PROCEEDINGS OF SCIENCE

Noncommutative Supergravity

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We present a noncommutative extension of first order Einstein-Hilbert gravity in the context of twist-deformed space-time, with a $\star$-product associated to a triangular Drinfeld twist. In particular the $\star$-product can be chosen to be the usual Groenewald-Moyal product. The action is geometric, invariant under diffeomorphisms and centrally extended Lorentz $GL(2,C)$ $\star$-gauge transformations. By imposing a charge conjugation condition on the noncommutative vielbein, the commutative limit reduces to ordinary gravity, with local Lorentz invariance and usual real vielbein. The theory is then coupled to fermions, by adding the analog of the Dirac action in curved space. A noncommutative Majorana condition can be imposed, consistent with the $\star$-gauge transformations.

We also discuss a noncommutative deformation of $D=4$, $N=1$ supergravity, reducing to the usual simple supergravity in the commutative limit. Its action is invariant under diffeomorphisms and local $GL(2,C)$ $\star$-gauge symmetry. The supersymmetry of the commutative action is broken by noncommutativity. Local $\star$-supersymmetry invariance can be realized in a noncommutative $D=4$, $N=1$ supergravity with chiral gravitino and complex vierbein.

Corfu Summer Institute on Elementary Particles and Physics - Workshop on Non Commutative Field Theory and Gravity,
September 8-12, 2010
Corfu Greece

*The results presented in these Proceedings are based on joint work with Paolo Aschieri.

PoS(CNCFG2010)013

1. Introduction

In the quest of a consistent quantum theory of gravity, noncommutativity of spacetime has been actively considered in the last two decades. The main historical motivation resided in uncertainty relations between coordinates, that could model "granularity" of spacetime at the Planck scale, and ensure a built-in regularization mechanism for the quantum theory. In string/brane theories, a framework that unifies particles and geometry as excitations of extended relativistic objects, quantum finiteness is believed to hold because of the "smearing" of interactions due to the spatial extension of the basic objects. A noncommuting (NC) scenario was seen to emerge from the string/brane framework when considering the low-energy limit of open strings in a background B-field [1]. Also the field theory effective description of the infinite tower of massive higher spins contained in the string spectrum seems to point towards noncommutative geometric structures [2]. It seems fair to say that noncommutative geometry plays a central role in theories that aim to quantize gravity.

Even lattice theories (for gauge fields and gravity) can be related to a NC geometric structure. Indeed discrete group lattices, for example, can be endowed with a natural NC differential geometry that enables to generalize (continuous) geometric quantities to the discrete case, thus allowing the formulation of gauge and gravity actions on these discrete lattices (see for example [3, 4, 5] and references therein).

Quantum groups have also been investigated as an interesting arena for NC gauge and gravity theories. Inhomogeneous quantum groups (including the quantum Poincaré group) and their NC differential geometries have been used to construct NC generalizations of gravity lagrangians (see for ex. the ref.s in [6]).

NC gravity theories have been constructed more recently in the twisted noncommutative geometry setting [7, 8, 9], that generalizes the Moyal deformation, where ordinary products between fileds are replaced by the noncommutative Moyal product. In this setting the deformed theory is invariant under $\star$-diffeomorphisms, but in [8] no gauge invariance on the tangent space (generalizing local Lorentz symmetry) is incorporated, and therefore coupling to fermions could not be implemented. A local symmetry, enlarging the local $SO(3,1)$ symmetry of $D=4$ Einstein gravity to $GL(2,C)$, has been considered in the approach of Chamseddine [7]. The resulting theory has a complicated classical limit, with two vielbeins (or, equivalently, a complex vielbein). Noncommutative gravities in lower dimensions have been studied in [10] (D=2) and in [11, 12] (D=3).

In these Proceedings we review the twisted NC deformations of gravity and supergravity theories constructed in ref.s [13, 14], where the noncommutativity is given by a $\star$-product associated to a very general class of twists. This $\star$-product can also be x-dependent. As a particular case we obtain noncommutative theories where noncommutativity is realized with the Moyal-Groenewald $\star$-product.

The topics reviewed here are:

1) a noncommutative gravity action, with a coupling to fermions, that reduces in the commutative limit to the action of ordinary gravity + fermions, without extra fields (in particular without an extra graviton). This is achieved by imposing a noncommutative charge conjugation condition on the bosonic fields, consistent with the $\star$-gauge transformations. We can also impose a noncommutative generalization of the Majorana condition on the fermions, compatible with the $\star$-gauge

transformations. The action is invariant under diffeomorphisms, $\star$-diffeomorphisms and a $GL(2,C)$ $\star$-gauge symmetry that becomes ordinary local Lorentz symmetry in the commutative limit.

2) an action for a noncommutative deformation of $N=1$ supergravity in $D=4$, invariant under diffeomorphisms and local $GL(2,C)$ $\star$-gauge transformations, but without $\star$-supersymmetry. In this case noncommutativity breaks the local supersymmetry of the commutative theory. The commutative limit is the usual $D=4$, $N=1$ simple supergravity, with a Majorana gravitino. We can obtain local $\star$-supersymmetry invariance of the noncommutative action if we impose a Weyl condition on the fermions, rather than a Majorana condition. This leads to a noncommutative supergravity whose commutative limit is a chiral $D=4$, $N=1$ supergravity with two vierbein fields (or a complex vierbein) and a left-handed gravitino.

In the Appendix we collect $D=4$ gamma matrices conventions and properties.

2. First order gravity coupled to fermions

2.1 Action

The usual action of first-order gravity coupled to fermions can be recast in an index-free form, convenient for generalization to the non-commutative case:

$$S=\int Tr\,(iR\wedge V\wedge V\gamma_5-[(D\psi)\bar{\psi}-\psi D\bar{\psi}]\wedge V\wedge V\wedge V\gamma_5) \tag{2.1}$$

The fundamental fields are the 1-forms Ω (spin connection), V (vielbein) and the fermionic 0-form ψ (spin 1/2 field). The curvature 2-form R and the exterior covariant derivative on ψ are defined by

$$R=d\Omega-\Omega\wedge\Omega, \qquad D\psi=d\psi-\Omega\psi \tag{2.2}$$

with

$$\Omega=\frac{1}{4}\omega^{ab}\gamma_{ab}, \qquad V=V^a\gamma_a \tag{2.3}$$

and thus are 4×4 matrices in the spinor representation. See Appendix A for $D=4$ gamma matrix conventions and useful relations. The Dirac conjugate is defined as usual: $\bar{\psi}=\psi^\dagger\gamma_0$. Then also $(D\psi)\bar{\psi}$, $\psi D\bar{\psi}$ are matrices in the spinor representation, and the trace Tr is taken on this representation. Using the $D=4$ gamma matrix identities:

$$\gamma_{abc}=i\varepsilon_{abcd}\gamma^d\gamma_5, \qquad Tr(\gamma_{ab}\gamma_c\gamma_d\gamma_5)=-4i\varepsilon_{abcd} \tag{2.4}$$

leads to the usual action:

$$S=\int R^{ab}\wedge V^c\wedge V^d\varepsilon_{abcd}+i[\bar{\psi}\gamma^a D\psi-(D\bar{\psi})\gamma^a\psi]\wedge V^b\wedge V^c\wedge V^d\varepsilon_{abcd} \tag{2.5}$$

with

$$R\equiv\frac{1}{4}R^{ab}\gamma_{ab}, \quad R^{ab}=d\omega^{ab}-\omega^a_{\ c}\wedge\omega^{cb} \tag{2.6}$$

2.2 Invariances

The action is invariant under local diffeomorphisms (it is the integral of a 4-form on a 4-manifold) and under the local Lorentz rotations:

$$\delta_\varepsilon V = -[V,\varepsilon], \quad \delta_\varepsilon \Omega = d\varepsilon - [\Omega,\varepsilon], \quad \delta_\varepsilon \psi = \varepsilon\psi, \quad \delta_\varepsilon \bar{\psi} = -\bar{\psi}\varepsilon \tag{2.7}$$

with

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} \tag{2.8}$$

The invariance can be directly checked on the action (2.1) noting that

$$\delta_\varepsilon R = -[R,\varepsilon] \quad \delta_\varepsilon D\psi = \varepsilon D\psi, \quad \delta_\varepsilon((D\psi)\bar{\psi}) = -[(D\psi)\bar{\psi},\varepsilon], \quad \delta_\varepsilon(\psi D\bar{\psi}) = -[\psi D\bar{\psi},\varepsilon], \tag{2.9}$$

using the cyclicity of the trace Tr (on spinor indices) and the fact that ε commutes with γ_5. The Lorentz rotations close on the Lie algebra:

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] = -\delta_{[\varepsilon_1,\varepsilon_2]} \tag{2.10}$$

2.3 Hermiticity and charge conjugation

Since the vielbein V^a and the spin connection ω^{ab} are real fields, the following conditions hold:

$$\gamma_0 V \gamma_0 = V^\dagger, \quad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \tag{2.11}$$

$$\gamma_0[(D\psi)\bar{\psi}]\gamma_0 = [\psi D\bar{\psi}]^\dagger, \quad \gamma_0[\psi D\bar{\psi}]\gamma_0 = [(D\psi)\bar{\psi}]^\dagger \tag{2.12}$$

and can be used to check that the action (2.1) is real.

Moreover, if C is the $D=4$ charge conjugation matrix (antisymmetric and squaring to -1), we have

$$CVC = V^T, \quad C\Omega C = \Omega^T \tag{2.13}$$

since the matrices $C\gamma_a$ and $C\gamma_{ab}$ are symmetric.

Similar relations hold for the gauge parameter $\varepsilon = (1/4)\varepsilon^{ab}\gamma_{ab}$:

$$-\gamma_0\varepsilon\gamma_0 = \varepsilon^\dagger, \quad C\varepsilon C = \varepsilon^T \tag{2.14}$$

ε^{ab} being real.

The charge conjugation of fermions:

$$\psi^C \equiv C(\bar{\psi})^T \tag{2.15}$$

can be extended to the bosonic fields V, Ω:

$$V^C \equiv -CV^TC, \quad \Omega^C \equiv C\Omega^T C \tag{2.16}$$

Then the relations (2.13) can be written as:

$$V^C = -V, \quad \Omega^C = \Omega \tag{2.17}$$

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and are the analogues of the Majorana condition for the fermions:

$$\psi^C = \psi \quad \to \bar{\psi} = \psi^T C \tag{2.18}$$

Note also that

$$(V\psi)^C = V^C \psi^C \tag{2.19}$$

In particular, if ψ is a Majorana fermion, $V\psi$ is anti-Majorana.

So far we have been treating ψ as a Dirac fermion, and therefore reality of the action requires both terms in square brackets in the action (2.1) or (2.5). If ψ is Majorana, the two terms give the same contribution, and only one of them is necessary.

2.4 Field equations

Using the cyclicity of Tr in (2.1), the variation of V , Ω and $\bar{\psi}$ yield respectively the Einstein equation, the torsion equation and the (massless) Dirac equation in index-free form:

$$Tr\Big(\gamma_a\gamma_5[iV\wedge R + iR\wedge V - X\wedge V\wedge V - V\wedge X\wedge V - V\wedge V\wedge X]\Big) = 0,$$

$$Tr\Big(\gamma_{ab}[iT\wedge V - iV\wedge T + \psi\bar{\psi}V\wedge V\wedge V - V\wedge V\wedge V\psi\bar{\psi}]\Big) = 0 \tag{2.20}$$

$$V\wedge V\wedge V\wedge D\psi - (T\wedge V\wedge V - V\wedge T\wedge V + V\wedge V\wedge T)\psi = 0 \tag{2.21}$$

with

$$X \equiv (D\psi)\bar{\psi} - \psi D\bar{\psi} \tag{2.22}$$

and where the torsion $T = T^a\gamma_a$ is given by:

$$T \equiv dV - \Omega\wedge V - V\wedge\Omega \tag{2.23}$$

The torsion equation can be solved, and yields the known result:

$$T^a = 6i\ \bar{\psi}\gamma_b\psi\ V^b\wedge V^a \tag{2.24}$$

The Dirac equation (2.21) contains an extra term proportional to the torsion: this is due to requiring a real action for gravity coupled to Dirac fermions. If one uses the (complex) Dirac action

$$S_{Dirac} = -\int Tr[(D\psi)\bar{\psi}\wedge V\wedge V\wedge V\gamma_5] \tag{2.25}$$

the torsion terms in the Dirac equation (2.21) are not present.

3. Twist differential geometry

The noncommutative deformation of the gravity theories we construct in the next Sections relies on the existence (in the deformation quantization context, see for ex [19]) of an associative $\star$-product between functions and more generally an associative $\wedge_\star$ exterior product between forms, satisfying the following properties:

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- Compatibility with the undeformed exterior differential:

$$d(\tau \wedge_\star \tau') = d(\tau) \wedge_\star \tau' = \tau \wedge_\star d\tau' \tag{3.1}$$

- Compatibility with the undeformed integral (graded cyclicity property):

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.2}$$

with $deg(\tau) + deg(\tau')$ =D=dimension of the spacetime manifold, and where here τ and τ' have compact support (otherwise stated we require (3.2) to hold up to boundary terms).

- Compatibility with the undeformed complex conjugation:

$$(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^* \tag{3.3}$$

We describe here a (quite wide) class of twists whose $\star$-products have all these properties. In this way we have constructed a wide class of noncommutative deformations of gravity theories. Of course as a particular case we have the Groenewold-Moyal $\star$-product

$$f \star g = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\}, \tag{3.4}$$

where the map μ is the usual pointwise multiplication: $\mu(f \otimes g) = fg$, and $\theta^{\rho\sigma}$ is a constant antisymmetric matrix.

3.1 Twist

Let Ξ be the linear space of smooth vector fields on a smooth manifold M, and $U\Xi$ its universal enveloping algebra. A twist $\mathscr{F} \in U\Xi \otimes U\Xi$ defines the associative twisted product

$$f \star g \ = \ \mu\{\mathscr{F}^{-1} f \otimes g\} \tag{3.5}$$

where the map μ is the usual pointwise multiplication: $\mu(f \otimes g) = fg$. The product associativity relies on the defining properties of the twist [8, 19, 20]. Using the standard notation

$$\mathscr{F} \equiv \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha, \quad \mathscr{F}^{-1} \equiv \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha \tag{3.6}$$

(sum over α understood) where $\mathrm{f}^\alpha, \mathrm{f}_\alpha, \bar{\mathrm{f}}^\alpha, \bar{\mathrm{f}}_\alpha$ are elements of $U\Xi$, the $\star$-product is expressed in terms of ordinary products as:

$$f \star g = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g) \tag{3.7}$$

Many explicit examples of twist are provided by the so-called abelian twists:

$$\mathscr{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b} \tag{3.8}$$

where $\{X_a\}$ is a set of mutually commuting vector fields, and θ^{ab} is a constant antisymmetric matrix. The corresponding $\star$-product is in general position dependent because the vector fields X_a are in general x-dependent. In the special case that there exists a global coordinate system on the manifold we can consider the vector fields $X_a = \frac{\partial}{\partial x^a}$. In this instance we have the Moyal twist, cf. (3.4):

$$\mathscr{F}^{-1} = e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} \tag{3.9}$$

3.2 Deformed exterior product

The deformed exterior product between forms is defined as

$$\tau \wedge_\star \tau' \equiv \bar{\mathrm{f}}^\alpha(\tau) \wedge \bar{\mathrm{f}}_\alpha(\tau') \tag{3.10}$$

where $\bar{\mathrm{f}}^\alpha$ and $\bar{\mathrm{f}}_\alpha$ act on forms via the Lie derivatives $\mathscr{L}_{\bar{\mathrm{f}}^\alpha}$, $\mathscr{L}_{\bar{\mathrm{f}}_\alpha}$ (Lie derivatives along products $uv\cdots$ of elements of Ξ are defined simply by $\mathscr{L}_{uv\cdots} \equiv \mathscr{L}_u\mathscr{L}_v\cdots$). This product is associative, and in particular satisfies:

$$\tau \wedge_\star h \star \tau' = \tau \star h \wedge_\star \tau', \quad h \star (\tau \wedge_\star \tau') = (h \star \tau) \wedge_\star \tau', \quad (\tau \wedge_\star \tau') \star h = \tau \wedge_\star (\tau' \star h) \tag{3.11}$$

where h is a 0-form, i.e. a function belonging to $Fun(M)$, the $\star$-product between functions and one-forms being just a particular case of (3.10):

$$h \star \tau = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(\tau), \quad \tau \star h = \bar{\mathrm{f}}^\alpha(\tau)\bar{\mathrm{f}}_\alpha(h) \tag{3.12}$$

3.3 Exterior derivative

The exterior derivative satisfies the usual (graded) Leibniz rule, since it commutes with the Lie derivative:

$$d(f \star g) = df \star g + f \star dg \tag{3.13}$$
$$d(\tau \wedge_\star \tau') = d\tau \wedge_\star \tau' + (-1)^{deg(\tau)}\, \tau \wedge_\star d\tau' \tag{3.14}$$

3.4 Integration: graded cyclicity

If we consider an abelian twist (3.8) given by globally defined commuting vector fields X_a, then the usual integral is cyclic under the $\star$-exterior products of forms, i.e., up to boundary terms,

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.15}$$

with $deg(\tau) + deg(\tau') =$D=dimension of the spacetime manifold. In fact we have

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge \tau = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.16}$$

For example at first order in θ,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' - \frac{i}{2}\theta^{ab} \int \mathscr{L}_{X_a}(\tau \wedge \mathscr{L}_{X_b}\tau') = \int \tau \wedge \tau' - \frac{i}{2}\theta^{ab} \int d i_{X_a}(\tau \wedge \mathscr{L}_{X_b}\tau') \tag{3.17}$$

where we used the Cartan formula $\mathscr{L}_{X_a} = di_{X_a} + i_{X_a}d$.

3.5 Complex conjugation

If we choose real fields X_a in the definition of the twist (3.8), it is immediate to verify that:

$$(f \star g)^* = g^* \star f^* \tag{3.18}$$

$$(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^* \tag{3.19}$$

since sending i into $-i$ in the twist (3.9) amounts to send θ^{ab} into $-\theta^{ab} = \theta^{ba}$, i.e. to exchange the order of the factors in the $\star$-product.

More in general we can consider twists $\mathscr{F}$ that satisfy the reality condition (cf. Section 8 in [8]) $\bar{\mathrm{f}}^{\alpha *} \otimes \bar{\mathrm{f}}_\alpha{}^* = S(\bar{\mathrm{f}}_\alpha) \otimes S(\bar{\mathrm{f}}^\alpha)$. The $\star$-products associated to these twists satisfy properties (3.18), (3.19).

4. Noncommutative gravity coupled to fermions

4.1 Action and symmetries

Here we generalize Section 2 to the noncommutative case, mostly by replacing exterior products by deformed exterior products. Thus the action becomes:

$$S = \int Tr\,(iR \wedge_\star V \wedge_\star V\gamma_5 - [(D\psi) \star \bar{\psi} - \psi \star D\bar{\psi}] \wedge_\star V \wedge_\star V \wedge_\star V\gamma_5) \tag{4.1}$$

with

$$R = d\Omega - \Omega \wedge_\star \Omega, \qquad D\psi = d\psi - \Omega \star \psi \tag{4.2}$$

Almost all formulae in Section 2 continue to hold, with $\star$-products and $\star$-exterior products. However, the expansion of the fundamental fields on the Dirac basis of gamma matrices must now include new contributions:

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1 + \tilde{\omega}\gamma_5, \qquad V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5 \tag{4.3}$$

Similarly for the curvature :

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + ir1 + \tilde{r}\gamma_5 \tag{4.4}$$

and for the gauge parameter:

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 + \tilde{\varepsilon}\gamma_5 \tag{4.5}$$

Indeed now the $\star$-gauge variations read:

$$\delta_\varepsilon V = -V \star \varepsilon + \varepsilon \star V, \quad \delta_\varepsilon \Omega = d\varepsilon - \Omega \star \varepsilon + \varepsilon \star \Omega, \quad \delta_\varepsilon \psi = \varepsilon \star \psi, \quad \delta_\varepsilon \bar{\psi} = -\bar{\psi} \star \varepsilon \tag{4.6}$$

and in the variations for V and Ω also anticommutators of gamma matrices appear, due to the noncommutativity of the $\star$-product. Since for example the anticommutator $\{\gamma_{ab}, \gamma_{cd}\}$ contains 1 and γ_5, we see that the corresponding fields must be included in the expansion of Ω. Similarly, V must contain a $\gamma_a\gamma_5$ term due to $\{\gamma_{ab}, \gamma_c\}$. Finally, the composition law for gauge parameters becomes:

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{\varepsilon_2 \star \varepsilon_1 - \varepsilon_1 \star \varepsilon_2} \tag{4.7}$$

so that ε must contain the 1 and γ_5 terms, since they appear in the composite parameter $\varepsilon_2 \star \varepsilon_1 - \varepsilon_1 \star \varepsilon_2$.

The invariance of the noncommutative action (4.1) under the $\star$-variations is demonstrated in exactly the same way as for the commutative case, noting that

$$\delta_\varepsilon R = -R \star \varepsilon + \varepsilon \star R, \quad \delta_\varepsilon D\psi = \varepsilon \star D\psi, \quad \delta_\varepsilon((D\psi) \star \bar{\psi}) = -(D\psi) \star \bar{\psi} \star \varepsilon + \varepsilon \star (D\psi) \star \bar{\psi} \tag{4.8}$$

etc., and using now, besides the cyclicity of the trace Tr and the fact that ε still commutes with γ_5, also the graded cyclicity of the integral.

The local $\star$-symmetry satisfies the Lie algebra of $GL(2,C)$, and centrally extends the $SO(1,3)$ Lie algebra of the commutative theory.

Finally, the $\star$-action (4.1) is invariant under diffeomorphisms generated by the Lie derivative, in the sense that

$$\int \mathscr{L}_v(4\text{-form}) = \int (i_v d + d i_v)(4\text{-form}) = \int d(i_v(4\text{-form})) = boundary\ term \tag{4.9}$$

since $d(4\text{-form}) = 0$ on a 4-dimensional manifold.

We have constructed a geometric lagrangian where the fields are exterior forms and the $\star$-product is given by the Lie derivative action of the twist on forms. The twist $\mathscr{F}$ in general is not invariant under the diffeomorphism $\mathscr{L}_v$. However we can consider the $\star$-diffeomorphisms of ref. [8] (see also [19], section 8.2.4), generated by the $\star$-Lie derivative. This latter acts trivially on the twist $\mathscr{F}$ but satisfies a deformed Leibniz rule. $\star$-Lie derivatives generate infinitesimal noncommutative diffeomorphisms and leave invariant the action and the twist. They are noncommutative symmetries of our action.

Finally in our geometric action no coordinate indices μ,ν appear, and this implies invariance of the action under (undeformed) general coordinate transformations. Otherwise stated every contravariant tensor index ${}^{\mu}$ is contracted with the corresponding covariant tensor index ${}_{\mu}$, for example $X_a = X_a^\mu \partial_\mu$ and $V^a = V^a_\mu dx^\mu$.

4.2 Field equations

Using the cyclicity of Tr and the graded cyclicity of the integral in (4.1), the variation of V , Ω and $\bar\psi$ yield respectively the noncommutative Einstein equation, torsion equation and Dirac equation in index-free form:

$$Tr[\Gamma_{a,a5}(iV \wedge_\star R + iR \wedge_\star V - X \wedge_\star V \wedge_\star V - V \wedge_\star X \wedge_\star V - V \wedge_\star V \wedge_\star X)] = 0$$

$$Tr[\Gamma_{ab,1,5}(iT \wedge_\star V - iV \wedge_\star T + \psi \star \bar\psi \star V \wedge_\star V \wedge_\star V - V \wedge_\star V \wedge_\star V \star \psi \star \bar\psi)] = 0 \tag{4.10}$$

$$V \wedge_\star V \wedge_\star V \wedge_\star D\psi - (T \wedge_\star V \wedge_\star V - V \wedge_\star T \wedge_\star V + V \wedge_\star V \wedge_\star T) \star \psi = 0$$

where $\Gamma_{a,a5}$ indicates γ_a and $\gamma_a\gamma_5$ (thus there are two distinct equations) and likewise for $\Gamma_{ab,1,5}$ (three equations corresponding to γ_{ab}, 1 and γ_5). The noncommutative torsion two-form is defined by:

$$T \equiv T^a \gamma_a + \tilde T^a \gamma_a \gamma_5 \equiv dV - \Omega \wedge_\star V - V \wedge_\star \Omega \tag{4.11}$$

The torsion equation (4.10) can be written as:

$$[iT \wedge_\star V - iV \wedge_\star T + \psi \star \bar\psi \star V \wedge_\star V \wedge_\star V - V \wedge_\star V \wedge_\star V \star \psi \star \bar\psi, \gamma_5]_+ = 0 \tag{4.12}$$

Indeed the anticommutator with γ_5 selects the γ_{ab}, 1 and γ_5 components. This equation can be solved for the torsion:

$$T = \frac{i}{2}[\psi \star \bar\psi \star V \wedge_\star V + V \wedge_\star \psi \star \bar\psi \star V + V \wedge_\star V \star \psi \star \bar\psi, \gamma_5]\gamma_5 \tag{4.13}$$

as can be verified by substitution into (4.12).

4.3 θ - dependent fields

We can rewrite the Moyal twist as:

$$\mathscr{F}^{-1} = e^{\frac{i}{2}\theta\Theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} \tag{4.14}$$

where θ is a dimensionful parameter (so that $\Theta^{\rho\sigma}$ is a numerical matrix). In the spirit of the Seiberg-Witten map [1], the fields and the gauge parameter can be considered functions of x and θ. Expanding fields ϕ and gauge parameter ε in powers of θ:

$$\phi_\theta(x) = \phi_0(x) + \theta\phi_1(x) + \theta^2\phi_2(x) + \ldots, \quad \varepsilon_\theta(x) = \varepsilon_0(x) + \theta\varepsilon_1(x) + \theta^2\varepsilon_2(x) + \ldots \tag{4.15}$$

introduces an infinite tower of x - dependent fields and gauge parameters: a finite number of them enters in the action (4.1) at each given order in θ. At 0-th order only the classical fields $\phi_0(x)$ contribute. The gauge variations of all ϕ_i are deduced by expanding the $\star$-gauge transformations in (4.6) in powers of θ. Clearly the classical fields ϕ_0 transform with the classical gauge variations δ^0_ε.

If one feels uncomfortable with these new fields ϕ_i, the Seiberg-Witten map can be used to relate the higher-order fields to the classical ones in a way consistent with the $\star$ - gauge transformations δ_ε:

$$\delta_\varepsilon\phi(\phi_0) = \phi(\delta^0_\varepsilon\phi_0) \tag{4.16}$$

so that the $\star$-deformed theory will contain only the ϕ_0 fields [1, 16].

All the fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$ contained in the action (4.1) are then θ-expanded, and the 0-th order action contains their $\theta \to 0$ limit.

4.4 Hermiticity and charge conjugation

Hermiticity conditions can be imposed on V, Ω and the gauge parameter ε:

$$\gamma_0 V\gamma_0 = V^\dagger, \quad -\gamma_0\Omega\gamma_0 = \Omega^\dagger, \quad -\gamma_0\varepsilon\gamma_0 = \varepsilon^\dagger \tag{4.17}$$

Moreover it is easy to verify the analogues of conditions (2.12):

$$\gamma_0[(D\psi)\star\bar{\psi}]\gamma_0 = [\psi\star D\bar{\psi}]^\dagger, \;\; \gamma_0[\psi\star D\bar{\psi}]\gamma_0 = [D\psi\star\bar{\psi}]^\dagger \tag{4.18}$$

These hermiticity conditions are consistent with the gauge variations, as in the commutative case, and can be used to check that the action (4.1) is real. On the component fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$, and on the component gauge parameters ε^{ab}, ε, and $\tilde{\varepsilon}$ the hermiticity conditions (4.17) imply that they are real fields.

The charge conjugation relations (2.13), however, cannot be exported to the noncommutative case as they are. Indeed they would imply the vanishing of the component fields $\tilde{V}^a$, ω, and $\tilde{\omega}$ (whose presence is necessary in the noncommutative case) and moreover would not be consistent with the $\star$-gauge variations.

An essential modification is needed, and makes use of the θ dependence of the noncommutative fields:

$$CV_\theta(x)C = V_{-\theta}(x)^T, \quad C\Omega_\theta(x)C = \Omega_{-\theta}(x)^T, \quad C\varepsilon_\theta(x)C = \varepsilon_{-\theta}(x)^T \tag{4.19}$$

These conditions can be checked to be consistent with the $\star$-gauge transformations. For example $CV_\theta(x)^T C$ can be shown to transform in the same way as $V_{-\theta}(x)$:

$$\begin{aligned}\delta_\varepsilon(CV_\theta^T C) &= C(\delta_\varepsilon V_\theta)^T C = C(-\varepsilon_\theta^T \star_{-\theta} V_\theta^T + V_\theta^T \star_{-\theta} \varepsilon_\theta^T)C = \\ &= \varepsilon_{-\theta} \star_{-\theta} V_{-\theta} - V_{-\theta} \star_{-\theta} \varepsilon_{-\theta} = \delta_\varepsilon V_{-\theta}\end{aligned} \tag{4.20}$$

where we have used $C^2 = -1$ and the fact that the transposition of a $\star$-product of matrix-valued fields interchanges the order of the matrices but not of the $\star$-multiplied fields. To interchange both it is necessary to use the "reflected" $\star_{-\theta}$ product obtained by changing the sign of θ, since

$$f \star_\theta g = g \star_{-\theta} f \tag{4.21}$$

for any two functions f, g.
For the component fields and gauge parameters the charge conjugation conditions imply:

$$V_\theta^a = V_{-\theta}^a, \quad \omega_\theta^{ab} = \omega_{-\theta}^{ab} \tag{4.22}$$

$$\tilde{V}_\theta^a = -\tilde{V}_{-\theta}^a, \quad \omega_\theta = -\omega_{-\theta}, \quad \tilde{\omega}_\theta = -\tilde{\omega}_{-\theta}, \tag{4.23}$$

Similarly for the gauge parameters:

$$\varepsilon_\theta^{ab} = \varepsilon_{-\theta}^{ab} \tag{4.24}$$

$$\varepsilon_\theta = -\varepsilon_{-\theta}, \quad \tilde{\varepsilon}_\theta = -\tilde{\varepsilon}_{-\theta} \tag{4.25}$$

Finally, let us consider the charge conjugate spinor:

$$\psi^C \equiv C(\bar{\psi})^T \tag{4.26}$$

It transforms under $\star$-gauge variations as:

$$\delta_\varepsilon \psi^C = C(\delta_\varepsilon \bar{\psi})^T = C(-\bar{\psi} \star \varepsilon)^T = C(-\varepsilon^T \star_{-\theta} \psi^*) = C\varepsilon^T C \star_{-\theta} C\psi^* = \varepsilon_{-\theta} \star_{-\theta} \psi^C \tag{4.27}$$

i.e. it transforms in the same way as $\psi_{-\theta}$. Then we can impose the noncommutative Majorana condition:

$$\psi_\theta^C = \psi_{-\theta} \quad \Rightarrow \quad \psi_\theta^\dagger \gamma_0 = \psi_{-\theta}^T C \tag{4.28}$$

4.5 Commutative limit $\theta \to 0$

In the commutative limit the action reduces to the usual action of gravity coupled to fermions of eq. (2.1). Indeed in virtue of the charge conjugation conditions on V and Ω, the component fields $\tilde{V}^a$, ω, and $\tilde{\omega}$ all vanish in the limit $\theta \to 0$ (see the second line of (4.23)), and only the classical spin connection ω^{ab}, vierbein V^a and Dirac fermion ψ survive. Similarly the gauge parameters ε, and $\tilde{\varepsilon}$ vanish in the commutative limit.

5. Classical $D=4$, $N=1$ supergravity

The $D=4$, $N=1$ simple supergravity action can be written in index-free notation as follows:

$$S=\int Tr\,[iR(\Omega)\wedge V\wedge V\gamma_5-2(\rho\wedge\bar{\psi}+\psi\wedge\bar{\rho})\wedge V\gamma_5] \tag{5.1}$$

The fundamental fields are the 1-forms Ω (spin connection), V (vielbein) and gravitino ψ. The curvature 2-form R and the gravitino curvature ρ are defined by

$$R=d\Omega-\Omega\wedge\Omega, \quad \rho\equiv D\psi=d\psi-\Omega\psi,\ \ \bar{\rho}=D\bar{\psi}=d\bar{\psi}-\bar{\psi}\wedge\Omega \tag{5.2}$$

with

$$\Omega=\frac{1}{4}\omega^{ab}\gamma_{ab}, \quad V=V^a\gamma_a \tag{5.3}$$

and thus are 4×4 matrices with spinor indices. See Appendix C for $D=4$ gamma matrix conventions and useful relations. The Dirac conjugate is defined as usual: $\bar{\psi}=\psi^{\dagger}\gamma_0$. Then also $\rho\wedge\bar{\psi}$ and $\psi\wedge\bar{\rho}$ are matrices in the spinor representation, and the trace Tr is taken on this representation. The gravitino field satisfies the Majorana condition:

$$\psi^{\dagger}\gamma_0=\psi^T C \tag{5.4}$$

where C is the $D=4$ charge conjugation matrix, antisymmetric and squaring to -1.

Using the $D=4$ gamma matrix trace identity:

$$Tr(\gamma_{ab}\gamma_c\gamma_d\gamma_5)=-4i\varepsilon_{abcd} \tag{5.5}$$

leads to the usual supergravity action in terms of the component fields V^a, ω^{ab} :

$$S=\int R^{ab}\wedge V^c\wedge V^d\varepsilon_{abcd}-4\bar{\psi}\wedge\gamma_5\gamma_a\rho\wedge V^a \tag{5.6}$$

with

$$R\equiv\frac{1}{4}R^{ab}\gamma_{ab}, \quad R^{ab}=d\omega^{ab}-\omega^a_{\ c}\wedge\omega^{cb} \tag{5.7}$$

We have also used

$$\bar{\rho}\gamma_5\gamma_a\psi=\bar{\psi}\gamma_5\gamma_a\rho \tag{5.8}$$

due to ψ and ρ being Majorana spinors [1].

5.1 Field equations and Bianchi identities

Using the cyclicity of the Tr in the action (5.1), the variation on V, Ω and ψ yield respectively the Einstein equation, the torsion equation and the gravitino equation in index-free form:

$$Tr[\gamma_a\gamma_5(-iV\wedge R-iR\wedge V+2(\rho\wedge\bar{\psi}+\psi\wedge\bar{\rho})]=0 \tag{5.9}$$

$$Tr[\gamma_{ab}\gamma_5(iT\wedge V-iV\wedge T+2\psi\wedge\bar{\psi}\wedge V-2V\wedge\psi\wedge\bar{\psi})]=0 \tag{5.10}$$

[1]Then the two addends in the fermionic part of the action (5.1) are equal, so that we could have used only one of them, with factor -4. However in the noncommutative extension both will be necessary.

$$V \wedge D\psi = 0 \tag{5.11}$$

where the torsion $T = T^a\gamma_a$ is defined as:

$$T \equiv dV - \Omega \wedge V - V \wedge \Omega \tag{5.12}$$

The solution of the torsion equation (5.10) is given by:

$$T = i[\psi \wedge \bar{\psi}, \gamma_5]\gamma_5 = i\psi \wedge \bar{\psi} - i\gamma_5 \psi \wedge \bar{\psi}\gamma_5 \tag{5.13}$$

Upon use of the Fierz identity for Majorana spinor one-forms:

$$\psi \wedge \bar{\psi} = \frac{1}{4}\gamma_a \bar{\psi}\gamma^a \wedge \psi - \frac{1}{8}\gamma_{ab}\bar{\psi}\gamma^{ab} \wedge \psi \tag{5.14}$$

the torsion is seen to satisfy the familiar condition

$$T \equiv T^a\gamma_a = \frac{i}{2}\bar{\psi}\gamma^a \wedge \psi\gamma_a \tag{5.15}$$

Finally, the Bianchi identities for the curvatures and the torsion are:

$$dR = -R \wedge \Omega + \Omega \wedge R \tag{5.16}$$

$$d\rho = -R \wedge \psi + \Omega \wedge \rho, \quad d\bar{\rho} = \bar{\psi} \wedge R - \bar{\rho} \wedge \Omega \tag{5.17}$$

$$dT = -R \wedge V + \Omega \wedge T - T \wedge \Omega + V \wedge R \tag{5.18}$$

The terms with the spin connection Ω reconstruct covariant derivatives of the curvatures and the torsion.

5.2 Invariances

We know that the classical supergravity action (5.6) is invariant under general coordinate transformations, under local Lorentz rotations and under local supersymmetry transformations. It is of interest to write the transformation rules of the fields in the index-free notation, so as to verify the invariances directly on the index-free action (5.1).

Local Lorentz rotations

$$\delta_\varepsilon V = -[V, \varepsilon], \quad \delta_\varepsilon \Omega = d\varepsilon - [\Omega, \varepsilon], \quad \delta_\varepsilon \psi = \varepsilon\psi, \quad \delta_\varepsilon \bar{\psi} = -\bar{\psi}\varepsilon \tag{5.19}$$

with

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} \tag{5.20}$$

The invariance can be directly checked on the action (5.1) noting that

$$\delta_\varepsilon R = -[R, \varepsilon], \quad \delta_\varepsilon D\psi = \varepsilon D\psi, \quad \delta_\varepsilon D\bar{\psi} = -(D\bar{\psi})\varepsilon \tag{5.21}$$

using the cyclicity of the trace Tr (on spinor indices) and the fact that ε commutes with γ_5. The Lorentz rotations close on the Lie algebra:

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{[\varepsilon_2, \varepsilon_1]} \tag{5.22}$$

Local supersymmetry

The supersymmetry variations are:

$$\delta_\varepsilon V = i[\varepsilon\overline{\psi} - \psi\overline{\varepsilon}, \gamma_5]\gamma_5, \quad \delta_\varepsilon \psi = D\varepsilon \equiv d\varepsilon - \Omega\varepsilon \tag{5.23}$$

where now ε is a spinorial parameter (satisfying the Majorana condition). Notice that again Ω is not varied since we work in 1.5 - order formalism, i.e. Ω satisfies its own equation of motion (5.10).

The commutator of $\varepsilon\overline{\psi} - \psi\overline{\varepsilon}$ with γ_5 in the supersymmetry variation of V eliminates the terms even in γ_a in the Fierz expansion of two generic anticommuting spinors (see Appendix C). Moreover, since ε and ψ are Majorana spinors, the combination $\varepsilon\overline{\psi} - \psi\overline{\varepsilon}$ ensures that only the γ_a component survives. Then (5.23) reproduce the usual supersymmetry variations (see below).

The variations (5.23) imply:

$$\delta_\varepsilon \overline{\psi} = D\overline{\varepsilon} \equiv d\overline{\varepsilon} + \overline{\varepsilon}\Omega, \quad \delta_\varepsilon \rho = -R\varepsilon, \quad \delta_\varepsilon \overline{\rho} = \overline{\varepsilon}R \tag{5.24}$$

Then the action varies as:

$$\begin{aligned}\delta_\varepsilon S = \int & 2\,Tr[R \wedge (\psi\overline{\varepsilon} - \varepsilon\overline{\psi}) \wedge V\gamma_5 + R \wedge V \wedge (\psi\overline{\varepsilon} - \varepsilon\overline{\psi})\gamma_5] - \\ & -2\,Tr[\Big(-R\varepsilon \wedge \overline{\psi} \wedge V + \rho \wedge (d\overline{\varepsilon} + \overline{\varepsilon}\Omega) \wedge V + (d\varepsilon - \Omega\varepsilon) \wedge \overline{\rho} \wedge V + \psi \wedge \overline{\varepsilon}R \wedge V \Big)\gamma_5] \\ & +2i\,Tr[(\rho \wedge \overline{\psi} + \psi \wedge \overline{\rho})(\psi\overline{\varepsilon} - \varepsilon\overline{\psi})\gamma_5 - (\rho \wedge \overline{\psi} + \psi \wedge \overline{\rho})\gamma_5(\psi\overline{\varepsilon} - \varepsilon\overline{\psi})]\end{aligned} \tag{5.25}$$

After integrating by parts the terms with $d\varepsilon$ and $d\overline{\varepsilon}$, and using the Bianchi identity (5.17) for $d\rho$ the variation becomes:

$$\begin{aligned}\delta_\varepsilon S = \int & 2\,Tr[R \wedge (\psi\overline{\varepsilon} - \varepsilon\overline{\psi}) \wedge V\gamma_5 + R \wedge V \wedge (\psi\overline{\varepsilon} - \varepsilon\overline{\psi})\gamma_5] - \\ & -2\,Tr[\Big(-R\varepsilon \wedge \overline{\psi} \wedge V + \rho \wedge \overline{\varepsilon}\Omega \wedge V - \Omega\varepsilon \wedge \overline{\rho} \wedge V + \psi \wedge \overline{\varepsilon}R \wedge V + \\ & +(R \wedge \psi - \Omega \wedge \rho)\overline{\varepsilon} \wedge V - \rho\overline{\varepsilon} \wedge (T + \Omega \wedge V + V \wedge \Omega) - \\ & -\varepsilon(-\overline{\rho} \wedge \Omega + \overline{\psi} \wedge \rho) \wedge V - \varepsilon\overline{\rho} \wedge (T + \Omega \wedge V + V \wedge \Omega)\Big)\gamma_5] + \\ & +2i\,Tr[(\rho \wedge \overline{\psi} + \psi \wedge \overline{\rho})(\psi\overline{\varepsilon} - \varepsilon\overline{\psi})\gamma_5 - (\rho \wedge \overline{\psi} + \psi \wedge \overline{\rho})\gamma_5(\psi\overline{\varepsilon} - \varepsilon\overline{\psi})]\end{aligned} \tag{5.26}$$

where we have substituted dV by $T + \Omega \wedge V + V \wedge \Omega$ (torsion definition). Using now the cyclicity of Tr , and the fact that γ_5 anticommutes with V and commutes with Ω, all terms can be easily checked to cancel, except those containing the torsion T and the last line (four-fermion terms).

Once we make use of the torsion equation ((5.13) to express T in terms of gravitino fields, the variation reduces to:

$$\begin{aligned}\delta_\varepsilon S = 2i \int & Tr[\rho\overline{\varepsilon} \wedge (\psi \wedge \overline{\psi}\gamma_5 - \gamma_5\psi \wedge \overline{\psi}) + \varepsilon\overline{\rho} \wedge (\psi \wedge \overline{\psi}\gamma_5 - \gamma_5\psi \wedge \overline{\psi}) \\ & + (\rho \wedge \overline{\psi} + \psi \wedge \overline{\rho}) \wedge (\psi\overline{\varepsilon} - \varepsilon\overline{\psi})\gamma_5 - (\rho \wedge \overline{\psi} + \psi \wedge \overline{\rho}) \wedge \gamma_5(\psi\overline{\varepsilon} - \varepsilon\overline{\psi})]\end{aligned} \tag{5.27}$$

Finally, carrying out the trace on spinor indices results in

$$\begin{aligned}\delta_\varepsilon S = 2i \int & (\overline{\psi}\varepsilon - \overline{\varepsilon}\psi) \wedge (\overline{\psi}\gamma_5 \wedge \rho - \overline{\rho}\gamma_5 \wedge \psi) + (\overline{\psi} \wedge \rho - \overline{\rho} \wedge \psi) \wedge (\overline{\psi}\gamma_5\varepsilon - \overline{\varepsilon}\gamma_5\psi) \\ & +(\overline{\varepsilon}\rho - \overline{\rho}\varepsilon) \wedge (\overline{\psi}\gamma_5 \wedge \psi) + (\overline{\rho}\gamma_5\varepsilon - \overline{\varepsilon}\gamma_5\rho) \wedge (\overline{\psi} \wedge \psi)\end{aligned} \tag{5.28}$$

Each factor between parentheses vanishes, due to all spinors being Majorana spinors. This proves the invariance of the classical supergravity action under the local supersymmetry variations (5.23).

On the component fields, the Lorentz transformations (5.19) read:

$$\begin{aligned} \delta_\varepsilon V^a &= \varepsilon^a_{\ b} V^b \\ \delta_\varepsilon \omega^{ab} &= d\varepsilon^{ab} + \varepsilon^{ac}\omega_c^{\ b} - \varepsilon^{bc}\omega_c^{\ a} \\ \delta_\varepsilon \psi &= \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi \end{aligned} \tag{5.29}$$

and the supersymmetry variations (5.23) become:

$$\begin{aligned} \delta_\varepsilon V^a &= i\bar{\varepsilon}\gamma^a\psi \\ \delta_\varepsilon \psi &= d\varepsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\varepsilon \end{aligned} \tag{5.30}$$

6. Noncommutative $D = 4$, $N = 1$ supergravity

6.1 Action and $GL(2,C)$ $\star$-gauge symmetry

A noncommutative generalization of the $D = 4$, $N = 1$ simple supergravity action is obtained by replacing exterior products by $\star$-exterior products in (5.1):

$$S = \int Tr\,[iR(\Omega)\wedge_\star V\wedge_\star V\gamma_5 + 2(\rho\wedge_\star\bar{\psi} + \psi\wedge_\star\bar{\rho})\wedge_\star V\gamma_5] \tag{6.1}$$

where the curvature 2-form R and the gravitino curvature ρ are defined as:

$$R = d\Omega - \Omega\wedge_\star\Omega, \qquad \rho \equiv D\psi = d\psi - \Omega\star\psi \tag{6.2}$$

Almost all formulae of the commutative case continue to hold, with ordinary products replaced by $\star$-products and $\star$-exterior products. However, the expansion of the fundamental fields on the Dirac basis of gamma matrices must now include new contributions; more precisely the spin connection contains all even gamma matrices and the vielbein contains all odd gamma matrices:

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1 + \tilde{\omega}\gamma_5, \qquad V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5 \tag{6.3}$$

The one-forms Ω and V are thus also 4×4 matrices with spinor indices. Similarly for the curvature :

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + ir1 + \tilde{r}\gamma_5 \tag{6.4}$$

and for the gauge parameter:

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 + \tilde{\varepsilon}\gamma_5 \tag{6.5}$$

Indeed now the $\star$-gauge variations read:

$$\delta_\varepsilon V = -V\star\varepsilon + \varepsilon\star V, \quad \delta_\varepsilon\Omega = d\varepsilon - \Omega\star\varepsilon + \varepsilon\star\Omega, \quad \delta_\varepsilon\psi = \varepsilon\star\psi, \quad \delta_\varepsilon\bar{\psi} = -\bar{\psi}\star\varepsilon \tag{6.6}$$

and in the variations for V and Ω also anticommutators of gamma matrices appear, due to the noncommutativity of the $\star$-product. Since for example the anticommutator $\{\gamma_{ab}, \gamma_{cd}\}$ contains 1 and γ_5, we see that the corresponding fields must be included in the expansion of Ω. Similarly, V must contain a $\gamma_a\gamma_5$ term due to $\{\gamma_{ab},\gamma_c\}$. Finally, the composition law for gauge parameters becomes:

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] = \delta_{\varepsilon_2\star\varepsilon_1-\varepsilon_1\star\varepsilon_2} \tag{6.7}$$

so that ε must contain the 1 and γ_5 terms, since they appear in the composite parameter $\varepsilon_2\star\varepsilon_1-\varepsilon_1\star\varepsilon_2$.

The invariance of the noncommutative action (6.1) under the $\star$-gauge variations is demonstrated in exactly the same way as for the commutative case, noting that

$$\delta_\varepsilon R = -R\star\varepsilon+\varepsilon\star R,\quad \delta_\varepsilon D\psi = \varepsilon\star D\psi,\quad \delta_\varepsilon((D\psi)\wedge_\star\bar{\psi}) = -(D\psi)\wedge_\star\bar{\psi}\star\varepsilon+\varepsilon\star(D\psi)\wedge_\star\bar{\psi} \tag{6.8}$$

and using now, besides the cyclicity of the trace Tr and the fact that ε still commutes with γ_5, also the graded cyclicity of the integral.

6.2 Local $\star$-supersymmetry

The $\star$-supersymmetry variations are obtained from the classical ones using $\star$-products:

$$\delta_\varepsilon V = i[\varepsilon\star\bar{\psi}-\psi\star\bar{\varepsilon},\gamma_5]\gamma_5 \quad \delta_\varepsilon\psi = d\varepsilon-\Omega\star\varepsilon \tag{6.9}$$

where ε is a spinorial parameter. Under these variations the noncommutative action varies as given in (5.28), with ordinary products substituted with $\star$-products. Indeed the algebra is identical, since γ_5 still anticommutes with V and commutes with Ω, and we can use the cyclicity of Tr and graded cyclicity of the integral.

The question is now: does this variation vanish? Classically it vanishes because of the Majorana condition on the spinors (gravitino and supersymmetry gauge parameter). We recall the noncommutative generalization of the Majorana condition, consistent with the $*$-gauge transformations [13]:

$$\psi^c_\theta = \psi_{-\theta}, \qquad \psi^c \equiv C(\bar{\psi})^T \tag{6.10}$$

This condition involves the θ dependence of the fields [2], and is consistent with the $\star$-gauge transformations only if the gauge parameter satisfies the charge conjugation condition [13]:

$$C\varepsilon_\theta C = \varepsilon^T_{-\theta} \tag{6.11}$$

The NC Majorana condition (6.10) is consistent also with $\star$-supersymmetry transformations if the supersymmetry parameter is Majorana, and the bosonic fields satisfy the charge conjugation conditions

$$C\Omega_\theta C = \Omega^T_{-\theta},\quad CV_\theta C = V^T_{-\theta} \tag{6.12}$$

Now consider the first term in the supersymmetry variation of the action (for the other three terms the reasoning is identical):

$$2i\int(\bar{\psi}\star\varepsilon-\bar{\varepsilon}\star\psi)\wedge_\star(\bar{\psi}\gamma_5\wedge_\star\rho-\bar{\rho}\gamma_5\wedge_\star\psi) \tag{6.13}$$

[2]The fields can be formally expanded in powers of θ: in principle this picture would introduce infinitely many fields, one for each power of θ. However the Seiberg-Witten map [1, 16] can be used to express all fields in terms of the classical one, ending up with a finite number of fields.

If ψ and ε are noncommutative Majorana fermions, they satisfy the relations:

$$\bar{\psi} \star \varepsilon = \bar{\varepsilon}_{-\theta} \star_{-\theta} \psi_{-\theta}, \quad \bar{\psi}\gamma_5 \wedge_\star \rho = \bar{\rho}_{-\theta}\gamma_5 \wedge_{-\theta} \psi_{-\theta} \tag{6.14}$$

and one sees that (6.13) does not vanish anymore (although it vanishes in the commutative limit). Thus the NC Majorana condition does not ensure the local $\star$-supersymmetry invariance of the action in (6.1). In fact, the local supersymmetry of the commutative action is broken by noncommutativity.

There is another condition that we can impose on fermi fields, the Weyl condition, still consistent with the $\star$-symmetry structure of the action:

$$\gamma_5 \psi = \psi, \quad \gamma_5 \varepsilon = \varepsilon \tag{6.15}$$

i.e. all fermions are left-handed (so that their Dirac conjugates $\bar{\psi}$ and $\bar{\varepsilon}$ are right-handed). In this case the local $\star$-supersymmetry variation vanishes because in all the fermion bilinears the γ_5 matrices can be omitted, and the product of a right-handed spinor with a left-handed spinor vanishes. Thus the noncommutative supergravity action (6.1) with Weyl fermions is locally supersymmetric.

Note that now we cannot impose the charge conjugation relations (6.12) on the bosonic fields: indeed $\star$-supersymmetry links together these relations with the NC Majorana condition, which is not compatible in $D = 4$ with the Weyl condition (as in the classical case).

The $\theta \to 0$ limit of this chiral noncommutative theory is a complex version of the so-called $D = 4$, $N = 1$ Weyl supergravity and is discussed in Section 4.6 below.

6.3 Hermiticity conditions and reality of the action

Hermiticity conditions can be imposed on V, Ω and the gauge parameter ε:

$$\gamma_0 V \gamma_0 = V^\dagger, \quad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \quad -\gamma_0 \varepsilon \gamma_0 = \varepsilon^\dagger \tag{6.16}$$

Moreover it is easy to verify that :

$$\gamma_0 [\rho \wedge_\star \bar{\psi}] \gamma_0 = [\psi \wedge_\star \bar{\rho}]^\dagger \tag{6.17}$$

These conditions are consistent with the $\star$-gauge and $\star$-supersymmetry variations (both for Majorana and chiral fermions), as in the commutative case, and can be used to check that the action (6.1) is real. The hermiticity conditions imply that the component fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$, and gauge parameters ε^{ab}, ε, and $\tilde{\varepsilon}$ are real fields.

6.4 Component analysis

Here we list the $\star$-gauge and supersymmetry variations of the component fields. In the supersymmetry variations we consider both Majorana and Weyl fermions.

6.4.1 $\star$-Gauge variations

$$\delta_\varepsilon V^a = \frac{1}{2}(\varepsilon^a_{\ b} \star V^b + V^b \star \varepsilon^a_{\ b}) + \frac{i}{4}\varepsilon^a_{\ bcd}(\tilde{V}^b \star \varepsilon^{cd} - \varepsilon^{cd} \star \tilde{V}^b)$$
$$+ \varepsilon \star V^a - V^a \star \varepsilon - \tilde{\varepsilon} \star \tilde{V}^a - \tilde{V}^a \star \tilde{\varepsilon} \tag{6.18}$$
$$\delta_\varepsilon \tilde{V}^a = \frac{1}{2}(\varepsilon^a_{\ b} \star \tilde{V}^b + \tilde{V}^b \star \varepsilon^a_{\ b}) + \frac{i}{4}\varepsilon^a_{\ bcd}(V^b \star \varepsilon^{cd} - \varepsilon^{cd} \star V^b)$$
$$+ \varepsilon \star \tilde{V}^a - \tilde{V}^a \star \varepsilon - \tilde{\varepsilon} \star V^a - V^a \star \tilde{\varepsilon} \tag{6.19}$$
$$\delta_\varepsilon \omega^{ab} = \frac{1}{2}(\varepsilon^a_{\ c} \star \omega^{cb} - \varepsilon^b_{\ c} \star \omega^{ca} + \omega^{cb} \star \varepsilon^a_{\ c} - \omega^{ca} \star \varepsilon^b_{\ c})$$
$$+ \frac{1}{4}(\varepsilon^{ab} \star \omega - \omega \star \varepsilon^{ab}) + \frac{i}{8}\varepsilon^{ab}_{\ \ cd}(\varepsilon^{cd} \star \tilde{\omega} - \tilde{\omega} \star \varepsilon^{cd})$$
$$+ \frac{1}{4}(\varepsilon \star \omega^{ab} - \omega^{ab} \star \varepsilon) + \frac{i}{8}\varepsilon^{ab}_{\ \ cd}(\tilde{\varepsilon} \star \omega^{cd} - \omega^{cd} \star \tilde{\varepsilon}) \tag{6.20}$$
$$\delta_\varepsilon \omega = \frac{1}{8}(\omega^{ab} \star \varepsilon_{ab} - \varepsilon_{ab} \star \omega^{ab}) + \varepsilon \star \omega - \omega \star \varepsilon + \tilde{\varepsilon} \star \tilde{\omega} - \tilde{\omega} \star \tilde{\varepsilon} \tag{6.21}$$
$$\delta_\varepsilon \tilde{\omega} = \frac{i}{16}\varepsilon_{abcd}(\omega^{ab} \star \varepsilon^{cd} - \varepsilon^{cd} \star \omega^{ab}) + \varepsilon \star \tilde{\omega} - \tilde{\omega} \star \varepsilon + \tilde{\varepsilon} \star \omega - \omega \star \tilde{\varepsilon} \tag{6.22}$$

6.4.2 Supersymmetry variations: Majorana fermions

$$\delta_\varepsilon V^a = \frac{i}{2} Tr[(\varepsilon \star \overline{\psi} - \psi \star \overline{\varepsilon})\gamma^a] \tag{6.23}$$
$$\delta_\varepsilon \tilde{V}^a = \frac{i}{2} Tr[(\varepsilon \star \overline{\psi} - \psi \star \overline{\varepsilon})\gamma^a \gamma_5] \tag{6.24}$$
$$\delta_\varepsilon \psi = d\varepsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\varepsilon - (i\omega + \tilde{\omega}\gamma_5)\varepsilon \tag{6.25}$$

6.4.3 Supersymmetry variations: Weyl fermions

$$\delta_\varepsilon V^a = \delta_\varepsilon \tilde{V}^a = \frac{i}{2} Tr[(\varepsilon \star \overline{\psi} - \psi \star \overline{\varepsilon})\gamma^a] \tag{6.26}$$
$$\delta_\varepsilon \psi = d\varepsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\varepsilon - (i\omega + \tilde{\omega})\varepsilon \tag{6.27}$$

6.4.4 Charge conjugation conditions

The charge conjugation relations (6.12) imply for the component fields:

$$V^a_\theta = V^a_{-\theta}, \quad \omega^{ab}_\theta = \omega^{ab}_{-\theta} \tag{6.28}$$
$$\tilde{V}^a_\theta = -\tilde{V}^a_{-\theta}, \quad \omega_\theta = -\omega_{-\theta}, \quad \tilde{\omega}_\theta = -\tilde{\omega}_{-\theta}, \tag{6.29}$$

and for the gauge parameters:

$$\varepsilon^{ab}_\theta = \varepsilon^{ab}_{-\theta} \tag{6.30}$$
$$\varepsilon_\theta = -\varepsilon_{-\theta}, \quad \tilde{\varepsilon}_\theta = -\tilde{\varepsilon}_{-\theta} \tag{6.31}$$

6.5 Field equations and Bianchi identities

Using the cyclicity of the integral and of the Tr in the action (6.1), the variation on V, Ω and ψ yield respectively the Einstein equation, the torsion equation and the gravitino equation in index-free form:

$$Tr[\Gamma_{a,a5}(-iV\wedge_\star R - iR\wedge_\star V + 2(\rho\wedge_\star \bar{\psi} + \psi\wedge_\star \bar{\rho})] = 0 \tag{6.32}$$

$$Tr[\Gamma_{ab,1,5}(iT\wedge_\star V - iV\wedge_\star T + 2\psi\wedge_\star \bar{\psi}\wedge V - 2V\wedge_\star \psi\wedge_\star \bar{\psi})] = 0 \tag{6.33}$$

$$V\wedge_\star D\psi - \frac{1}{2}T\wedge_\star \psi = 0 \tag{6.34}$$

where $\Gamma_{ab,1,5}$ indicates γ_{ab}, 1 and γ_5 (thus there are three distinct equations) and likewise for $\Gamma_{a,a5}$ (two equations corresponding to γ_a and $\gamma_a\gamma_5$). The torsion $T = T^a\gamma_a + \tilde{T}^a\gamma_a\gamma_5$ is defined as:

$$T \equiv dV - \Omega\wedge_\star V - V\wedge_\star \Omega \tag{6.35}$$

The torsion equation can be written as:

$$[iT\wedge_\star V - iV\wedge_\star T + 2\psi\wedge_\star \bar{\psi}\wedge_\star V - 2V\wedge_\star \psi\wedge_\star \bar{\psi}, \gamma_5] = 0 \tag{6.36}$$

since the anticommutator with γ_5 selects the γ_{ab}, 1 and γ_5 components. This equation can be solved for the torsion:

$$T = i[\psi\wedge_\star \bar{\psi}, \gamma_5]\gamma_5 = i\psi\wedge_\star \bar{\psi} - i\gamma_5\psi\wedge_\star \bar{\psi}\gamma_5 \tag{6.37}$$

For chiral gravitini:

$$T = 2i\psi\wedge_\star \bar{\psi} \tag{6.38}$$

The Bianchi identities for the curvatures and the torsion are obtained from the commutative ones simply by replacing exterior products by $\star$-exterior products.

6.6 Commutative limit

The nonsupersymmetric NC theory with NC Majorana gravitino, and charge conjugation conditions (6.12), reduces in the $\theta \to 0$ limit to the usual $D = 4$, $N = 1$ supergravity. Indeed the charge conjugation conditions on V and Ω imply that the component fields $\tilde{V}^a$, ω, and $\tilde{\omega}$ all vanish in the limit $\theta \to 0$ (see the second line of (6.29)), and only the classical spin connection ω^{ab}, vierbein V^a and Majorana fermion ψ survive. Similarly the gauge parameters ε, and $\tilde{\varepsilon}$ vanish in the commutative limit.

In the chiral case, the extra vielbein $\tilde{V}^a$ cannot vanish in the commutative limit, since its supersymmetry variation is equal to that of V^a. Then one obtains a commutative limit that is a (locally) supersymmetric version of gravity with a complex vielbein studied by Chamseddine, or a bigravity-like theory (in our case a super-bigravity theory). For a discussion on chiral supergravity see for ex. [17]. A detailed study of this commutative limit will not be carried out in the present paper.

7. Conclusions

The index-free notation, based on Clifford algebra expansion of the bosonic fields (see for ex. ref.s [17, 7]), allows to study invariances with simple algebraic manipulations. This framework is ideally suited to study noncommutative generalizations of field theories containing gravity, cf. ref.s [7], where a complex noncommutative gravity was proposed. In these Proceedings we have reviewed our construction of a a NC gravity with a commutative limit coinciding with the usual Einstein-Cartan theory. We proved that a NC charge conjugation condition on the vierbein and on the spin connection yields a real vierbein in the commutative limit. The theory can also be coupled to (Majorana) fermion zero-forms (spin 1/2).

We have then presented noncommutative supergravity in $D = 4$: if we use the NC Majorana condition for the gravitino, the action is not $\star$-supersymmetric. However also in this case we can impose charge conjugation conditions on the vierbein and spin connection, so that the commutative limit of the theory reproduces usual $D = 4$, $N = 1$ supergravity.

We recover $\star$-local supersymmetry of the action when the gravitino is chiral. In this case we cannot impose the charge conjugation condition on the vierbein (because then $\star$-supersymmetry requires the NC Majorana condition on the gravitino), and therefore the commutative limit does not involve only one real vierbein, but reduces to a chiral $D = 4$, $N = 1$ supergravity with a complex vierbein.

Note that the $\star$-products deformations considered in this paper are associated to a very general triangular Drinfeld twist $\mathscr{F}$, a particular case being the Groenewold-Moyal $\star$-product. In our general framework one could consider promoting the twist $\mathscr{F}$ itself to a dynamical field, see [18] for an example in the flat case.

Finally, we briefly comment on a class of solutions [21, 22] for the NC gravity and supergravity theories we have reviewed. These solutions can be simply obtained by considering *classical* solutions for the vielbein and their classical Killing vectors, i.e. solutions of the undeformed theory and their symmetries. Using a subset K of these Killing vectors to *define* a star product, all star products involving the vielbein reduce to ordinary products, since V can always be chosen to satisfy $\mathscr{L}_K V = 0$. Then V is a solution also for the $\star$ - equations of motion of the deformed theory.

Acknowledgements It is a pleasure to thank the Corfu Summer Institute of elementary particle physics, that hosted this Workshop. It provided a perfect setting for discussions, in an enchanting and stimulating atmosphere.

8. Appendix : gamma matrices in $D = 4$

We summarize in this Appendix our gamma matrix conventions in $D = 4$.

$$\eta_{ab} = (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \tag{8.1}$$

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \tag{8.2}$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \quad \gamma_5^\dagger = \gamma_5 \tag{8.3}$$

$$\gamma_a^T = -C\gamma_a C^{-1}, \quad \gamma_5^T = C\gamma_5 C^{-1}, \quad C^2 = -1, \quad C^T = -C \tag{8.4}$$

8.1 Useful identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \quad (8.5)$$
$$\gamma_{ab}\gamma_5 = \frac{i}{2}\varepsilon_{abcd}\gamma^{cd} \quad (8.6)$$
$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (8.7)$$
$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (8.8)$$
$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (8.9)$$
$$\gamma^{ab}\gamma_{cd} = -i\varepsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd} \quad (8.10)$$

8.2 Charge conjugation and Majorana condition

$$\text{Dirac conjugate} \;\; \bar\psi \equiv \psi^\dagger\gamma_0 \quad (8.11)$$
$$\text{Charge conjugate spinor} \;\; \psi^c = C(\bar\psi)^T \quad (8.12)$$
$$\text{Majorana spinor} \;\; \psi^c = \psi \;\; \Rightarrow \bar\psi = \psi^T C \quad (8.13)$$

8.3 Fierz identities for two spinor one-forms

$$\psi\wedge\bar\chi = \frac{1}{4}[(\bar\chi\wedge\psi)1 + (\bar\chi\gamma_5\wedge\psi)\gamma_5 + (\bar\chi\gamma^a\wedge\psi)\gamma_a + (\bar\chi\gamma^a\gamma_5\wedge\psi)\gamma_a\gamma_5 - \frac{1}{2}(\bar\chi\gamma^{ab}\wedge\psi)\gamma_{ab}] \quad (8.14)$$

Noncommutative Fierz identities

$$\psi\wedge_\star\bar\chi = \frac{1}{4}[Tr(\psi\wedge_\star\bar\chi)1 + Tr(\psi\gamma_5\wedge_\star\bar\chi)\gamma_5 + Tr(\psi\gamma^a\wedge_\star\bar\chi)\gamma_a + Tr(\psi\gamma^a\gamma_5\wedge_\star\bar\chi)\gamma_a\gamma_5 - \frac{1}{2}Tr(\psi\gamma^{ab}\wedge_\star\bar\chi)\gamma_{ab}] \quad (8.15)$$

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