

Definition of Instantons in Noncommutative Gauge Theory in Higher Dimensions

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Theories with noncommutative space-time coordinates represent alternative candidates of grand unified theories. We discuss $U(1)$ gauge theory in 2 and 4 dimensions on a lattice with N sites. The mapping to a $U(N)$ plaquette model in the sense of Eguchi and Kawai can be used for computer simulations. In 2D it turns out that the formulation of the topological charge leads to the imaginary part of the plaquette. Concerning 4D, the definition of instantons seems straightforward. We show that one can transcribe the plaquette and hypercube formulation to the matrix theory. It will be interesting to measure the topological charge on a noncommutative hypercube.

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1. Motivation

In noncommutative geometry, where the coordinate operators \hat{x}_μ satisfy the commutation relation $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$, a mixing between ultraviolet and infrared degrees of freedom takes place [1]. So lattice simulations are a promising tool to get deeper insight into noncommutative quantum field theories. For noncommutative U(1) gauge there exists an equivalent matrix model which makes numerical calculations feasible [2].

The main topic of the underlying contribution is to discuss the topological charge of noncommutative U(1) gauge theory in two and in higher dimensions. In two dimensions the instanton configurations carry a topological charge q which was shown being non-integer [3, 4]. We work out the definition of instantons in four and higher dimensions.

2. Topology and Instantons in QCD

The Lagrangian of pure gluodynamics (the Yang-Mills theory with no matter fields) in Euclidean spacetime can be written as

$$\mathcal{L} = \frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a \quad (2.1)$$

where $G_{\mu\nu}^a$ is the gluon field strength tensor

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \quad (2.2)$$

and f^{abc} are structure constants of the gauge group considered. The classical action of the Yang-Mills fields can be identically rewritten as

$$S = \frac{1}{8g^2} \int dx^4 (G_{\mu\nu}^a \pm \tilde{G}_{\mu\nu}^a)^2 \mp \frac{8\pi^2}{g^2} Q \quad (2.3)$$

where Q denotes the topological charge

$$Q = \frac{1}{32\pi^2} \int dx^4 G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a \quad (2.4)$$

with

$$\tilde{G}_{\mu\nu}^a = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}^a \quad (2.5)$$

3. Definition of the Topological Charge in Two Dimensions

3.1 Lattice Regularization of Noncommutative Two-Dimensional U(1) Gauge Theory

The lattice regularized version of the theory can be defined by an analog of Wilson's plaquette action

$$S = -\beta \sum_x \sum_{\mu < \nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^\dagger \star U_\nu(x)^\dagger + \text{c.c.} \quad (3.1)$$

where the symbol $\hat{\mu}$ represents a unit vector in the μ -direction and we have introduced the lattice spacing a . The link variables $U_\mu(x)$ are complex fields on the lattice satisfying the star-unity

condition. The star-product [1] on the lattice can be obtained by rewriting its definition within non-commutative derivatives in terms of Fourier modes and restricting the momenta to the Brillouin zone.

Let us define the topological charge for a gauge field configuration on the discretized two-dimensional torus. In the language of fields, we define the topological charge as

$$q = \frac{1}{4\pi i} \sum_x \sum_{\mu\nu} \varepsilon_{\mu\nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^\dagger \star U_\nu(x)^\dagger \quad (3.2)$$

which reduces to the usual definition of the topological charge in 2d gauge theory

$$q = \frac{1}{4\pi} \int d^2x \varepsilon_{\mu\nu} G_{\mu\nu} \quad (3.3)$$

in the continuum limit.

3.2 Matrix-Model Formulation

It is much more convenient for computer simulations to use an equivalent formulation, in which one maps functions on a noncommutative space to operators so that the star-product becomes nothing but the usual operator product, which is noncommutative. The action (3.1) can then be written as

$$\begin{aligned} S &= -N\beta \sum_{\mu \neq \nu} \text{tr} \left\{ \hat{U}_\mu (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) (\Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger) \hat{U}_\nu^\dagger \right\} + 2\beta N^2 \\ &= -N\beta \sum_{\mu \neq \nu} \mathcal{L}_{\nu\mu} \text{tr} \left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right) + 2\beta N^2 \end{aligned} \quad (3.4)$$

where $V_\mu \equiv \hat{U}_\mu \Gamma_\mu$ is a $U(N)$ matrix and N is the linear extent of the original lattice. An explicit representation of Γ_μ in the $d = 2$ case shall be given in Sec. 3.3. This is the twisted Eguchi-Kawai (TEK) model [5], which appeared in history as a matrix model equivalent to the large N gauge theory [6]. We have added the constant term $2\beta N^2$ to what we would obtain from (3.1) in order to make the absolute minimum of the action zero.

By using the map between fields and matrices, the topological charge (3.2) can be represented in terms of matrices as

$$\begin{aligned} q &= \frac{1}{4\pi i} N \sum_{\mu\nu} \varepsilon_{\mu\nu} \text{tr} \left\{ \hat{U}_\mu (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) (\Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger) \hat{U}_\nu^\dagger \right\} \\ &= \frac{1}{4\pi i} N \sum_{\mu\nu} \varepsilon_{\mu\nu} \mathcal{L}_{\nu\mu} \text{tr} \left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right) \end{aligned} \quad (3.5)$$

3.3 Classical Solutions

The classical equation of motion was worked out in the literature [7, 3] for the action (3.4)

$$V_\mu^\dagger (W - W^\dagger) V_\mu = W - W^\dagger \quad (3.6)$$

with the unitary matrix W

$$W = \mathcal{L}_{\nu\mu} V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \quad (3.7)$$

General solutions to this equation can be written in a block-diagonal form [7]

$$V_\mu = \begin{pmatrix} \Gamma_\mu^{(1)} & & & \\ & \Gamma_\mu^{(2)} & & \\ & & \ddots & \\ & & & \Gamma_\mu^{(k)} \end{pmatrix} \quad (3.8)$$

by an appropriate $SU(N)$ transformation, where $\Gamma_\mu^{(j)}$ are $n_j \times n_j$ unitary matrices, $j = 1, \dots, k$, satisfying the 't Hooft-Weyl algebra

$$\Gamma_\mu^{(j)} \Gamma_\nu^{(j)} = Z_{\mu\nu}^{(j)} \Gamma_\nu^{(j)} \Gamma_\mu^{(j)} \quad (3.9)$$

$$Z_{12}^{(j)} = Z_{21}^{(j)*} = \exp\left(2\pi i \frac{m_j}{n_j}\right) \quad (3.10)$$

$$m_j = \frac{n_j + 1}{2} \quad (3.11)$$

An explicit representation is given by the clock and shift operators, Q and P

$$\Gamma_1^{(j)} = P_{n_j}, \quad \Gamma_2^{(j)} = (Q_{n_j})^{m_j} \quad (3.12)$$

The action and the topological charge can then be evaluated as [3, 7]

$$S = 4N\beta \sum_j n_j \sin^2 \left\{ \pi \left(\frac{m_j}{n_j} - \frac{M}{N} \right) \right\} \quad (3.13)$$

$$q = \frac{N}{2\pi} \sum_j n_j \sin \left\{ 2\pi \left(\frac{m_j}{n_j} - \frac{M}{N} \right) \right\} \quad (3.14)$$

In general, the topological charge q is not an integer. If we require the action to be less than of order N the argument of the sine has to vanish for all j . In that case the topological charge approaches an integer

$$q \simeq N \left(\sum_j m_j - M \right) \quad (3.15)$$

being a multiple of N .

4. Definition of the Topological Charge in Four and Higher Dimensions

The lattice action (3.1) taking into account the star-product can be used in any dimension. The field-theoretic definition of the topological charge (3.2) can be extended in two ways.

One can rely on the so-called plaquette definition which then yields a product of two plaquettes

$$q^{(P)} = \frac{-1}{32\pi^2} \sum_x \sum_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^\dagger \star U_\nu(x)^\dagger \star U_\rho(x) \star U_\sigma(x + a\hat{\rho}) \star U_\rho(x + a\hat{\sigma})^\dagger \star U_\sigma(x)^\dagger \quad (4.1)$$

which reduces to the definition of the topological charge in 4d gauge theory

$$q^{(P)} = \frac{1}{32\pi^2} \int d^4x \varepsilon_{\mu\nu\rho\sigma} G_{\mu\nu} \star G_{\rho\sigma} \quad (4.2)$$

in the continuum limit.

By using the map between fields and matrices, the topological charge (4.1) can be represented in terms of matrices as

$$\begin{aligned} q^{(P)} &= \frac{-1}{32\pi^2} N \sum_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} \text{tr} \left\{ \hat{U}_\mu (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) (\Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger) \hat{U}_\nu^\dagger \right\} \left\{ \hat{U}_\rho (\Gamma_\rho \hat{U}_\sigma \Gamma_\rho^\dagger) (\Gamma_\sigma \hat{U}_\rho^\dagger \Gamma_\sigma^\dagger) \hat{U}_\sigma^\dagger \right\} \\ &= \frac{-1}{32\pi^2} N \sum_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} \mathcal{L}_{\nu\mu} \mathcal{L}_{\rho\sigma} \text{tr} \left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger V_\rho V_\sigma V_\rho^\dagger V_\sigma^\dagger \right) \end{aligned} \quad (4.3)$$

Alternatively, one can rely on the so-called hypercube definition which leads to a star-product of matrices winding along the edges of the hypercube

$$\begin{aligned} q^{(H)} &= \frac{-1}{32\pi^2} \sum_x \sum_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\rho(x + a\hat{\mu} + a\hat{\nu}) \star U_\sigma(x + a\hat{\mu} + a\hat{\nu} + a\hat{\rho}) \\ &\quad \star U_\mu(x + a\hat{\nu} + a\hat{\rho} + a\hat{\sigma})^\dagger \star U_\nu(x + a\hat{\rho} + a\hat{\sigma})^\dagger \star U_\rho(x + a\hat{\sigma})^\dagger \star U_\sigma(x)^\dagger \end{aligned} \quad (4.4)$$

which reduces to the definition of the topological charge in 4d gauge theory

$$q^{(H)} = \frac{1}{32\pi^2} \int d^4x \varepsilon_{\mu\nu\rho\sigma} G_{\mu\nu} \star G_{\rho\sigma} \quad (4.5)$$

in the continuum limit.

By using the map between fields and matrices, the topological charge (4.1) can be represented in terms of matrices as

$$\begin{aligned} q^{(H)} &= \frac{-1}{32\pi^2} N \sum_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} \text{tr} \left\{ \hat{U}_\mu (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) (\Gamma_\nu \Gamma_\mu \hat{U}_\rho \Gamma_\mu^\dagger \Gamma_\nu^\dagger) (\Gamma_\rho \Gamma_\nu \Gamma_\mu \hat{U}_\sigma \Gamma_\mu^\dagger \Gamma_\nu^\dagger) \Gamma_\rho^\dagger \right. \\ &\quad \left. (\Gamma_\sigma \Gamma_\rho \Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger \Gamma_\rho^\dagger) \Gamma_\sigma^\dagger (\Gamma_\sigma \Gamma_\rho \hat{U}_\nu^\dagger \Gamma_\rho^\dagger \Gamma_\sigma^\dagger) (\Gamma_\sigma \hat{U}_\rho^\dagger \Gamma_\sigma^\dagger) \hat{U}_\sigma^\dagger \right\} \\ &= \frac{-1}{32\pi^2} N \sum_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} \mathcal{L}_{\nu\mu} \mathcal{L}_{\rho\nu} \mathcal{L}_{\sigma\rho} \mathcal{L}_{\rho\mu} \mathcal{L}_{\sigma\nu} \mathcal{L}_{\sigma\mu} \text{tr} \left(V_\mu V_\nu V_\rho V_\sigma V_\mu^\dagger V_\nu^\dagger V_\rho^\dagger V_\sigma^\dagger \right) \end{aligned} \quad (4.6)$$

The extension to higher dimensions is straight-forward. In practical studies, one can choose one or more planes noncommutative while leaving the others commutative [8].

5. Conclusion and Outlook

Today there exist several investigations of the topological sector of the two-dimensional non-commutative U(1) theory [3, 4]. Also classical solutions are available. The situation with the field-theoretic definition of instantons is reminiscent of lattice QCD where quantum gauge field configurations are topologically trivial and one needs to apply some smoothing procedure onto the gauge fields to unhide instantons.

In this contribution we worked out the field-theoretic definition to four and higher dimensions. We demonstrated that both the plaquette and hypercube definition can be taken over from the commutative gauge theory by respecting the star-multiplication and applying the map to the matrix model.

It would be interesting to adapt cooling techniques from QCD to the four-dimensional non-commutative U(1) theory [8]. At present we are working on this. It would be desirable to send the

noncommutativity parameter θ of the four-dimensional noncommutative gauge theory to zero in order to obtain a realistic comparison of its topological content with the well-studied topological objects like instantons and monopoles in QCD.

Unfortunately, the transcription of a monopole observable seems to be difficult. The analogy to commutative $U(1)$ theory of summing up the phases of the field over an elementary cube does not obviously transfer to the $U(N)$ theory in the matrix model. Finding a reasonable definition one could be able to measure the monopole number on a noncommutative hypercube.

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