

## Background field dependence from the Slavnov-Taylor identity in (non-perturbative) Yang-Mills theory

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We show that in Yang-Mills theory the Slavnov-Taylor (ST) identity, extended in the presence of a background gauge connection, allows to fix in a unique way the dependence of the vertex functional on the background, once the 1-PI amplitudes at zero background are known. The reconstruction of the background dependence is carried out by purely algebraic techniques and therefore can be applied in a non-perturbative scheme (e.g. on the lattice or in the Schwinger-Dyson approach), provided that the latter preserves the ST identity. The field-antifield redefinition, which replaces the classical background-quantum splitting when quantum corrections are taken into account, is considered on the example of an instanton background in SU(2) Yang-Mills theory.

*International Workshop on QCD Green's Functions, Confinement and Phenomenology,  
September 05-09, 2011  
Trento Italy*

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## 1. Introduction

In a recent paper [1] the background field method (BFM) [2, 3] has been reformulated within the Batalin-Vilkovisky formalism [4] as a prescription for handling the quantization of a gauge theory in the presence of a topologically non-trivial background  $\hat{A}_\mu$ . The background-dependent amplitudes are recovered via a canonical transformation. The latter guarantees the fulfillment of the relevant Slavnov-Taylor (ST) identity of the model at non-zero background.

Since the approach only relies on the ST identity associated with the BRST symmetry of the theory (extended in the presence of the background connection  $\hat{A}_\mu$ ), it can be applied in any non-perturbative symmetric framework which preserves the relevant functional identities of the theory, like e.g. the approach to QCD based on the Schwinger-Dyson (SD) equations. Moreover our formalism provides a consistent strategy for the implementation of the BFM on the lattice. The relevant composite operators, needed to control the background dependence at the quantum level, are identified in terms of the set of antifields of the theory plus an additional anticommuting source  $\Omega_\mu$ , coupled to the covariant derivative of the antighost field. Once this set of operators is introduced on the lattice, one can reconstruct the background-dependent amplitudes in a unique way. We remark that, if one is only interested to the gauge sector, this set of operators boils down to just two: the covariant derivative w.r.t. the ghost, coupled to the antifield  $A_\mu^*$  (i.e. the source of the BRST transformation of the gauge field), and the already mentioned covariant derivative w.r.t. the antighost.

The implementation of the BFM on the lattice (for whatever value of the gauge fixing parameter) would be a long awaited leap forward [5]. For instance, the understanding of the behaviour of the ghost and gluon propagators in the deep IR, where the existence of massive solutions has been firmly established in the Landau gauge by recent lattice data [6, 7] supported by SD-computations [8]-[13], might benefit from the extension of these investigations to different gauges and to the effects due to the presence of non-trivial backgrounds.

The existence of a canonical transformation, implementing at the quantum level the deformation of the background-quantum splitting, has several important consequences. On the one hand, it shows that the BFM can be made consistent with the fundamental symmetries of the theory also in non-perturbative approaches to Yang-Mills theory. On the other hand, it allows us to reconstruct the full dependence of the vertex functional on the background connection by purely algebraic means.

This in turn provides a separation between the integration over the quantum fluctuations around the classical background (controlled by the vertex functional  $\mathbb{I}\Gamma_0$  at zero background), and the further reconstruction of the background-dependent amplitudes. The latter can be represented in a compact way by using homotopy techniques [1].

Another interesting consequence is that the background splitting at the quantum level implies a (gauge background-dependent) redefinition also of the ghosts and the antifields, which, to the best of our knowledge, was not pointed out before in the literature.

The paper is organized as follows. In Sect. 2 we introduce our notation. In Sect. 3 we summarize the results of the implementation of the BFM via a canonical transformation, culminating in the homotopy formula for the vertex functional. In Sect. 4 we show that the canonical transformation, governing the BFM splitting, gives rise to a field and antifield redefinition both in the gauge and the ghost sector. In Sect. 5 we discuss as an example the explicit deformation at one loop level

of the instanton solution in the singular gauge for SU(2) pure Yang-Mills theory. Conclusions are finally presented in Sect. 6.

## 2. Classical Action, BV Bracket and ST identity

We consider Yang-Mills theory based on a semisimple gauge group  $G$  with generators  $T_a$  in the adjoint representation, satisfying

$$[T_a, T_b] = if_{abc}T_c. \quad (2.1)$$

The Yang-Mills action  $S_{YM}$  is

$$S_{YM} = -\frac{1}{4g^2} \int d^4x G_{a\mu\nu} G_a^{\mu\nu} \quad (2.2)$$

where  $g$  is the coupling constant and  $G_{a\mu\nu}$  is the Yang-Mills field strength

$$G_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + f_{abc}A_{b\mu}A_{c\nu}. \quad (2.3)$$

We adopt a (background)  $R_\xi$ -gauge-fixing condition [14] by adding to  $S_{YM}$  the gauge-fixing term

$$\begin{aligned} S_{g.f.} &= \int d^4x s \left[ \bar{c}_a \left( \frac{\xi}{2} B_a - D_\mu [\hat{A}] (A - \hat{A})_a \right) \right] \\ &= \int d^4x \left( \frac{\xi}{2} B_a^2 - B_a D_\mu [\hat{A}] (A - \hat{A})_a \right. \\ &\quad \left. + \bar{c}_a D_\mu [\hat{A}] (D^\mu [A] c)_a + (D_\mu [A] \bar{c})_a \Omega_a^\mu \right). \end{aligned} \quad (2.4)$$

In the above equation  $\xi$  is the gauge parameter (the Landau gauge used in [1] is obtained for  $\xi = 0$ ) and  $\hat{A}_{a\mu}$  denotes the background connection.  $\bar{c}_a, c_a$  are the antighost and ghost fields respectively and  $B_a$  is the Nakanishi-Lautrup multiplier field.

We will sometimes use the notation  $A_\mu = A_{a\mu} T_a$  and similarly for  $\hat{A}_\mu, \bar{c}, c, B$ .

The BRST differential  $s$  acts on the fields of the theory as follows

$$\begin{aligned} sA_{a\mu} &= D_\mu [A] c_a \equiv \partial_\mu c_a + f_{abc} A_{b\mu} c_c, \\ sc_a &= -\frac{1}{2} f_{abc} c_b c_c, \\ s\bar{c}_a &= B_a, \quad sB_a = 0. \end{aligned} \quad (2.5)$$

$s$  is nilpotent.  $\Omega_{a\mu}$  is an external source [15]-[17] with ghost number +1 pairing with the background connection  $\hat{A}_{a\mu}$  into a BRST doublet [18]

$$s\hat{A}_{a\mu} = \Omega_{a\mu}, \quad s\Omega_{a\mu} = 0. \quad (2.6)$$

$\Omega_{a\mu}$  was introduced in [15], where it was shown that no new  $\hat{A}_\mu$ - and  $\Omega_\mu$ -dependent anomalies can arise, as a consequence of the pairing in eq.(2.6).

Since the BRST transformations of the fields  $A_{a\mu}$  and  $c_a$  in eq.(2.5) are non-linear in the quantum fields, we need a suitable set of sources, known as antifields [19, 20], in order to control

	$A_{a\mu}$	$c_a$	$\bar{c}_a$	$B_a$	$A_{a\mu}^*$	$c_a^*$	$\bar{c}_a^*$	$B_a^*$	$\widehat{A}_{a\mu}$	$\Omega_{a\mu}$
Ghost charge	0	1	-1	0	-1	-2	0	-1	0	1
Statistics	B	F	F	B	F	B	B	F	B	F
Dimension	1	0	2	2	3	4	2	2	1	1

**Table 1:** Ghost charge, statistics (B for Bose, F for Fermi), and mass dimension of both the Yang-Mills conventional fields and anti-fields as well as background fields and sources.

their quantum corrections. For that purpose we finally add to the classical action the following antifield-dependent term

$$S_{a.f.} = \int d^4x \left( A_{a\mu}^* D^\mu [A] c_a - c_a^* \left( -\frac{1}{2} f_{abc} c_b c_c \right) - \bar{c}_a^* B_a \right). \quad (2.7)$$

Although it is not necessary for renormalization purposes, we have included in eq.(2.7) the antifield  $\bar{c}_a^*$  for  $\bar{c}_a$ . This will allow us to treat on an equal footing all the fields of the theory by a single Batalin-Vilkovisky (BV) bracket [4, 20].

We summarize in Table 1 the ghost charge, statistics and dimension of the fields and antifields of the theory. We end up with the tree-level vertex functional given by

$$\Pi^{(0)} = S_{YM} + S_{g.f.} + S_{a.f.}. \quad (2.8)$$

$\Pi^{(0)}$  fulfills several functional identities [1]:

- the Slavnov-Taylor (ST) identity

The ST identity encodes in functional form the invariance under the BRST differential  $s$  in eqs.(2.5) and (2.6). In order to set up the formalism required for the consistent treatment of the quantum deformation for the background-quantum splitting, it is convenient to write the ST identity within the BV formalism.

We adopt for the BV bracket the same conventions as in [20]; then, using only left derivatives, one can write

$$(X, Y) = \int d^4x \sum_{\phi} \left[ (-1)^{\varepsilon_{\phi}(\varepsilon_X + 1)} \frac{\delta X}{\delta \phi} \frac{\delta Y}{\delta \phi^*} - (-1)^{\varepsilon_{\phi^*}(\varepsilon_X + 1)} \frac{\delta X}{\delta \phi^*} \frac{\delta Y}{\delta \phi} \right] \quad (2.9)$$

where the sum runs over the fields  $\phi = \{A_{a\mu}, c_a, \bar{c}_a, B_a\}$  and the antifields  $\phi^* = \{A_{a\mu}^*, c_a^*, \bar{c}_a^*, B_a^*\}$ . In the equations above,  $\varepsilon_{\phi}$ ,  $\varepsilon_{\phi^*}$  and  $\varepsilon_X$  represent respectively the grading of the field  $\phi$ , the antifield  $\phi^*$  and the functional  $X$ .

The extended ST identity arising from the invariance of  $\Pi^{(0)}$  under the BRST differential in eq.(2.5) and eq.(2.6), in the presence of a background field, can now be written as

$$\int d^4x \Omega_a^\mu(x) \frac{\delta \Pi^{(0)}}{\delta \widehat{A}_\mu^a(x)} = -\frac{1}{2} (\Pi^{(0)}, \Pi^{(0)}). \quad (2.10)$$

- the B-equation

$$\frac{\delta \Pi^{(0)}}{\delta B_a} = \xi B_a - D_\mu [\hat{A}] (A - \hat{A})_a - \bar{c}_a^*. \quad (2.11)$$

The B-equation guarantees the stability of the gauge-fixing condition under radiative corrections. Notice that the r.h.s. of the above equation is linear in the quantum fields and thus no new external source is needed in order to define it. It does not receive any quantum corrections.

- the antighost equation

$$\frac{\delta \Pi^{(0)}}{\delta \bar{c}_a} = D[\hat{A}]_\mu \frac{\delta \Pi^{(0)}}{\delta A_{a\mu}^*} - D_\mu [A] \Omega_{a\mu}. \quad (2.12)$$

In the background Landau gauge one can also write an equation for the derivative of the effective action w.r.t. the ghost  $c_a$  (also sometimes called antighost equation) [21]. This was introduced in [22] in the context of the BFM formulation of Yang-Mills theory for semi-simple gauge groups in the background 't Hooft gauge.

- the background Ward identity

By using the background gauge-fixing condition in eq.(2.4), the vertex functional  $\Pi^{(0)}$  becomes invariant under a simultaneous gauge transformation of the quantum fields, external sources and the background connection, i.e.

$$\begin{aligned} \mathcal{W}_a \Pi^{(0)} = & -\partial_\mu \frac{\delta \Pi^{(0)}}{\delta \hat{A}_{a\mu}} + f_{acb} \hat{A}_{b\mu} \frac{\delta \Pi^{(0)}}{\delta \hat{A}_{c\mu}} - \partial_\mu \frac{\delta \Pi^{(0)}}{\delta \hat{A}_{a\mu}} + f_{acb} A_{b\mu} \frac{\delta \Pi^{(0)}}{\delta A_{c\mu}} + \sum_{\Phi \in \{B, c, \bar{c}\}} f_{acb} \Phi_b \frac{\delta \Pi^{(0)}}{\delta \Phi_c} \\ & + f_{acb} A_{b\mu}^* \frac{\delta \Pi^{(0)}}{\delta A_{c\mu}^*} + f_{acb} c_b^* \frac{\delta \Pi^{(0)}}{\delta c_c^*} + f_{acb} \bar{c}_b^* \frac{\delta \Pi^{(0)}}{\delta \bar{c}_c^*} = 0. \end{aligned} \quad (2.13)$$

Several comments are in order here. First we remark that the ST identity (2.10) is bilinear in the vertex functional, unlike the background Ward identity (2.13). Thus the relations between 1-PI amplitudes, derived by functional differentiation of the ST identity in eq.(2.10), are bilinear, in contrast with the linear ones generated by functional differentiation of the background Ward identity (2.13). The linearity of the background Ward identity explains why the BFM has been advantageously used in several applications, ranging from perturbative calculations in Yang-Mills theories [3, 23] and in the Standard Model [24, 25] to gravity and supergravity calculations [26].

One should however notice that the background Ward identity is no substitute to the ST identity: physical unitarity stems from the validity of the ST identity and does not follow from the background Ward identity alone [17].

Since the theory is non-anomalous, in perturbation theory all the functional identities in eqs. (2.10), (2.11), (2.12) and (2.13) are fulfilled also for the full vertex functional  $\Pi$  [27, 28]. This can be proven in a regularization-independent way by standard methods in Algebraic Renormalization [16, 17, 29]. In what follows we assume that the same identities hold true for the vertex functional of the theory in the non-perturbative regime.

### 3. Canonical Transformation for the Background Dependence

In order to control the dependence on the background connection we start from eq.(2.10) for the full vertex functional  $\Pi$ :

$$\int d^4x \Omega_{a\mu}(x) \frac{\delta \Pi}{\delta \hat{A}_{a\mu}(x)} = -\frac{1}{2}(\Pi, \Pi). \quad (3.1)$$

By taking a derivative w.r.t.  $\Omega_{a\mu}(x)$  and then setting  $\Omega_{a\mu} = 0$  we get

$$\left. \frac{\delta \Pi}{\delta \hat{A}_{a\mu}(x)} \right|_{\Omega_{a\mu}=0} = - \left( \left. \frac{\delta \Pi}{\delta \Omega_{a\mu}(x)} \right|_{\Omega_{a\mu}=0}, \Pi|_{\Omega_{a\mu}=0} \right). \quad (3.2)$$

This equation states that the derivative of the full vertex functional  $\Pi$  w.r.t.  $\hat{A}_{a\mu}$  at  $\Omega_{a\mu} = 0$  equals the variation of  $\Pi$  w.r.t. to a canonical transformation [20] generated by the fermionic functional  $\left. \frac{\delta \Pi}{\delta \Omega_{a\mu}(x)} \right|_{\Omega_{a\mu}=0}$ .

This is a crucial observation. First of all it shows that the source  $\Omega_{a\mu}$  has a clear geometrical interpretation, being the source of the fermionic functional which governs the canonical transformation giving rise to the background field dependence. Moreover, the dependence of the vertex functional on the background field is designed in such a way to preserve the validity of the ST identity (since the transformation is canonical).

In a non-perturbative setting, we can use eq.(3.2) in order to control the background-dependent amplitudes. For that purpose one needs a method for solving eq.(3.2). An effective recursive procedure is based on cohomological techniques. Let us introduce the auxiliary BRST differential  $\omega$  given by [1]

$$\omega \hat{A}_{a\mu} = \Omega_{a\mu}, \quad \omega \Omega_{a\mu} = 0, \quad (3.3)$$

while  $\omega$  does not act on the other variables of the theory. Clearly  $\omega^2 = 0$  and, since the pair  $(\hat{A}_{a\mu}, \Omega_{a\mu})$  forms a BRST doublet [18] under  $\omega$ , the cohomology of  $\omega$  in the space of local functionals spanned by  $\hat{A}_{a\mu}, \Omega_{a\mu}$  is trivial.

This allows us to introduce the homotopy operator  $\kappa$  according to

$$\kappa = \int d^4x \int_0^1 dt \hat{A}_{a\mu}(x) \lambda_t \frac{\delta}{\delta \Omega_{a\mu}(x)} \quad (3.4)$$

where the operator  $\lambda_t$  acts as follows on a functional  $X(\hat{A}_{a\mu}, \Omega_{a\mu}; \zeta)$

$$\lambda_t X(\hat{A}_{a\mu}, \Omega_{a\mu}; \zeta) = X(t\hat{A}_{a\mu}, t\Omega_{a\mu}; \zeta) \quad (3.5)$$

depending on  $\hat{A}_{a\mu}, \Omega_{a\mu}$  and on other variables collectively denoted by  $\zeta$ . The operator  $\kappa$  obeys the relation

$$\{\omega, \kappa\} = \mathbf{1}_{\hat{A}, \Omega} \quad (3.6)$$

where  $\mathbf{1}_{\hat{A}, \Omega}$  denotes the identity in the space of functionals containing at least one  $\hat{A}_{a\mu}$  or  $\Omega_{a\mu}$ .

Then we can rewrite the ST identity (3.1) as

$$\omega\mathbb{\Gamma} = \Upsilon, \quad (3.7)$$

where

$$\Upsilon = -\frac{1}{2}(\mathbb{\Gamma}, \mathbb{\Gamma}). \quad (3.8)$$

By the nilpotency of  $\omega$

$$\omega\Upsilon = 0. \quad (3.9)$$

Since  $\Upsilon|_{\Omega=0} = 0$ , we have from eq.(3.6)

$$\Upsilon = \{\omega, \kappa\}\Upsilon = \omega\kappa\Upsilon \quad (3.10)$$

Thus from eq.(3.7) we have the identity

$$\omega(\mathbb{\Gamma} - \kappa\Upsilon) = 0, \quad (3.11)$$

which has the general solution

$$\mathbb{\Gamma} = \mathbb{\Gamma}_0 + \omega\Xi + \kappa\Upsilon \quad (3.12)$$

with  $\Xi$  an arbitrary functional with ghost number  $-1$ . In the above equation  $\mathbb{\Gamma}_0$  denotes the vertex functional evaluated at  $\hat{A}_\mu = \Omega_\mu = 0$  (i.e. the set of 1-PI amplitudes with no background insertions and no  $\Omega_\mu$ -legs). The second term vanishes at  $\Omega_\mu = 0$  but is otherwise unconstrained. I.e. the extended ST identity is unable to fix the sector where  $\Omega_\mu \neq 0$ . However this ambiguity is irrelevant if one is interested in the 1-PI amplitudes with no  $\Omega_\mu$ -legs, which are those needed for physical computations.

In practical applications it is convenient to expand the term  $\kappa\Upsilon$  in eq.(3.12) according to the number of background legs. Then one can write a tower of equations allowing to solve for the dependence on  $\hat{A}_{a\mu}$  recursively down to the vertex functional at zero background  $\mathbb{\Gamma}_0$  [30].

In the zero background ghost sector  $\Omega_\mu = 0$ , the  $\omega\Xi$  term in Eq. (3.12) drops out, and one is left with the result (notice that  $\mathbb{\Gamma}_{\bar{c}_a^*} = -B_a$ )

$$\begin{aligned} \mathbb{\Gamma}|_{\Omega=0} &= \kappa\Upsilon + \mathbb{\Gamma}_0 \\ &= - \int d^4x \hat{A}_\mu^a(x) \int_0^1 dt \lambda_t \frac{\delta}{\delta \Omega_\mu^a(x)} \int d^4y \left[ \mathbb{\Gamma}_{A_b^{*v}}(y) \mathbb{\Gamma}_{A_v^b}(y) \right. \\ &\quad \left. + \mathbb{\Gamma}_{c^{*b}}(y) \mathbb{\Gamma}_{c^b}(y) + B^b(y) \mathbb{\Gamma}_{\bar{c}^b}(y) \right] \Big|_{\Omega_\mu=0} \\ &\quad + \mathbb{\Gamma}_0. \end{aligned} \quad (3.13)$$

In the above equation we have used the short-hand notation  $\mathbb{\Gamma}_\varphi = \frac{\delta \mathbb{\Gamma}}{\delta \varphi}$ .

#### 4. Field and Antifield Redefinition in the BFM

By taking a derivative w.r.t.  $\Omega_{a\mu}$  of eq.(2.10) and then setting  $\Omega_\mu = 0$  we get (from now on we denote by  $\Gamma$  the vertex functional where  $\Omega_\mu$  is set to zero)

$$\begin{aligned} \frac{\delta\Gamma}{\delta\widehat{A}_{a\mu}} = & - \int d^4x \left( \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta A_{bv}^*} \Big|_{\Omega_\mu=0} \frac{\delta\Gamma}{\delta A_{bv}} - \frac{\delta\Gamma}{\delta A_{bv}^*} \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta A_{bv}} \Big|_{\Omega_\mu=0} \right. \\ & - \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta c_b^*} \Big|_{\Omega_\mu=0} \frac{\delta\Gamma}{\delta c_b} - \frac{\delta\Gamma}{\delta c_b^*} \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta c_b} \Big|_{\Omega_\mu=0} \\ & \left. - \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta \bar{c}_b^*} \Big|_{\Omega_\mu=0} \frac{\delta\Gamma}{\delta \bar{c}_b} - \frac{\delta\Gamma}{\delta \bar{c}_b^*} \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta \bar{c}_b} \Big|_{\Omega_\mu=0} \right). \end{aligned} \quad (4.1)$$

Suppose that one can find a set of field and antifield redefinitions

$$\begin{aligned} A_{av} & \rightarrow A_{av} - \mathcal{G}_{av}, \quad A_{av}^* \rightarrow A_{av}^* - \mathcal{G}_{av}^*, \\ c_a & \rightarrow c_a + \mathcal{C}_a, \quad c_a^* \rightarrow c_a^* + \mathcal{C}_a^*, \\ \bar{c}_a & \rightarrow \bar{c}_a + \bar{\mathcal{C}}_a, \quad \bar{c}_a^* \rightarrow \bar{c}_a^* + \bar{\mathcal{C}}_a^*, \end{aligned} \quad (4.2)$$

such that

$$\begin{aligned} \frac{\delta\mathcal{G}_{bv}}{\delta\widehat{A}_{a\mu}} &= \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta A_{bv}^*} \Big|_{\Omega_\mu=0}, & \frac{\delta\mathcal{G}_{bv}^*}{\delta\widehat{A}_{a\mu}} &= \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta A_{bv}} \Big|_{\Omega_\mu=0}, \\ \frac{\delta\mathcal{C}_b}{\delta\widehat{A}_{a\mu}} &= \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta c_b^*} \Big|_{\Omega_\mu=0}, & \frac{\delta\mathcal{C}_b^*}{\delta\widehat{A}_{a\mu}} &= \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta c_b} \Big|_{\Omega_\mu=0}, \\ \frac{\delta\bar{\mathcal{C}}_b}{\delta\widehat{A}_{a\mu}} &= \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta \bar{c}_b^*} \Big|_{\Omega_\mu=0}, & \frac{\delta\bar{\mathcal{C}}_b^*}{\delta\widehat{A}_{a\mu}} &= \frac{\delta^2\Pi}{\delta\Omega_{a\mu}\delta \bar{c}_b} \Big|_{\Omega_\mu=0}. \end{aligned} \quad (4.3)$$

Then the solution to eq.(4.1) is obtained by carrying out the field and antifield redefinition in eq.(4.2) on the vertex functional at zero background  $\Gamma[A_\mu, c, \bar{c}, A_\mu^*, c^*, \bar{c}^*; 0]$  according to

$$\Gamma[A_\mu, c, \bar{c}, A_\mu^*, c^*, \bar{c}^*; \widehat{A}_\mu] = \Gamma[A_\mu - \mathcal{G}_\mu, c + \mathcal{C}, \bar{c} + \bar{\mathcal{C}}_a, A_\mu^* - \mathcal{G}_\mu^*, c^* + \mathcal{C}^*, \bar{c}^* + \bar{\mathcal{C}}_a^*; 0]. \quad (4.4)$$

The background-dependent field and antifield redefinition in eq.(4.2) generalizes the classical background-quantum splitting and is the correct mapping when quantum corrections are taken into account. This result directly follows from the requirement of the validity of the ST identity. We remark that the redefinition in eq.(4.2) also involves the ghosts and the antifields.. This is in sharp contrast with the classical background-quantum splitting, which is limited to the gauge field.

The existence of the field and antifield redefinitions in eq.(4.2) requires a careful check of the corresponding integrability conditions. This has been done for the case of the gauge field in Ref. [1] and requires an extensive use of the relations among 1-PI amplitudes encoded in the ST identity. The analysis of the general case will be deferred to a later work. Here we only wish to remark that the field and antifield redefinitions are related to the deformation of the canonical variables controlled by the canonical transformation generated by  $\frac{\delta\Pi}{\delta\Omega_\mu} \Big|_{\Omega_\mu=0}$ .



## 5. One-loop Deformed Instanton Profile

As an example, we sketch the one-loop corrections to the classical instanton [31] profile function. For a detailed treatment we refer the reader to [30]. To lowest order, the background-dependent field redefinition for  $A_\mu$  in the first of Eqs. (4.2) implies that the background field will be deformed according to

$$V_\mu^a(x) = \widehat{A}_\mu^a(x) + \int d^4y \Gamma_{\Omega_\mu^a A_\nu^{*b}}(y, x) \widehat{A}_\nu^b(y), \quad (5.1)$$

where the auxiliary function  $\Gamma_{\Omega_\mu^a}$  is evaluated at  $\widehat{A} = 0$ ; equivalently, in momentum space one has

$$V_\mu^a(p) = \left[ g_{\mu\nu} \delta^{ab} + \Gamma_{\Omega_\mu^a A_\nu^{*b}}(p) \right] \widehat{A}_\nu^b(p). \quad (5.2)$$

Notice that the formulas above are totally general and not limited to the instanton case we are considering here; thus the calculation of  $\Gamma_{\Omega_\mu^a A_\nu^{*b}}^{(1)}$  performed below will determine the universal (lowest-order) deformation of any background at one loop level.

The function  $\Gamma_{\Omega_\mu^a A_\nu^{*b}}$  can be decomposed according to the following form factors:

$$\Gamma_{\Omega_\mu^a A_\nu^{*b}}(p) = -\delta^{ab} \frac{g^2 C_A}{16\pi^2} \left[ A(p) g_{\mu\nu} + B(p) \frac{p_\mu p_\nu}{p^2} \right], \quad (5.3)$$

we see that in the instanton case the  $B$  form factor does not contribute, and we finally get the one-loop corrected background field

$$V_\mu^a(p) = \overline{\eta}_{\mu\nu}^a p_\nu [f_0(p) + f_1(p)]; \quad f_1(p) = -\frac{g^2 C_A}{16\pi^2} A^{(1)}(p) f_0(p), \quad (5.4)$$

and  $f_0$  is the classical instanton profile

$$\begin{aligned} f_0(p) &= (-8\pi^2 i \rho) \frac{1}{p^3} \left[ -\frac{2}{p\rho} + K_1(p\rho) - (p\rho) K_1'(p\rho) \right] \\ &= (-8\pi^2 i \rho) \frac{1}{p^3} \left[ -\frac{2}{p\rho} + (p\rho) K_2(p\rho) \right], \end{aligned} \quad (5.5)$$

with  $K_i$  the modified Bessel functions of the second kind.

Choosing the Landau gauge (which is the appropriate choice in the instanton case) one has at the one-loop level

$$\Gamma_{\Omega_\mu^a A_\nu^{*b}}^{(1)}(p) = -g^2 C_A \delta^{ab} \int_k \frac{1}{k^2 (k+p)^2} P_{\mu\nu}(k), \quad (5.6)$$

where  $C_A$  is the Casimir eigenvalue of the adjoint representation [ $C_A = N$  for  $SU(N)$ ]; a straightforward calculation gives (Euclidean space)

$$\begin{aligned} A^{(1)}(p) &= -\frac{3}{2} \frac{1}{d-4} + \frac{3}{2} - \frac{3}{4} \log \left( \frac{p^2}{\mu^2} \right) \\ B^{(1)}(p) &= -\frac{1}{2}. \end{aligned} \quad (5.7)$$

The divergence in  $A^{(1)}$  in the  $d \rightarrow 4$  limit is removed by adding a counterterm controlled by the invariant  $\mathcal{S}_0(A_{a\mu}^* \widehat{A}_{a\mu})$ , where  $\mathcal{S}_0$  is the linearized ST operator  $\mathcal{S}_0 = (\mathbb{I}^{(0)}, \cdot)$ .

Eq.(5.7) does not change in the one-loop approximation if fermions are added to the theory.

It is now convenient to have a representation of the instanton profile in position space; therefore we need to find the inverse Fourier transform of  $f_1$ . Let us set

$$V_\mu^a(x) = \bar{\eta}_{\mu\nu}^a x_\nu [f_0(x) + f_1(x)]; \quad f_1(x) = \frac{i}{4\pi^2} \frac{x_\nu}{r^2} \frac{\partial}{\partial x_\nu} \int_0^\infty dp p^3 f_1(p) \frac{1}{pr} J_1(pr). \quad (5.8)$$

The evaluation of  $f_1(x)$  can be performed analytically, and we find<sup>1</sup>

$$f_1(x) = -3 \frac{g^2 C_A}{16\pi^2} \left[ \frac{1}{\rho^2} \frac{1 + \log \rho \mu}{\lambda^2 (1 + \lambda^2)} - \frac{x_\nu}{r^2} \frac{\partial}{\partial x_\nu} \int_0^\infty dt F(t, \lambda) \right], \quad (5.9)$$

where we have  $t = p\rho$  and

$$F(t, \lambda) = \log t \left[ -\frac{2}{t} + t K_2(t) \right] \frac{1}{\lambda t} J_1(\lambda t). \quad (5.10)$$

The integral in  $t$  yields

$$\begin{aligned} \int_0^\infty dt F(t, \lambda) &= \frac{1}{8\lambda^2} \left\{ \log^2(1 + \lambda^2) \lambda^2 - 4 \left( \log \frac{\lambda}{4} + 2\gamma_E - 1 \right) \lambda^2 \log \lambda + 2\lambda^2 \text{Li}_2 \left( \frac{1}{1 + \lambda^2} \right) \right. \\ &\quad \left. + \left[ -2\lambda^2 \log \frac{\lambda^2}{1 + \lambda^2} + (-2 + 4\gamma_E - 4 \log 2) \lambda^2 - 2 \right] \log(1 + \lambda^2) \right\}, \quad (5.11) \end{aligned}$$

where  $\gamma_E$  is the Euler-Mascheroni constant ( $\gamma_E = 0.57721\dots$ ); thus one has

$$\frac{x_\nu}{r^2} \frac{\partial}{\partial x_\nu} \int_0^\infty dt F(t, \lambda) = \frac{1}{\rho^2} \left[ -\frac{\gamma_E - \log 2}{\lambda^2 (1 + \lambda^2)} - \frac{\log \lambda}{\lambda^2} + \frac{1 + \lambda^4}{2\lambda^4 (1 + \lambda^2)} \log(1 + \lambda^2) \right], \quad (5.12)$$

which gives for  $f_1$  the final result

$$f_1(\lambda) = -3 \frac{g^2 C_A}{16\pi^2} \frac{1}{\rho^2} \left[ \frac{1 + \log \rho \mu}{\lambda^2 (1 + \lambda^2)} + \frac{\gamma_E - \log 2}{\lambda^2 (1 + \lambda^2)} + \frac{\log \lambda}{\lambda^2} - \frac{1 + \lambda^4}{2\lambda^4 (1 + \lambda^2)} \log(1 + \lambda^2) \right]. \quad (5.13)$$

$f_1$  shows a log enhancement w.r.t. the classical profile both for  $\lambda \rightarrow 0$  and for  $\lambda \rightarrow \infty$ , i.e. both for small and large instanton sizes. Clearly the one-loop corrected instanton is neither self-dual nor it reduces to pure gauge as  $r \rightarrow \infty$ .

## 6. Conclusions

We have shown that there is a close connection between the quantization of Yang-Mills theory in a topologically non-trivial background and the ST identity of the theory (extended in the presence of a background connection).

If the ST identity is fulfilled, the dependence of the vertex functional on the background can be uniquely reconstructed (in the relevant sector at  $\Omega_\mu = 0$ ) by algebraic techniques, starting from 1-PI amplitudes evaluated at zero background.

<sup>1</sup>This is only true in the singular gauge. In the regular gauge the integral over  $p$  does not converge.

The procedure amounts to perform a field and antifield redefinition, controlled by a canonical transformation w.r.t. the BV bracket associated with the ST identity. Moreover, a compact homotopy formula for the full vertex functional at non-zero background has been derived.

As an example of this technique, we have explicitly worked out in lowest order in the background field the one-loop redefinition of the SU(2) instanton profile, induced by the canonical transformation responsible for the quantum deformation of the classical background-quantum splitting.

These results could be applied to a variety of problems, like e.g. the implementation of the BFM on the lattice or SD-computations in the presence of a topologically non-trivial background.

Several aspects should be further investigated. We only mention a few of them here. The fulfillment of the integrability conditions for eqs.(4.3) has to be further analyzed. Although it is plausible that these conditions are indeed fulfilled (since they are a consequence of the existence of a canonical transformation governing the dependence on the background field), it would be very useful to obtain a more explicit form for the field and antifield redefinition.

The SD equations for the  $\Omega$ -dependent kernels in eqs.(4.3) should be studied. Finally one could also investigate whether the present approach, derived for the 1-PI vertex functional  $\Pi$ , can be extended to the well-known 2-PI formalism of [32].

## Acknowledgments

Financial support from INFN and ECT\* is gratefully acknowledged.

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