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Supertranslations call for superrotations

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We review recent results on symmetries of asymptotically flat spacetimes at null infinity. In higher dimensions, the symmetry algebra realizes the Poincaré algebra. In three and four dimensions, besides the infinitesimal supertranslations that have been known since the sixties, the algebras are evenly balanced because there are also infinitesimal superrotations. We provide the classification of central extensions of $\mathfrak{bms}_3$ and $\mathfrak{bms}_4$. Applications and consequences as well as directions for future work are briefly indicated.

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1. The $\mathfrak{bms}_n$ algebra in higher dimensions

When studying asymptotic symmetries, a fast way to get an idea of what the eventual algebra might be is to solve the Killing equation for the background metric to leading order. When this is done for asymptotically flat space-times at null infinity in n spacetime dimensions [1], one finds that the symmetry algebra consists of the semi-direct sum of conformal Killing vectors of the $n-2$ sphere acting on the ideal of infinitesimal supertranslations, which are parametrised by arbitrary functions on the $n-2$ sphere.

If x^A, $A = 2,\dots,n$ are coordinates on the $n-2$ sphere, $\bar{D}_A$ the associated covariant derivative, $Y^A(x^B)\partial_A$ the conformal Killing vectors and $T(x^A)$ the functions parametrising the infinitesimal supertranslations, the $\mathfrak{bms}_n$ algebra is explicitly defined through the commutation relations $[(Y_1,T_1),(Y_2,T_2)] = (\widehat{Y},\widehat{T})$ where

$$\begin{cases} \widehat{Y}^A = Y_1^B\partial_B Y_2^A - Y_2^B\partial_B Y_1^A\,, \\ \widehat{T} = Y_1^A\partial_A T_2 - Y_2^A\partial_A T_1 + \frac{1}{n-2}(T_1\bar{D}_A Y_2^A - T_2\bar{D}_A Y_1^A)\,. \end{cases} \tag{1.1}$$

For $n > 4$, the first factor is isomorphic to the $n(n-1)/2$ dimensional algebra $\mathfrak{so}(n-1,1)$ of infinitesimal conformal transformations of Euclidean space in $n-2$ dimensions and also to the Lorentz algebra in n dimensions.

When making a more detailed analysis taking the precise definitions of asymptotically flat spacetimes in higher dimensions into account, it turns out that the supertranslations collapse to ordinary translations so that the resulting symmetry algebra is just the Poincaré algebra [2, 3].

In the realizations of asymptotic symmetries in general relativity in higher dimensions, there thus remain only standard rotations, including the hyperbolic ones, and translations. In three and four dimensions however, the asymptotic symmetry algebras are infinite-dimensional and thus yield much more information on the system.

Before turning to the algebra in three and four dimensions, we make a couple of remarks on how to actually compute the asymptotic symmetry algebra and on its realizations.

2. Asymptotic versus complete gauge fixations. Realizations

There are basically two attitudes to the problem. On the one hand, one can fix the coordinate freedom only asymptotically in which case the asymptotic symmetry algebra appears as the quotient algebra of allowed, modulo an ideal of trivial, infinitesimal transformations. The advantage of this approach is that it is easier to show that specific solutions to the equations of motion are admissible, i.e., asymptotically flat in the case of interest here. When one chooses to fix the coordinate freedom completely on the other hand, the asymptotic symmetry algebra appears as the residual "global" symmetry algebra after gauge fixing and no longer depends on arbitrary functions of the bulk spacetime. The advantage of this "reduced phase space" approach is that only physical degrees of freedom remain. A standard example illustrating this difference is the Brown-Henneaux [4] versus the Feffermann-Graham [5, 6] definition of asymptotically anti-de Sitter spacetimes in three dimensions.

Recently in [7, 8], we have followed the latter approach in the asymptotically flat case by using a Bondi-Metzner-Sachs type of gauge in four dimensions [9, 10, 11] and a reasonable analog thereof in three. In particular, the asymptotic symmetry algebras to be discussed below have explicitly been shown to be the same whether one fixes the gauge completely or only asymptotically. Furthermore, as suggested by Penrose's conformal approach to asymptotically flat spacetimes [12, 13, 14], we have considered classes of gauge fixations differing by a choice of the conformal factor for the degenerate metric on Scri and have investigated the behavior of the theory under changes of such gauges.

From this point of view, the Newman-Unti (NU) approach to asymptotically flat spacetimes [15] corresponds to a different gauge choice for the radial coordinate. In view of its embedding in the widely used Newman-Penrose formalism [16] and its direct relevance in many applications, see e.g. the review article [17], it is worthwhile to show that the asymptotic symmetry algebra is unchanged and to provide explicit formulae for the realization of the algebra in this gauge. This has been done in [18].

A novel result in our study concerns the realization of the asymptotic algebra not only on the boundary Scri but in the bulk gauge fixed spacetime by using a natural modification of the Lie bracket for vector fields that depend on the metric and is related to the theory of Lie algebroids [19]. Furthermore, this modified bracket is also needed for the realization on Scri in order to disentangle the gauge transformations from the residual global symmetries when allowing for changes of the conformal factor. We have also studied in detail how the symmetry algebra is realized on the arbitrary functions parametrizing solution space.

3. The $\mathfrak{bms}_3$ algebra

The $\mathfrak{bms}_3$ algebra consists of the algebra of vector fields on the circle acting on the functions of the circle and has been originally derived in the context of a symmetry reduction of four dimensional gravitational waves [20, 21].

More precisely, let $y = Y\frac{\partial}{\partial\phi} \in \text{Vect}(S^1)$ be the vector fields on the circle and $T(d\phi)^{-\lambda} \in \mathscr{F}_\lambda(S^1)$ tensor densities of degree λ, which form a module of the Lie algebra $\text{Vect}(S^1)$ for the action

$$\rho(y)t = (YT' - \lambda Y'T)d\phi^{-\lambda}. \tag{3.1}$$

The algebra $\mathfrak{bms}_3$ is the semi-direct sum of $\text{Vect}(S^1)$ with the abelian ideal $\mathscr{F}_1(S^1)$, the bracket between elements of $\text{Vect}(S^1)$ and elements $t = Td\phi^{-1} \in \mathscr{F}_1(S^1)$ being induced by the module action, $[y,t] = \rho(y)t$.

Consider the associated complexified Lie algebra and let $z = e^{i\phi}$, $m,n,k\ldots \in \mathbb{Z}$. Expanding into modes, $y = a^n l_n$, $t = b^n t_n$, where

$$l_n = e^{in\phi}\frac{\partial}{\partial\phi} = iz^{n+1}\frac{\partial}{\partial z}, \quad t_n = e^{in\phi}(d\phi)^{-1} = iz^{n+1}(dz)^{-1},$$

the commutation relations read explicitly

$$i[l_m, l_n] = (m-n)l_{m+n}, \quad i[l_m, t_n] = (m-n)t_{m+n}, \quad i[t_m, t_n] = 0. \tag{3.2}$$

The non-vanishing structure constants of $\mathfrak{bms}_3$ are thus entirely determined by the structure constants $[l_m, l_n] = -if^k_{mn} l_k$, $f^k_{mn} = \delta^k_{m+n}(m-n)$ of the Witt subalgebra $\mathfrak{w}$ defined by the linear span of the l_n.

Up to equivalence, the most general central extension of $\mathfrak{bms}_3$ is given by

$$\begin{cases} i[l_m, l_n] = (m-n)l_{m+n} + \frac{c_1}{12} m(m+1)(m-1)\delta^0_{m+n}, \\ i[l_m, t_n] = (m-n)t_{m+n} + \frac{c_2}{12} m(m+1)(m-1)\delta^0_{m+n}, \\ i[t_m, t_n] = 0\,. \end{cases} \tag{3.3}$$

The proof follows by generalizing the one for the Witt algebra $\mathfrak{w}$, which is textbook material, see e.g [22, 23, 24]. Nevertheless, in order to be self-contained, we give a complete derivation in the appendix.

The associated classical charge algebra of asymptotically flat three dimensional space-times has been constructed in [1] with central charges[1]

$$c_1 = 0, \quad c_2 = \frac{3}{G}. \tag{3.4}$$

When one considers the extension of the $\mathfrak{bms}_3$ algebra obtained by replacing the vanishing commutators of the t_m's in (3.3) through

$$i[t_m, t_n] = \frac{1}{l^2}(m-n)l_{m+n}\,, \tag{3.5}$$

and defines $l^\pm_m = \frac{1}{2}(lt_{\pm m} \pm l_{\pm m})$, the resulting algebra turns into two copies of the Virasoro algebra with central charges $c^\pm = \frac{3l}{2G}$,

$$i[l^\pm_m, l^\pm_n] = (m-n)l^\pm_{m+n} + \frac{c^\pm}{12} m(m+1)(m-1)\delta^0_{m+n}, \quad i[l^\pm_m, l^\mp_n] = 0\,, \tag{3.6}$$

which is precisely the value of the classical central extensions in the charge algebra of asymptotically anti-de Sitter spacetimes [4]. In other words, starting from the charge algebra (3.6) in asymptotically anti-de Sitter space-times, the flat result is obtained by first writing the algebra in terms of the new generators $l_m = l^+_m - l^-_{-m}$, $t_m = \frac{1}{l}(l^+_m + l^-_{-m})$ and then taking $l \to \infty$.

An important question is a complete understanding of the physically relevant representations of $\mathfrak{bms}_3$. Note that in the present gravitational context, the Hamiltonian is associated with t_0, so that one is especially interested in representations with a lowest eigenvalue of t_0. This question should be tractable, given all that is known on both the Poincaré and Virasoro subalgebras of $\mathfrak{bms}_3$.

It turns out that $\mathfrak{bms}_3$ is isomorphic to the Galilean conformal algebra in 2 dimensions $\mathfrak{gca}_2$ [25]. In a different context, a class of non-unitary representations of $\mathfrak{gca}_2$ have been studied in some details [26].

4. The $\mathfrak{bms}_4$ algebra

In four dimensions, the infinitesimal Lorentz transformations appear as the conformal Killing vectors of the 2 sphere. By the standard argument, when focusing on infinitesimal local transformations that are not required to be everywhere regular, the conformal Killing vectors are given by two copies of the Witt algebra, so that besides supertranslations, there now also are superrotations.

[1]Note that in the equivalent gauge fixed derivation given in [8], there is a misprint in the last line of (3.18), where Θ has to be replaced by $\Theta + 1$.

More precisely, in stereographic coordinates $\zeta = e^{i\phi}\cot\frac{\theta}{2}$ and $\bar{\zeta}$ for the 2 sphere with $\varphi_0 = \ln\frac{1}{2}(1+\zeta\bar{\zeta})$, the algebra may be realized through the vector fields $y = Y(\zeta)\partial$, $\bar{y} = \bar{Y}(\bar{\zeta})\bar{\partial}$ where $\partial = \frac{\partial}{\partial\zeta}$, $\bar{\partial} = \frac{\partial}{\partial\bar{\zeta}}$. If $T(\zeta,\bar{\zeta}) = \widetilde{T}(\zeta,\bar{\zeta})e^{-\varphi_0}$, they act on tensor densities $\mathscr{F}_{\frac{1}{2},\frac{1}{2}}$ of degree $(\frac{1}{2},\frac{1}{2})$,

$$t = \widetilde{T}(\zeta,\bar{\zeta})e^{-\varphi_0}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}}, \tag{4.1}$$

through

$$\rho(y)t = (Y\partial\widetilde{T} - \frac{1}{2}\partial Y\widetilde{T})e^{-\varphi_0}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}}, \tag{4.2}$$

$$\rho(\bar{y})t = (\bar{Y}\bar{\partial}\widetilde{T} - \frac{1}{2}\bar{\partial}\bar{Y}\widetilde{T})e^{-\varphi_0}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}}. \tag{4.3}$$

The algebra $\mathfrak{bms}_4$ is then the semi-direct sum of the algebra of vector fields $y,\bar{y}$ with the abelian ideal $\mathscr{F}_{\frac{1}{2},\frac{1}{2}}$, the bracket being induced by the module action, $[y,t] = \rho(y)t$, $[\bar{y},t] = \rho(\bar{y})t$. When expanding $y = a^n l_n$, $\bar{y} = \bar{a}^n\bar{l}_n$, $t = b^{m,n}T_{m,n}$, with

$$l_n = -\zeta^{n+1}\partial, \quad \bar{l}_n = -\bar{\zeta}^{n+1}\bar{\partial}, \quad T_{m,n} = \zeta^m\bar{\zeta}^n e^{-\varphi_0}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}}, \tag{4.4}$$

the enhanced symmetry algebra reads

$$\begin{gathered}[l_m,l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m,\bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_m,\bar{l}_n] = 0,\\ [l_l,T_{m,n}] = (\tfrac{l+1}{2}-m)T_{m+l,n}, \quad [\bar{l}_l,T_{m,n}] = (\tfrac{l+1}{2}-n)T_{m,n+l}, \quad [T_{m,n},T_{o,p}] = 0,\end{gathered} \tag{4.5}$$

where $m,n\ldots \in \mathbb{Z}$. The Poincaré algebra is the subalgebra spanned by the generators $T_{0,0}$, $T_{0,1}$, $T_{1,0}$, $T_{1,1}$ for ordinary translations and $l_{-1},l_0,l_1,\bar{l}_{-1},\bar{l}_0,\bar{l}_1$ for ordinary (Lorentz) rotations.

The quotient algebra of $\mathfrak{bms}_4$ by the abelian ideal of infinitesimal supertranslations is no longer given by the Lorentz algebra but by two copies of the Witt algebra. It follows that the problem with angular momentum in general relativity [27], at least in its group theoretical formulation, disappears as now the choice of an infinite number of conditions is needed to fix an infinite number of rotations. For a complete analysis, the associated charges and their algebra is needed. This will be discussed in detail elsewhere.

In the appendix, we will show that the only non trivial central extensions of $\mathfrak{bms}_4$ are the usual central extensions of the 2 copies of the Witt algebra, i.e., they appear in the commutators $[l_m,l_{-m}]$ and $[\bar{l}_m,\bar{l}_{-m}]$. Contrary to three dimensions, there are no central extensions involving the generators for supertranslations.

5. Outlook

The most obvious questions to be addressed next are a complete study of the physically relevant representations of $\mathfrak{bms}_3$ and of $\mathfrak{bms}_4$ and the construction of the surface charge algebra associated with supertranslations and superrotations in 4 dimensions. We have recently made progress on the latter problem and will report on these results elsewhere. It turns out that the extension in the charge algebra depends explicitly on the fields characterizing solution space, it is a Lie algebroid 2-cocycle rather than a Lie algebra 2 cocycle.

The ultimate hope of this program is to use the powerful apparatus of 2 dimensional conformal field theory in the context of quantum 4 dimensional general relativity, for instance in an S-matrix approach between $\mathscr{I}^+$ and $\mathscr{I}^-$.

Appendix 1: Central extensions of $\mathfrak{bms}_3$

In order to get rid of the overall i in (3.2), we redefine the generators as $l'_m = il_m$. Inequivalent central extensions of $\mathfrak{bms}_3$ are classified by the cohomology space $H^2(\mathfrak{bms}_3)$. More explicitly, the Chevally-Eilenberg differential is given by

$$\gamma = -\frac{1}{2}C^m C^{k-m}(2m-k)\frac{\partial}{\partial C^k} - C^m \xi^{k-m}(2m-k)\frac{\partial}{\partial \xi^k}\,, \tag{5.1}$$

in the space $\Lambda(C,\xi)$ of polynomials in the anticommuting "ghost" variables C^m,ξ^m. The grading is given by the eigenvalues of the ghost number operator, $N_{C,\xi} = C^m\frac{\partial}{\partial C^m} + \xi^m\frac{\partial}{\partial \xi^m}$, the differential γ being homogeneous of degree 1 and $H^2(\mathfrak{bms}_3) \cong H^2(\gamma,\Lambda(C,\xi))$. Furthermore, when counting only the ghosts ξ^m associated with supertranslations, $N_\xi = \xi^m\frac{\partial}{\partial \xi^m}$, the differential γ is homogeneous of degree 0, so that the cohomology decomposes into components of definite N_ξ degree. The cocycle condition then becomes

$$\gamma(\omega^0_{m,n}C^mC^n) = 0, \quad \gamma(\omega^1_{m,n}C^m\xi^n) = 0, \quad \gamma(\omega^2_{m,n}\xi^m\xi^n) = 0, \tag{5.2}$$

with $\omega^0_{m,n} = -\omega^0_{n,m}$ and $\omega^2_{m,n} = -\omega^2_{n,m}$. The coboundary condition reads

$$\omega^0_{m,n}C^mC^n = \gamma(\eta^0_m C^m), \quad \omega^1_{m,n}C^m\xi^n = \gamma(\eta^1_m\xi^m). \tag{5.3}$$

We have $\{\frac{\partial}{\partial C^0},\gamma\} = \mathcal{N}_{C,\xi}$ with $\mathcal{N}_{C,\xi} = m(C^m\frac{\partial}{\partial C^m} + \xi^m\frac{\partial}{\partial \xi^m})$. It follows that all cocycles of $\mathcal{N}_{C,\xi}$ degree different from 0 are coboundaries, $\gamma\omega_N = 0$, $\mathcal{N}_{C,\xi}\omega_N = N\omega_N$, $N \neq 0$ implies that $\omega_N = \gamma(\frac{1}{N}\frac{\partial}{\partial C^0}\omega_N)$. Without loss of generality we can thus assume that $\omega^0_{m,n}C^mC^n = \omega^0_m C^m C^{-m}$ with $\omega^0_m = -\omega^0_{-m}$ and in particular $\omega^0_0 = 0$; $\omega^1_{m,n}C^m\xi^n = \omega^1_m C^m\xi^{-m}$; $\omega^2_{m,n}\xi^m\xi^n = \omega^2_m\xi^m\xi^{-m}$ with $\omega^2_m = -\omega^2_{-m}$ and in particular $\omega^2_0 = 0$. By applying $\frac{\partial}{\partial C^0}$ to the coboundary condition $\omega^0_m C^m C^{-m} = \gamma(\eta^0_m C^m)$ we find that $0 = m\eta^0_m C^m$. The coboundary condition then gives $\omega^0_m C^m C^{-m} = \gamma(\eta^0_0 C^0) = -m\eta^0_0 C^m C^{-m}$. By adjusting η^0_0, we can thus assume without loss of generality that $\omega^0_1 = 0$ and that the coboundary condition has been entirely used. In the same way $\omega^1_m C^m\xi^{-m} = \gamma(\eta^1_m\xi^m)$ implies first that $\eta^1_m = 0$ for $m \neq 0$ and then that one can assume that $\omega^1_1 = 0$, with no coboundary condition left.

Taking into account the anticommuting nature of the ghosts, the cocycle conditions become explicitly, $\omega^0_m(2n+m) - \omega^0_n(2m+n) + \omega^0_{m+n}(n-m) = 0$, $\omega^1_m(2n-m) + \omega^1_n(n-2m) + \omega^1_{m-n}(n+m) = 0$, $\omega^2_m(2n+m) + \omega^2_{m+n}(n-m) = 0$. Putting $m = 0$ in the last relation gives $\omega^2_m = 0$, for $m \neq 0$ and thus for all m, putting $m = 1 = n$ in the second relation gives $\omega^1_0 = 0$, while $m = 0$ gives $\omega^1_n n = -\omega^1_{-n}n$ and thus that $\omega^1_n = -\omega^1_{-n}$ for all n. Changing m to $-m$ and using this symmetry property, the cocycle conditions for ω^0_m and ω^1_m give the same constraints. Putting $m = 1$, one finds the recurrence relation $\omega^{0,1}_{n+1} = \frac{n+2}{n-1}\omega^{0,1}_n$, which gives a unique solution in terms of $\omega^{0,1}_2$. The result follows by setting $c_{1,2} = \frac{1}{2}\omega^{0,1}_2$ and checking that the constructed solution does indeed satisfy the cocycle condition.

Appendix 2: Central extensions of $\mathfrak{bms}_4$

For $\mathfrak{bms}_4$, the Chevally-Eilenberg differential is given by

$$\gamma = -\frac{1}{2}C^m C^{k-m}(2m-k)\frac{\partial}{\partial C^k} - \frac{1}{2}\bar{C}^m\bar{C}^{k-m}(2m-k)\frac{\partial}{\partial \bar{C}^k}$$

$$-C^m\xi^{k-m,n}(\frac{3m+1}{2}-k)\frac{\partial}{\partial\xi^{k,n}}-\bar{C}^n\xi^{m,k-n}(\frac{3n+1}{2}-k)\frac{\partial}{\partial\xi^{m,k}}\,, \tag{5.4}$$

in the space $\Lambda(C,\bar{C},\xi)$ of polynomials in the anticommuting "ghost" variables $C^m,\bar{C}^n,\xi^{m,n}$. The grading is given by the eigenvalues of the ghost number operator, $N_{C,\xi}=C^m\frac{\partial}{\partial C^m}+\bar{C}^m\frac{\partial}{\partial\bar{C}^m}+\xi^{m,n}\frac{\partial}{\partial\xi^{m,n}}$, the differential γ being homogeneous of degree 1 and $H^2(\mathfrak{bms}_4)\cong H^2(\gamma,\Lambda(C,\bar{C},\xi))$. Furthermore, when counting only the ghosts ξ^m associated with supertranslations, $N_\xi=\xi^{m,n}\frac{\partial}{\partial\xi^{m,n}}$, the differential γ is homogeneous of degree 0, so that the cohomology decomposes into components of definite N_ξ degree. The cocycle condition then becomes

$$\gamma(\omega^0_{m,n}C^mC^n+\bar{\omega}^0_{m,n}\bar{C}^m\bar{C}^n+\omega^{-1}_{m,n}C^m\bar{C}^n)=0,\qquad \gamma(\omega^1_{k,mn}C^k\xi^{m,n}+\bar{\omega}^1_{k,mn}\bar{C}^k\xi^{m,n})=0,$$
$$\gamma(\omega^2_{mn,kl}\xi^{m,n}\xi^{k,l})=0, \tag{5.5}$$

with $\omega^0_{m,n}=-\omega^0_{n,m}$, $\bar{\omega}^0_{m,n}=-\bar{\omega}^0_{n,m}$ and $\omega^2_{mn,kl}=-\omega^2_{kl,mn}$. The coboundary condition reads

$$\omega^0_{m,n}C^mC^n+\bar{\omega}^0_{m,n}\bar{C}^m\bar{C}^n+\omega^{-1}_{m,n}C^m\bar{C}^n=\gamma(\eta^0_mC^m+\bar{\eta}^0_m\bar{C}^m),$$
$$\omega^1_{k,mn}C^k\xi^{m,n}+\bar{\omega}^1_{k,mn}\bar{C}^k\xi^{m,n}=\gamma(\eta^1_{mn}\xi^{m,n}). \tag{5.6}$$

We have $\{\frac{\partial}{\partial C^0},\gamma\}=\mathcal{N}_{C,\xi}$ with $\mathcal{N}_{C,\xi}=mC^m\frac{\partial}{\partial C^m}+(m-\frac{1}{2})\xi^{m,n}\frac{\partial}{\partial\xi^{m,n}}$ and also $\{\frac{\partial}{\partial\bar{C}^0},\gamma\}=\bar{\mathcal{N}}_{\bar{C},\xi}$ with $\bar{\mathcal{N}}_{\bar{C},\xi}=n\bar{C}^n\frac{\partial}{\partial\bar{C}^n}+(n-\frac{1}{2})\xi^{m,n}\frac{\partial}{\partial\xi^{m,n}}$. It follows again that all cocycles of either $\mathcal{N}_{C,\xi}$ or $\bar{\mathcal{N}}_{\bar{C},\xi}$ degree different from 0 are coboundaries. Without loss of generality we can thus assume that $\omega^0_{m,n}C^mC^n+\bar{\omega}^0_{m,n}\bar{C}^m\bar{C}^n+\omega^{-1}_{m,n}C^m\bar{C}^n=\omega^0_mC^mC^{-m}+\bar{\omega}^0_m\bar{C}^m\bar{C}^{-m}+\omega^{-1}_{0,0}C^0\bar{C}^0$ with $\omega^0_m=-\omega^0_{-m}$, $\bar{\omega}^0_m=-\bar{\omega}^0_{-m}$ and in particular $\omega^0_0=0=\bar{\omega}^0_0$; none of monomials with one $\xi^{m,n}$ and either on C^k or one $\bar{C}^k$ can be of degree 0, so $\omega^1_{k,mn}=0=\bar{\omega}^1_{k,mn}$; $\omega^2_{mn,kl}\xi^{m,n}\xi^{k,l}=\omega^2_{m,n}\xi^{m,n}\xi^{-m+1,-n+1}$ with $\omega^2_{m,n}=-\omega^2_{-m+1,-n+1}$. Both the cocycle and the coboundary condition for $\omega^0_mC^mC^{-m}+\bar{\omega}^0_m\bar{C}^m\bar{C}^{-m}+\omega^{-1}_{0,0}C^0\bar{C}^0$ split. For $\omega^{-1}_{0,0}C^0\bar{C}^0$ there is no coboundary condition, while the cocycle condition implies $\omega^{-1}_{0,0}=0$. The rest of the analysis proceeds as in the previous subsection, separately for $\omega^0_mC^mC^{-m}$ and $\bar{\omega}^0_m\bar{C}^m\bar{C}^{-m}$, with the standard central extension for $[l_m,l_{-m}]$ and $[\bar{l}_m,\bar{l}_{-m}]$.

We still have to analyze $\gamma(\omega^2_{m,n}\xi^{m,n}\xi^{-m+1,-n+1})=0$. This condition gives $\omega^2_{m,n}(\frac{3l-1}{2}+m)+\omega^2_{l+m,n}(\frac{l+1}{2}-m)=0$ and also $\omega^2_{m,n}(\frac{3l-1}{2}+n)+\omega^2_{m,l+n}(\frac{l+1}{2}-n)=0$. Putting $m=0$ in the first relation gives $\omega^2_{0,n}(\frac{3l-1}{2})+\omega^2_{l,n}(\frac{l+1}{2})=0$. Putting $l=-1$ then implies $\omega^2_{0,n}=0$ and then also $\omega^2_{l,n}=0$ for $l\neq-1$. But $\omega^2_{-1,n}=-\omega^2_{2,-n+1}=0$ which shows that $\omega^2_{m,n}=0$ for all m,n and concludes the proof.

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