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Gauge Invariant Parton Subamplitudes: A Simple Example

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We explain how the pinch technique can be used to absorb non conformal logarithms. The procedure cannot be extend to general maximally helicity violating amplitudes.

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1. Introduction

The determination of the scale for processes is major problem in pQCD. To a given order in pQCD, predictions depend will depend on the renormalisation scheme. An appropriate chosen scale would absorb logarithms involving the β function. The remaining terms are the ones of a conformal theory. This is called the Principe of Maximal Conformality (PMC) [1].

In QED the scale is simply the virtuality of the virtual photon. The process $e^+e^- \to e^+e^-$ involves $s-$ and $t-$channels at leading order. Weighting the corresponding diagram by $\alpha(s)$ or $\alpha(t)$ sums all vacuum polarization logarithms.

However in QCD the diagrammatic representation is misleading. The non linearities in the BRST transformation induce subtle kinematics cancelations between vertices and propagators. The non conformal terms are shared between distinct diagrams and one has to isolated them before summing into a running coupling. This procedure of isolating well-defined kinematics subset of an amplitude is called the Pinch Technique [2, 3]. Once this procedure is applied the non conformal logarithms show up in a natural way as in QED.

We explain the philosophy in Section 2 and apply it to the quark gluon scattering in Section 3. A first attempt to generalisation to n gluon amplitude is presented in Section 4 where some problems of gauge invariance are mentioned.

2. General methodology

We illustrate the general procedure with the simplest example, the quark scattering. At the one loop level non conformal terms are hidden in various diagrams but they all arise from longitudinal momenta that will eventually pinch propagators. The quickest way to get the final answer is to work in the Feynman gauge. In this gauge there is no pinching momentum in the gluon propagator and the only contributions we have to extract comes from the gluon vertices. Of course the final answer is gauge independent, we refer the reader interested of the procedure in a general covariant gauge to ref. [3].

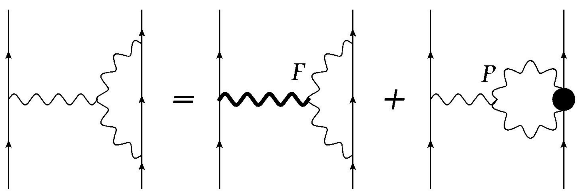

Figure 1: Vertex decomposition $\Gamma = \Gamma^F + \Gamma^P$. Γ^P pinches the quark propagator and gives rise to a propagator-like term. The latter is finally added to the gluon self-energy.

The starting point is the decomposition of the vertex into its regular and pinching (longitudinal) parts:

$$\begin{aligned}\Gamma_{\alpha\mu\beta}(k,q) &= \Gamma^F_{\alpha\mu\beta}(k,q) + \Gamma^P_{\alpha\mu\beta}(k,q) \\ \Gamma^F_{\alpha\mu\beta}(k,q) &= -(2k+q)_\mu g_{\alpha\beta} + 2q_\alpha g_{\mu\beta} - 2q_\beta g_{\mu\alpha} \\ \Gamma^P_{\alpha\mu\beta}(k,q) &= k_\alpha g_{\mu\beta} + (k+q)_\beta g_{\alpha\mu}\end{aligned} \tag{2.1}$$

The Γ^P pinches the internal propagators thanks to an elementary Ward identity $\not{k} = (\not{k} + \not{p} - m) - (\not{p} - m)$ and gives rise to a propagator-like piece (the factor 2 comes from the mirror diagram)

$$i\Pi^P_{\mu\nu}(q) = 2Ng^2 q^2 t_{\mu\nu}(q) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(k+q)^2}, \tag{2.2}$$

with $t_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu / q^2$ the transverse projector. When this contribution is added to the usual

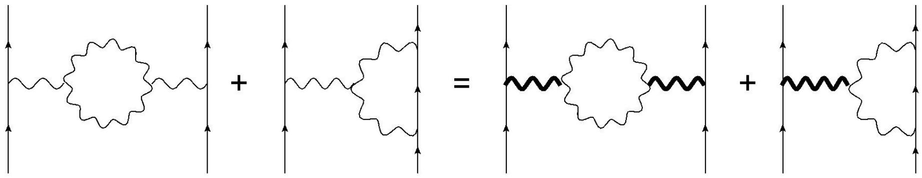

Figure 2: The pinch technique procedure. The mirror vertex graph has been omitted. Thick lines are background gluons.

one-loop gluon propagator in the Feynman gauge, the non conformal logarithm appear in the new gluon self-energy as it should

$$i\widehat{\mathbf{\Pi}}(q^2) = \beta_0 (g\mu^\varepsilon)^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(k+q)^2} = i\beta_0 \frac{\alpha(\mu)}{4\pi} \left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon \frac{1}{\varepsilon} = i\beta_0 \frac{\alpha(\mu)}{4\pi} \left(\frac{1}{\varepsilon} - \log(q^2/\mu^2)\right),$$

where $\beta_0 = (11N - 2n_f)/3$ is the first coefficient of the β function and $n = 4 - 2\varepsilon$ is the space-time dimension. The Lorentz structure has been factorized, $\Pi_{\mu\nu}(q) = q^2 t_{\mu\nu}(q) \mathbf{\Pi}(q^2)$ and constant terms were omitted.

We have traded the conventional representation for a new one where each subsets represent now well-defined kinematics (propagators, vertices and boxes) and are gauge independent. Interestingly enough, the new Green's functions (i.e gluon propagator and vertex) obey Abelian-like Ward identities. For this reason, the subsets are individually gauge invariant. Moreover, the new Green's functions generated coincide with the ones computed in the Background Field Method (BFM) in the Feynman gauge. It can be indeed check that Γ^F in (2.1) correspond to the background-quantum vertex in the (background) Feynman gauge. The thick lines in Fig. 1 and 2 represent background gluons. The (background) gluon propagator $\widehat{\Delta}_{\mu\nu}(q) = t_{\mu\nu}\widehat{\Delta}(q^2)$ with $\widehat{\Delta}^{-1}(q^2) = q^2(1 - \widehat{\mathbf{\Pi}}(q^2))$ has still a $\mu-$dependence which is canceled in the renormalisation group invariant (RGI) combination

$$\alpha(\mu)\widehat{\Delta}(q^2, \mu^2) = \frac{1}{q^2} \frac{\alpha(\mu^2)}{1 + (\beta_0/4\pi)\alpha(\mu)\log(q^2/\mu^2)} = \frac{\alpha(q^2)}{q^2}, \tag{2.3}$$

thanks to the Abelian-like relation $Z_g^2 = Z_A$ satisfied by the PT Green's functions. This relation is depicted in Fig. 3.

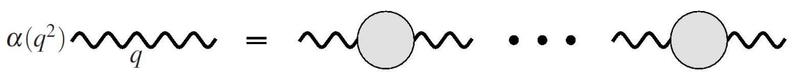

Figure 3: In the PT-BFM framework $\beta \neq 0$ logarithms are absorb into the running of the coupling constant.

The amplitudes is now proportional to the tree level one with a background exchanged gluon. We can safely use the background gluon virtuality as the scale for the running coupling for the tree level amplitude to encode all $\beta \neq 0$ terms. Of course, since the gluon quark is the same for quantum or background gluon, the generation of background gluon is trivial at tree level in the case of quark scattering but the procedure holds for any on-shell processes.

This idea is a generalisation of the method presented in [4]. In this reference the non conformal logarithms are identified, for processes that do not involve gluon self-coupling, with the n_f dependence of the expression. However, as the author pointed out, for reaction with gluon-gluon couplings, quark loop appear in the first-order correction to the three gluon vertex as well as in propagator insertions. It is therefore difficult to separate the divergent part of the part which renormalizes α. Fortunately, the pinch technique solves this ambiguity by an appropriate identification of the desired part of the vertex.

Squaring the amplitude for the quark scattering, we obtain the well-know result

$$\mathscr{M}^2(qQ \to qQ) = \alpha^2(t)\frac{s^2+u^2}{t^2} = \alpha_\mu^2\frac{s^2+u^2}{t^2}\left(1+2\beta_0\frac{\alpha_\mu}{4\pi}\log(q^2/\mu^2)\right). \tag{2.4}$$

Expanding $\alpha(x) = \alpha_0[1+(\beta/4\pi)\alpha_0\log(x)]^{-1}$ gives the leading non conformal logarithms. At next-to-leading order (NLO) the remaining terms are the ones of a conformal theory.

If the quark are identical the $u-$channel is worked out in a similar fashion and we find

$$\mathscr{M}^2(qq \to qq) = \alpha^2(t)\frac{s^2+u^2}{t^2} + \alpha^2(u)\frac{s^2+t^2}{u^2} - 2\alpha(t)\alpha(t)\frac{s^2}{ut}. \tag{2.5}$$

3. Quark gluon scattering

The quark gluon scattering involves the three s,t,u channels. In this case also we generate for free a background gluon in the $s-$channel thanks to current conservation. The $s-$diagram will generate non conformal logarithms that we absorb by weighting this graph with $\alpha(s)$ as explained in section 2. The other channels do not generate those terms are at NLO.

We are interested in the cross section and would like to square the amplitude. Although we can use any gauge for the gluon polarisation vectors it is more instructive to work with undetermined $\eta_{1,2}$ gauges [5]. Let us denote $\mathscr{T}_s^F$ the $s-$channel with a background gluon, $\mathscr{T}_t$ the sum of $t-$ and $u-$channels and $P^{\mu\sigma}(k,\eta) = -g_{\mu\nu} + \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta\cdot k} + \eta^2\frac{k_\mu k_\nu}{(\eta\cdot k)^2}$. The cross section reads

$$\begin{aligned}\mathscr{M}^2(q\bar{q} \to gg) &= (\mathscr{T}_s^F + \mathscr{T}_t)_{\mu\nu} P^{\mu\sigma}(k_1,\eta_1) P^{\nu\lambda}(k_2,\eta_2) (\mathscr{T}_s^F + \mathscr{T}_t)^*_{\sigma\lambda}, &(3.1)\\ &= \mathscr{T}_t\mathscr{T}_t^* + (\mathscr{T}_s^F\mathscr{T}_s^{F*} - 8SS^*) + (\mathscr{T}_s^F\mathscr{T}_t^* + \mathscr{T}_t\mathscr{T}_s^{F*}), &(3.2)\end{aligned}$$

By virtue of the Ward identity $k_1^\mu \mathscr{T}_{\mu\nu} = 2k_{2\nu}S$ with $\mathscr{T} = \mathscr{T}_s^F + \mathscr{T}_t^F$ and $S = \frac{1}{2}(k_1-k_2)_\alpha\bar{v}\gamma^\alpha u$ a ghost contribution, the gauges cancel out leading to three gauge invariant subamplitudes squared shown in Fig. 4. This procedure avoids choosing a gauge for the gluons and avoids using the tensor $P^{\mu\nu}(k,\eta)$ which simplify the evaluation of the cuts.

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The final result is

$$\mathcal{M}^2(q\bar{q}\to gg) = 2\alpha_0^2 C_F \frac{u^2+t^2}{ut} - C_A\alpha^2(s)\frac{7s^2+(t-u)^2}{s^2} - 2C_A\alpha_0\alpha(s)\frac{(t-s)+(u-s)}{s}$$
$$= 2\alpha_0^2(t^2+u^2)\left(\frac{C_F}{ut}-\frac{N}{s^2}\right) - 2\alpha_0^2 C_A \log(s/\mu^2)\left(\frac{5t^2+5u^2+6ut}{s^s}\right). \quad (3.3)$$

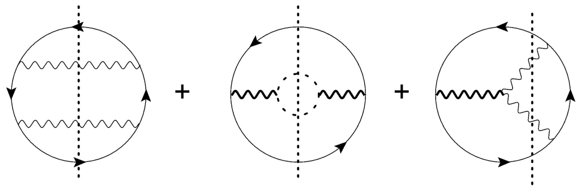

Figure 4: Three gauge invariant subamplitudes squared for the quark gluon scattering. The dashed blob denotes gluon and ghost loops.

In practice, when evaluating gluon cuts in a cross section, the gauges of internal and external gluons cancel and lead to the background Feynman gauge for the internal gluons and the Feynman gauge for the cut gluons. This method should be generalized for multi scales problems.

4. Helicity structure

The above procedure although straightforward requires a dedicated calculation for each process. There are however general expressions for n gluons amplitudes decomposed into colour-ordered partial amplitudes multiplied by an associated colour trace. Summing over all non cyclic permutations reconstructs the full amplitude [6]

$$\mathcal{A}_n(\{k_i,\lambda_i,a_i\}) = g^{n-2}\sum_{\sigma\in S_n/Z_n}\mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\mathcal{A}_n\left(k_{\sigma(1)}^{\lambda_{\sigma(1)}},\ldots,k_{\sigma(n)}^{\lambda_{\sigma(n)}}\right), \quad (4.1)$$

where k_i, λ_i and a_i are respectively the momentum, helicity and colour index of the $i-$th external (outgoing) gluon, g is the coupling constant and S_n/Z_n is the set of non cyclic permutations of $\{1,\ldots,n\}$. The $SU(N)$ generators T^a are normalized so that $\mathrm{Tr}(T^aT^b)=\delta^{ab}$.

In a supersymmetric theory, amplitudes with all helicities identical, or all but one identical, vanish due to supersymmetric Ward identities. Tree-level gluon amplitudes in super-Yang-Mills and in purely gluonic Yang-Mills are identical, so that

$$\mathcal{A}_n(1^\pm,2^+,\ldots,n^+) = 0. \quad (4.2)$$

The first non vanishing amplitudes are called Maximally Helicity Violating (MHV) amplitudes. Tree-level MHV amplitudes have simple expressions for n gluons scattering

$$\mathcal{A}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i\frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}. \quad (4.3)$$

We used the following notations

$$\psi_{\pm}(p) = \frac{1}{2}(1\pm\gamma_5)\psi(p), \tag{4.4}$$

$$|p\pm\rangle = \psi_{\pm}(p), \qquad \langle p\pm| = \bar{\psi}_{\pm}(p), \tag{4.5}$$

$$\langle pq\rangle = \langle p+|q-\rangle, \quad [pq] = \langle p-|q+\rangle. \tag{4.6}$$

Thanks to the colour ordering the MHV amplitudes have only collinear poles and are gauge independent. It would be of great interested to split them into gauge independent poles, each would be finally weighted by its appropriated scale to sum non conformal logarithms, but unfortunately, as a simple example show us, it is impossible. For the 4 gluons scattering, $\mathscr{A}(1,2,3,4)$ can be split, for each helicity configurations, in two contributions coming from the s and t poles. We have schematically

$$\mathscr{A}(1^-,2^-,3^+,4^+) = S_1+T_1, \quad \mathscr{A}(1^-,2^+,3^-,4^+) = S_2+T_2, \quad \ldots \tag{4.7}$$

Although for each helicity configurations the combination S_i+T_i is gauge independent, the coefficient S_i and T_i depend on the gauge chosen for the gluons. This gauge dependence cancel also in some particular combination like $\sum_i S_i^2$ involved in the cross section.

At the level of the amplitude, the cancelation proceeds though the use of the Schouten identity,

$$\langle ij\rangle\langle kl\rangle = \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle. \tag{4.8}$$

As a consequence, the pole contribution merge into a single gauge independent expression. For instance denoting g the gauge momentum of gluon 1, the following combination appear in the calculation

$$\frac{1}{\langle g1\rangle}\left(\frac{\langle 3g\rangle}{\langle 31\rangle}+\frac{\langle 2g\rangle}{\langle 21\rangle}\right) = \frac{\langle 32\rangle\langle g1\rangle}{\langle g1\rangle\langle 31\rangle\langle 12\rangle} \tag{4.9}$$

It is therefore no longer possible to identify each pole contribution separately.

5. Conclusion

We showed how the pinch technique by triggering dynamically the background Feynman gauge allows us to choose the appropriate scale for the coupling constant. All non conformal logarithms are therefore absorbed into the definition of a running coupling.

The pinch technique can be apply to any processes. Once background gluons are generated, when computing the cross section, the sum over physical gluon polarisations can be replaced formally by cuts of ghost and gluon propagators in the Feynman (background) gauge.

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