PROCEEDINGS OF SCIENCE

Canonical Formalism of 2D, Osp(1|2) Supergravity

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The Jackiw-Teitelboim model (JT) describes gravity with a cosmological constant in bidimensional spacetime. This model can be equivalently described by a topological action, the BF model. With the aim of studying the bidimensional gravity from the point of view of Loop Quantum Gravity, coupled to matter, we propose a supersymmetric extension of the BF model. We make a canonical analysis of super BF theory and show that the theory is invariant under supersymmetry gauge transformations; determine the constraints and their algebra, showing that they form a closed Lie algebra, allowing us to conclude that these are first class constraints, and as such, are the generators of the gauge transformations of the theory. Our gauge symmetry group is the supergroup OSP$(1\,|\,2)$, and its Lie superalgebra, osp$(1\,|\,2)$, has three bosonic generators J_i, and two fermionic generators Q_α.

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1. The Jackiw-Teitelboim Model

Since the Hilbert-Einstein action for the bidimensional gravity is topological, the equation of motion is zero identicaly, Jackiw and Teitelboim propose the Liouville equation, $R-2K=0$, as an alternative to the Einstein equation of motion, that be can derived from the action,

$$S_{JT}=\frac{1}{2}\int d^2x\sqrt{-g}\psi(R-2K), \tag{1.1}$$

here ψ is a scalar field, known as the dilaton, that acts as a Lagrange multiplier, R the scalar curvature and K the cosmological constant[1]. This action is known as the Jackiw-Teitelboim action and is invariant under the diffeomorphisms of spacetime.

2. The 2-Dimensional BF Model

In the metric formalism the pure gravitation action is given by (1.1). This action can be obtained from a topological action of the BF type[2].

We study the gravitation from the point of view of the first order formalism, where the fundamental fields are the 2-bein or zweibein e^I_μ and the spin connection ω^{IJ}_μ, ($\mu,\nu,\dots=t,x$; the spacetime coordinates, $I,J,\dots=0,1$; the tangent spacetime coordinates in the base defined by the zweibien e^I_μ). The BF model corresponding to the bidimensional gravity has as base the connection 1-form A, with the generators of group given by the spacetime translations, P_I, $(I=0,1)$ and the Lorentz boost generators Λ: $A(x)=e^I(x)P_I+\omega(x)\Lambda$. Considering the cosmological constant K nonzero, the generators of the group, P_I and Λ, satisfy the (anti-)deSitter algebra (A)dS, $\mathrm{SO}(2,1)$ (or $\mathrm{SO}(1,2)$),

$$[\Lambda,P_I]=\varepsilon_I{}^J P_J,\quad [P_I,P_J]=K\varepsilon_{IJ}\Lambda,\quad (\varepsilon_{IJ}=-\varepsilon_{JI},\,\varepsilon_{12}=1,\,\varepsilon^J_I=\eta^{JK}\varepsilon_{IK})$$

and be can written in a general form as $[J_i,J_i]=f_{ij}{}^kJ_k$, defining the generators of the Ad(S) algebra, as $\{J_i\}=\{J_0,J_1,J_2\}=\{P_0,P_1,\Lambda\}$, where lowercase indices of the half of the Latin alphabet $i,j=0,1,2$, represent the indices of the Ad(S) group, $\eta_{IJ}=\mathrm{diag}(\sigma,1)$, with the nonzero structure constants $f_{ij}{}^k$ given by $f_{01}{}^2=K$, $f_{12}{}^0=\sigma$ and $f_{20}{}^1=1$, (σ=1(-1) for the Riemann(Minkowski) space). This algebra is equipped with a Killing metric $k_{ij}=-\frac{\sigma}{2}f_{ik}{}^lf_{jl}{}^k$.

$$(k_{ij})=\begin{pmatrix}K\eta_{IJ} & 0\\ 0 & 1\end{pmatrix}. \tag{2.1}$$

Then, the classical action for the BF theory is:

$$S_{BF}[A,\phi]=\int\langle\phi,F\rangle \tag{2.2}$$

where ϕ is a scalar – "the dilaton", A is the connection, and $F=dA+A\wedge A$ the curvature, all being, in the adjoint representation, written as, $A=A^iJ_i$, $\phi=\phi^iJ_i$, and $F=F^iJ_i$. The quadratic form is defined by $\langle J_i,J_j\rangle=\mathrm{Tr}(\mathrm{ad(J_i)ad(J_j)})=\frac{-\sigma}{2}\mathrm{k_{ij}}$. The equations of motion are $\frac{\delta S_{BF}}{\delta A}=D\phi=0$ and $\frac{\delta S_{BF}}{\delta\phi}=F=0$, where $D=d+[A,\ ]$ is the covariant derivative. In order to couple matter we propose a supersymmetric extension of (2.2).

3. Supersymmetric Extension of the BF Model

Let g_s be a super Lie algebra, and $X,Y,Z,\cdots$ its elements. The super Lie bracket denoted by $[\,,\,\}$, satisfy graded commutation relations and a super Jacobi identity

$$[X,Y\} = XY - (-1)^{|X||Y|}YX, \quad [X,Y\} = -(-1)^{|X||Y|}[Y,X\},$$
$$[X,[Y,Z\}\}(-1)^{|X||Z|} + [Y,[Z,X\}\}(-1)^{|Y||X|} + [Z,[X,Y\}\}(-1)^{|Z||Y|} = 0.$$

Here X, Y and Z are elements that have a definite parity $|X| = 0(1)$ for even(odd) elements.

Making a supersymmetric extension of the (A)dS group, we choose, as basis of the Lie superalgebra g_s, the generators $T_A = \{T_i, Q_\alpha\}$, where T_i are even and Q_α odd elements,

$$[T_A, T_B\} = f_{AB}{}^C T_C\,,$$

where uppercase latin letters $(A,B,..)$ represent superalgebra indices, $A,B,.. = i,j,..;\alpha,\beta,..$, lowercase latin letters $i,j,.. = 0,1,2$, are indices of the bosonic generator and lowercase letters of the beginning of the Greek alphabet $\alpha,\beta,.. = 1,2$, label indices of the fermionic generators. Thus,

$$[T_i,T_j] = f_{ij}{}^k T_k, \quad [T_i,Q_\alpha] = f_{i\alpha}{}^\beta Q_\beta, \quad \{Q_\alpha,Q_\beta\} = f_{\alpha\beta}{}^i T_i, \tag{3.1}$$

with structure constants given by $f_{ij}{}^k = f_{ij}{}^k$, $f_{i\alpha}{}^\beta = -(J_i)_\alpha{}^\beta$, $f_{\alpha\beta}{}^i = -(J^i)_{\alpha\beta}$, $(J_i)_\alpha{}^\beta$ are elements of matrix of the generators of (A)dS algebra, in the fundamental representation,

$$J_0 = -\frac{i}{2}\sqrt{K}\tau_z\,, \quad J_1 = -\frac{i}{2}\sqrt{\sigma}\sqrt{K}\tau_x\,, \quad J_2 = -\frac{i}{2}\sqrt{\sigma}\tau_y\,. \tag{3.2}$$

where τ_x, τ_y and τ_z are Pauli matrices. The generators $T_A(=T_i,Q_\alpha)$, elements of the osp$(1,2)$ superalgebra [10, 11], written in the fundamental representation have the form:

$$(T_0) = \frac{-i\sqrt{K}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, (T_1) = \frac{-i\sqrt{\sigma}\sqrt{K}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, (T_2) = \frac{-i\sqrt{\sigma}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{3.3}$$

$$(Q_1) = \alpha\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (Q_2) = \alpha\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

The supersymmetric extension of the BF action (2.2), is then

$$\mathscr{S}[\Phi,\mathscr{A}] = \int \langle \Phi, \mathscr{F} \rangle\,, \tag{3.5}$$

where Φ is a scalar – "the superdilaton", $\mathscr{F}[\mathscr{A}]$ is the curvature of the superconnetion $\mathscr{A}$. Φ, $\mathscr{A}$ and $\mathscr{F}[\mathscr{A}]$ are valued in the Lie superalgebra, $\mathscr{A} = A^i J_i + \psi^\alpha Q_\alpha$, $\Phi = \phi^i J_i + \chi^\alpha Q_\alpha$, here A and ϕ are the connection and the dilaton respectively, bosonic fields, while, ψ and χ represent fermionic fields. The curvature of the superconnection $\mathscr{A}$ is given by:

$$\mathscr{F} = d\mathscr{A} + \mathscr{A}\wedge\mathscr{A} = F^i J_i + F^\alpha Q_\alpha, \tag{3.6}$$

with $F^i = dA^i + \frac{1}{2}A^j \wedge A^k f_{jk}{}^i - \frac{1}{2}\psi^\alpha \wedge \psi^\beta f_{\alpha\beta}{}^i$, $\quad F^\alpha = d\psi^\alpha + \frac{1}{2}A^i \wedge \psi^\beta f_{i\beta}{}^\alpha$. The quadratic form $\langle , \rangle$ defined by $\langle T_A, T_B \rangle = \mathcal{K}_{AB} = \mathrm{Str}(\mathrm{ad}(T_A)\mathrm{ad}(T_B))$, Str is the supertrace, $\mathcal{K}_{AB}$ defines a Killing form bilinear invariant [5],

$$(\mathcal{K}_{AB}) = \begin{pmatrix} (\mathcal{K}_{ij}) & 0 \\ 0 & (\mathcal{K}_{\alpha\beta}) \end{pmatrix}; \quad \mathcal{K}_{ij} = \frac{-3\sigma}{2} k_{ij}, \quad \mathcal{K}_{\alpha\beta} = 6\alpha^2 \varepsilon_{\alpha\beta}, \quad (\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$\mathcal{K}_{AB}$, k_{ij} and $k_{\alpha\beta}$ lower the indices: $X_B = X^A \mathcal{K}_{AB}$ (so, $X_i = X^j k_{ji}$ and $X_\alpha = X^\beta k_{\beta\alpha}$).

From the action (3.5) we get the superfield equation

$$\frac{\delta S}{\delta \Phi} = \mathcal{F} = 0, \quad \frac{\delta S}{\delta \mathcal{A}} = \mathcal{D}\Phi = 0, \tag{3.7}$$

where, $\mathcal{D} = d + [\mathcal{A}, \,]$ are the extended covariant derivative corresponding to the superconnection $\mathcal{A}$. As a theory of gravitation, the action (3.5) must be invariant under the diffeomorphism transformations.

The infinitesimal gauge transformations of the superfields $\mathcal{A}$ and Φ are given by:

$$\delta_\varepsilon \Phi = [\varepsilon, \Phi\} \longrightarrow \delta_\varepsilon \Phi_C = (-1)^{|C||A|} \varepsilon^A \Phi_B f_{CA}{}^B,$$
$$\delta_\varepsilon \mathcal{A} = -\mathcal{D}\varepsilon \longrightarrow \delta_\varepsilon \mathcal{A}^C = -d\varepsilon^C - (-1)^{|A||B|} \mathcal{A}^A \varepsilon^B f_{AB}{}^C$$

with $\varepsilon = \varepsilon^A T_A = \varepsilon^i J_i + \varepsilon^\alpha Q_\alpha$. The Lie derivative, $\mathcal{L}_\xi$, along the field ξ is defined by, $\mathcal{L}_\xi := i_\xi d + d i_\xi$, where $i_{(.)}$ is the interior derivative[12], so, the infinitesimal diffeomorphisms transformations of $\mathcal{A}$ and Φ, are

$$\mathcal{L}_\xi \mathcal{A} = i_\xi \mathcal{F} + \mathcal{D}(i_\xi \mathcal{A}) = i_\xi \frac{\delta S}{\delta \Phi} - \delta_{i_\xi \mathcal{A}} \mathcal{A}, \quad \mathcal{L}_\xi \Phi = i_\xi \mathcal{D}\Phi - \left[i_\xi \mathcal{A}, \Phi \right] = i_\xi \frac{\delta S}{\delta \mathcal{A}} - \delta_{i_\xi \mathcal{A}} \Phi.$$

We recognize that these transformations are, modulo equations of motion, gauge transformation with $\varepsilon = i_\xi \mathcal{A}$ as infinitesimal parameter. This means that the invariance under diffeomorphisms follows from gauge invariance, on shell.

3.1 Canonical Formalism

As usual in canonical gravity[6], we assume that the manifold $\mathcal{M}$ of our spacetime, admits a foliation: $\mathcal{M} = \Re \times \Sigma$, so $x^\mu = (t, x)$, with $t \in \Re$ and x the coordinate of the 1-dimensional space Σ. Under this foliation, we write, $\mathcal{A}_\mu = (\mathcal{A}_t, \mathcal{A}_x)$ ($\mathcal{A}_t = (A_t, \psi_t)$, $\mathcal{A}_x = (A_x, \psi_x)$). Thus, the action (2.2) takes the form:

$$S = (-1)^{|A||B|} \int dx^2 \Phi^A \Big(\partial_t \mathcal{A}_x^B - \partial_x \mathcal{A}_t^B + (-1)^{|C||D|} \mathcal{A}_t^C \mathcal{A}_x^D f_{CD}{}^B \Big) \mathcal{K}_{AB}, \tag{3.8}$$

From the definition of the canonical momentum we get

$$\Pi_B^{\mathcal{A}_x}(x) := \frac{\partial \mathcal{L}}{\partial \partial_t \mathcal{A}_x^B} = \Phi_B, \quad \Pi_B^{\mathcal{A}_t}(x) := \frac{\partial \mathcal{L}}{\partial \partial_t \mathcal{A}_t^B} \approx 0, \tag{3.9}$$

see that we have a singular Lagrangian, "$\approx$" indicates "weak equations" (becoming equality after that all Poisson bracket algebra is performed), $\Pi_B^{\mathcal{A}_t}$ are primary constraints [7, 8, 9]. So are the $\Pi_B^{\mathcal{A}_x}(x)$, but we directly reduce phase space by identifying Φ_B as the conjugate momentum of $\mathcal{A}_x$.

The Poisson brackets for two superfunctions F and G, at equal time[1] [8], are defined by

$$[F(x),G(y)\}:=\int dz(-1)^{|F||B|}\left(\frac{\delta F(x)}{\delta\mathscr{A}_\mu^B(z)}\frac{\delta G(y)}{\delta\Pi_B^{\mathscr{A}_\mu}(z)}-(-1)^{|B|}\frac{\delta F(x)}{\delta\Pi_B^{\mathscr{A}_\mu}(z)}\frac{\delta G(y)}{\delta\mathscr{A}_\mu^B(z)}\right),$$

where $[,\}$ represent the superpoisson brackets, $|F|$ define the parity of F; and by convention, we take the left derivative. The nonzero superpoisson brackets of the fundamental fields are:

$$\left[\mathscr{A}_\mu^A(x),\Pi_B^{\mathscr{A}_\nu}(y)\right\}=(-1)^{|A||B|}\delta_B^A\delta_\mu^\nu\delta(x-y)=-(-1)^{|A|}\left[\Pi_B^{\mathscr{A}_\nu}(y),\mathscr{A}_\mu^A(x)\right\}. \tag{3.10}$$

The total Hamiltonian of the theory takes the form[7];

$$H = H_c+\int dx\lambda_t^B\Pi_B^{\mathscr{A}_t}, \text{ where,}\quad \mathrm{H_c}=-\int \mathrm{dz}\mathscr{A}_\mathrm{t}^\mathrm{A}\mathscr{D}_\mathrm{z}\Phi_\mathrm{A} \tag{3.11}$$

H_c is the canonical Hamiltonian, λ_t^A's the Lagrange multipliers, $\mathscr{D}_x\Phi_A=\partial_x\Phi_A+(-1)^{|A||B|}\mathscr{A}_x^B\Phi_C f_{AB}{}^C$. After an analysis of constraints, we conclude that the constraints $\Pi_i^{\mathscr{A}_t}$ can be considered zero strongly; so the total Hamiltonian will be equivalent to $H=H_c$. For consistence, the constraints, eq.(3.9), should not evolve in time, so must satisfy the relation,

$$\dot{\Pi}_B^{\mathscr{A}_t}=\left[\Pi_B^{\mathscr{A}_t},H\right]\approx 0, \tag{3.12}$$

from which we obtain the secondary constraint [7], $\mathscr{G}_A(x):=\mathscr{D}_x\Phi_A\approx 0$. We can see that the Hamiltonian of the theory is completely constrained,

$$H=-\int dz\mathscr{A}_t^A\mathscr{G}_A(z), \tag{3.13}$$

the constraints $\mathscr{G}_A$ have even or odd parity depending if the index A is equal to i or α respectively.

For convenience we can smear the constraints in the form:

$$\mathscr{G}(\varepsilon)=\int dx\varepsilon^A\mathscr{G}_A(x),\longrightarrow\mathscr{G}_A(x)=\frac{\delta\mathscr{G}(\varepsilon)}{\delta\varepsilon^A(x)}, \tag{3.14}$$

where the ε^A's are arbitrary smooth function. The algebra of constraints then are

$$[\mathscr{G}(\varepsilon),\mathscr{G}(\eta)\} = \int dx[\varepsilon,\eta\}^A\mathscr{G}_A(x)=\mathscr{G}([\varepsilon,\eta\}) \tag{3.15}$$

with $[\varepsilon,\eta\}^C=(-1)^{|A||B|}\varepsilon^A\eta^B f_{AB}{}^C$; we can see from (3.15) that the constraints $\mathscr{G}_A$ are of first class. Thus, since the Hamiltonian is a combination of them, this ensures that the constraints are conserved in time.

4. Gauge Symmetries

Being $\mathscr{G}_A$ first class constraints, they are the generators of the gauge transformations. On the basis field, they give

$$\begin{aligned}\left[\mathscr{A}_x^A(x),\mathscr{G}(\varepsilon)\right\} &= -\partial_x\varepsilon^A-(-1)^{|D||C}\mathscr{A}_x^C\varepsilon^D f_{CD}{}^A=-\mathscr{D}_x\varepsilon^A=\delta_\varepsilon\mathscr{A}_A,\\ [\Phi_A(x),\mathscr{G}(\varepsilon)\} &= (-1)^{|A||C}\varepsilon^C\Phi_D f_{AC}{}^D=\{\varepsilon,\Psi]_A=\delta_\varepsilon\Phi_A,\end{aligned}$$

From these last expressions, we can see that indeed these constraints, $\mathscr{G}_A$, are the generators of gauge transformations.

[1] All the operation with brackets are given at the same time t.

5. Conclusion

We have studied classically the canonical formalism of the supersymmetric extension of the BF theory in 2D. We have shown that the model is theoretically consistent. The next step will be to quantize the theory, through the formalism of loops. In order to do this we have to construct the superholonomies, after that, the cylindrical functions and the Hilbert space [13].

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