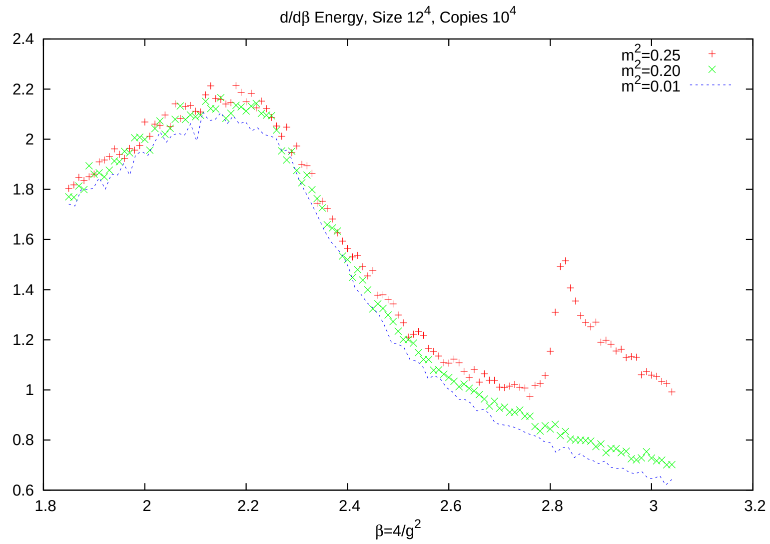

Figure 5: Scan of β derivative of E. I

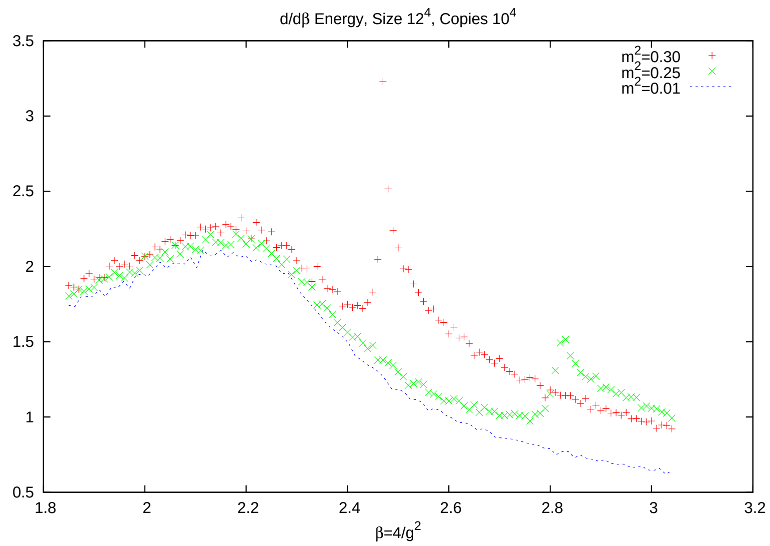

Figure 6: Scan of β derivative of E. II

Numerical simulations support the selection rules

$$\begin{aligned} &C_{0b,\mu\nu}(t) = 0 \\ &C_{ab,\mu\nu}(t)\Big|_{a\neq b} = 0, \quad a,b = 1,2,3 \end{aligned} \tag{9.3}$$

imposed by the global $SU(2)_R$ invariance.

The spin analysis is done by decomposing the correlators into a spin one and spin zero parts

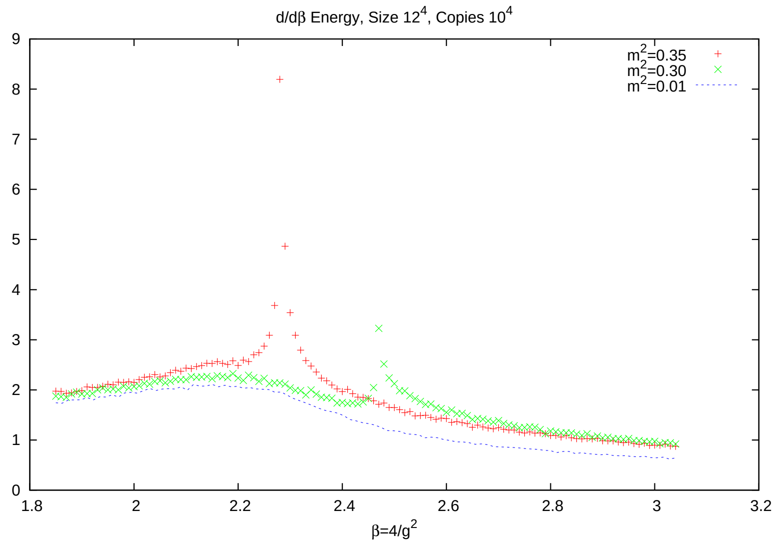

Figure 7: Scan of β derivative of E. III

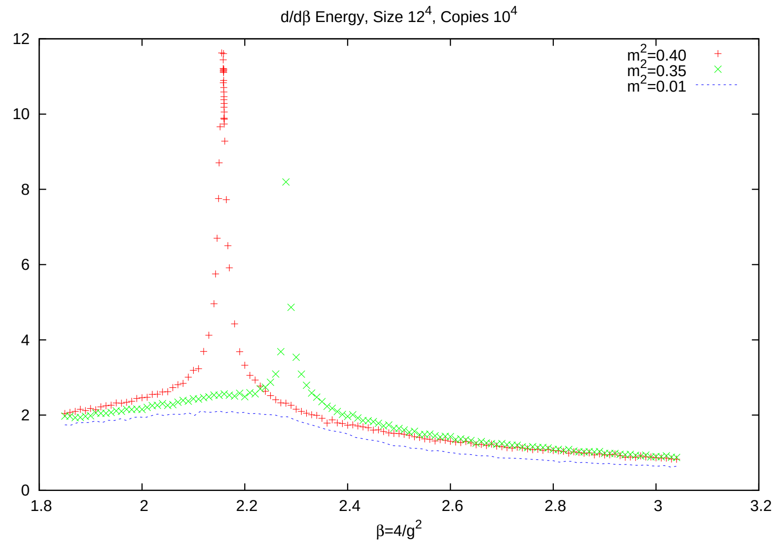

Figure 8: Scan of β derivative of E. IV.

(dots stand for 00 or 11)

$$C_{\dots,\mu\nu}(t) = V_{\dots}(t)(\delta_{\mu\nu} - \delta_{4\mu}\delta_{4\nu}) + S_{\dots}(t)\delta_{4\mu}\delta_{4\nu}. \tag{9.4}$$

We fit the amplitudes by a single exponential form

$$F(t) = a + be^{-t\Delta}. \tag{9.5}$$

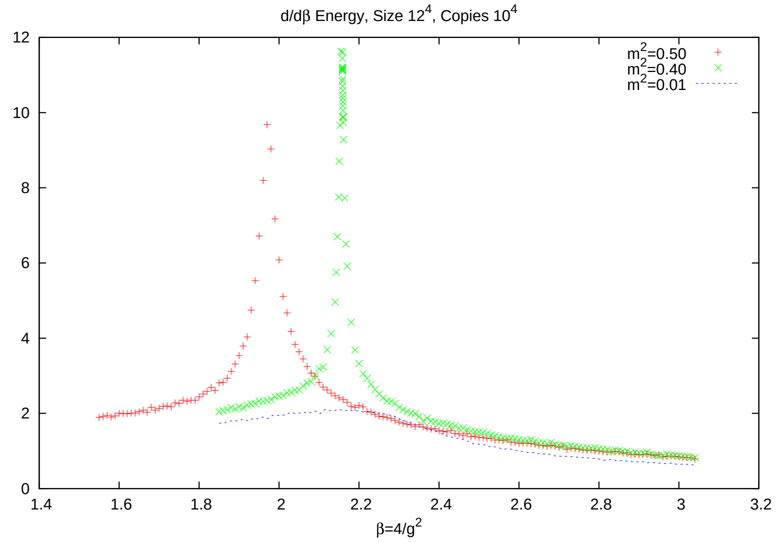

Figure 9: Scan of β derivative of E. V

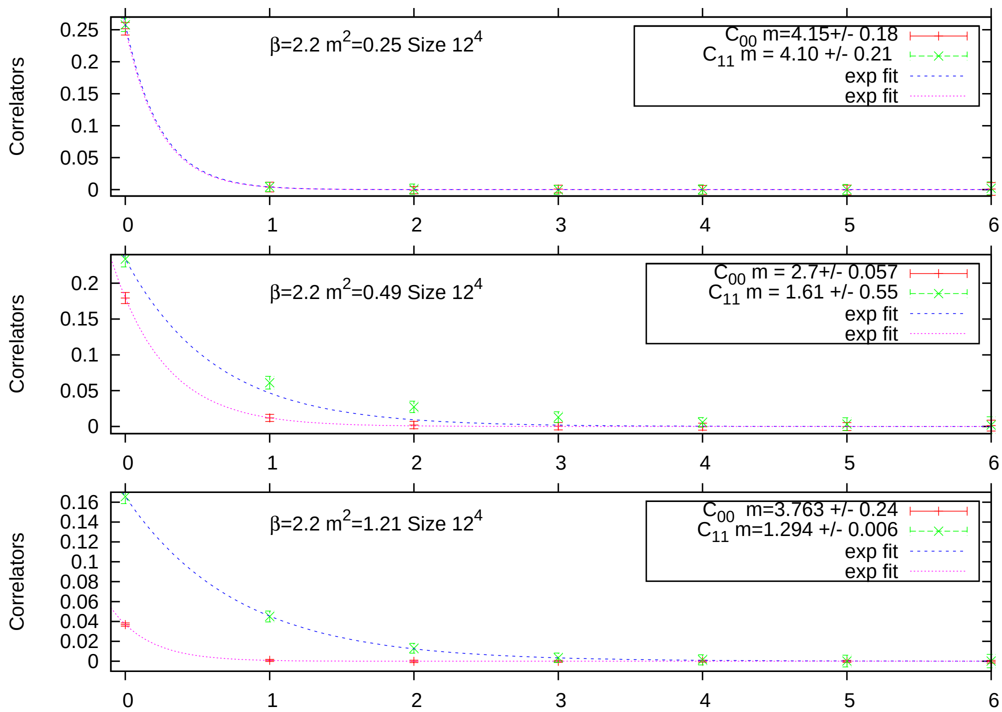

Figure 10: Time correlators. The polarizations are the same (Spin = 1)

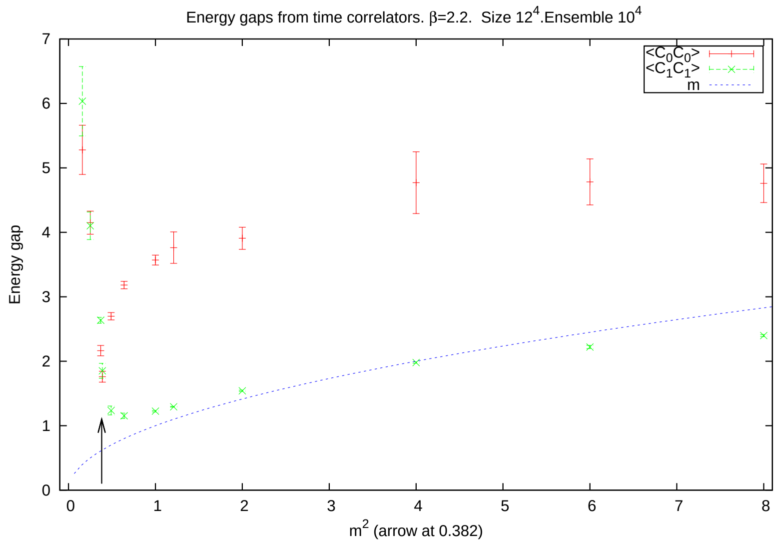

Figure 11: Energy gaps for $\beta = 2.2$

10. Below the Phase Transition Curve

Near the $m^2 = 0$ line one can integrate over Ω and then

$$\left\langle C(x,\mu)C(y,\nu)^{\dagger}\right\rangle = 1\delta_{xy}\delta_{\mu\nu}. \tag{10.1}$$

Thus we really test only the screening.

11. Above the Phase Transition Curve

In Fig. 11 the isovector part of the meson fields describes particles of Spin 1.
Far away from the transition line the energy gap is approximatively given by

$$\Delta \simeq \sqrt{|m^2|}. \tag{11.1}$$

Moreover Fig. 11 shows that also the isoscalar part of $C(x,\mu)$ describes a spin one state. The energy gap is compatible with the threshold of the isovector spin one fields.

12. Conclusions

- Lattice gauge theory is very promising as a regulator for massive Yang-Mils theory. It is important to learn how to relate dimensionally regularized amplitudes with those from the lattice.
- Simulations support the conjecture that above the transition line and $m^2 \sim 0$ the lattice has different phases. In the massless case Goldstone bosons and longitudinal modes decouple.

- In lattice gauge theory the evaluation of amplitudes near the transition line is at reach.

Finally we can conclude that the lattice simulation supports the conjecture made on the massive Yang-Mills theory with no Higgs (eq. (1.2)): at very high energy a phase transition separates the low energy regime from the high energy limit where longitudinal polarizations and Goldstone modes decouple.

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