PROCEEDINGS OF SCIENCE

# A Linear Collider as Top Factory

**Adrian Signer***
*IPPP Durham*
*E-mail:* adrian.signer@durham.ac.uk

The main aim of this lecture is to explain how to obtain reliable theoretical predictions for a $t\bar{t}$ threshold scan at a future linear collider. After explaining the basic ideas in some detail, a short overview over the current status of the theory is given, followed by a glance at physics related to $t\bar{t}$ production in a wider context.

*Third Linear Collider Physics School 2009 - LCPS2009*
*August 17 - 23 2009*
*Ambleside, UK*

*Speaker.

 

## 1. Introduction

The physics programme of a future linear collider (LC) with centre-of-mass energy of several hundred GeV is extremely rich. Among the large number of vitally important measurements that could be done at such a collider, this article focusses on one particular one, the $t\bar{t}$ threshold scan [1]. In the same way that critical information about the charm and bottom quarks has been obtained by considering the process $e^+e^- \to$ hadrons at $\sqrt{s} \simeq 2m_c$ and $\sqrt{s} \simeq 2m_b$ respectively, measurements of the top quark mass $m$ and its width $\Gamma$ with unprecedented accuracy could be made using the process $e^+e^- \to t\bar{t}$ at $\sqrt{s} \simeq 2m$. The reason why the threshold region is particularly interesting is the following: slightly below threshold, i.e. for $\sqrt{s} < 2m$, there is not enough energy to create a top quark pair, whereas above threshold, i.e. for $\sqrt{s} > 2m$, a top quark pair can be created. Thus, in the threshold region the cross section for producing a top pair varies very strongly. With precise theoretical input matched to a precise measurement it is therefore possible to extract information about the top quark that cannot be obtained by any other means.

Because the top quarks plays a special role in most extensions of the Standard Model, a precise determination of its parameters are particularly important. We will first discuss how to determine $m$ and $\Gamma$ from a threshold scan. Given that this is a lecture for a summer[1] school, I will spend most of my time explaining in quite some detail the basic idea, followed by only a short overview of the current status of the theory and no attempt is made to give a complete list of references. After the discussion of the threshold scan I also briefly mention a possible measurement of the Yukawa coupling of the top. Finally we conclude by making a quick comparison to top quark pair production at the LHC.

## 2. Top threshold scan

The quantity we will mainly be concerned with is the $R$ ratio, which is nothing but the cross section $\sigma(e^+e^- \to t\bar{t})$ normalized for convenience by $\sigma(e^+e^- \to \mu^+\mu^-)$. To compute $R$ naively at leading order, we have to calculate a single $s$-channel diagram $e^+e^- \to \gamma^* \to t\bar{t}$, square it and integrate the result over the two-body phase space. We find

$$R^{(0)} = \frac{3}{2} N_c e_t^2 \beta \left(1 - \frac{\beta^2}{3}\right) \tag{2.1}$$

where $e_t = 2/3$ is the electric charge of the top, $N_c = 3$ is a colour factor and the velocity of the top is given by $\beta = \sqrt{1 - 4m^2/s}$. Of course, there is also the process where the photon is replaced by a $Z$ boson. However, since we are considering the $R$ ratio at leading order, this does not change Eq. (2.1). The process where the photon is replaced by a Higgs does not contribute because we consider the electrons to be massless. Hence they do not couple to the Higgs boson.

Considering Eq. (2.1) we note that near threshold the term $\beta^2/3 \ll 1$, since $\beta \to 0$ in the limit $\sqrt{s} \to 2m$. Hence near threshold this term can be neglected. While it seems pointless to do this, these simplifications become crucial when going to higher orders. The corresponding technical simplifications will allow us to resum a certain class of contributions to all orders in the strong coupling $\alpha_s$. But we have to keep in mind that (far) above threshold $\beta \to 1$ and the second term

[1] Or whatever goes under the name "summer" in the Lake District.

in Eq. (2.1) is as important as the first. Finally we note that an alternative way to obtain Eq. (2.1) is to consider the one-loop forward scattering amplitude $e^+e^- \to \gamma^* \to t\bar{t} \to \gamma^* \to e^+e^-$. Ignoring the trivial electroweak part $e^+e^- \to \gamma^*$ this is nothing but the photon vacuum polarization with a top quark loop. According to the optical theorem, the imaginary part of the forward scattering amplitude equals the total cross section up to a trivial prefactor.

The usual procedure for improving the theoretical result is to compute higher order corrections. At next-to-leading order (NLO) in $\alpha_s$ we either have to add the separately divergent virtual corrections for $\gamma^* \to t\bar{t}$ and real corrections $\gamma^* \to t\bar{t}g$, or compute the forward scattering amplitude at $\mathscr{O}(\alpha_s)$. This has been done a long time ago [2]. The precise form of the result is not important for us, but schematically it reads

$$R^{(1)} = R^{(0)} \left(1 + \alpha_s \left[\sim \frac{1}{\beta} + \mathrm{cst} + \beta^2 \log\beta + \ldots\right]\right) \tag{2.2}$$

What is important is that at NLO we would find corrections that behave as $\alpha_s/\beta$. The problem with these terms is that in the region $\beta \to 0$ they are not small compared to 1, i.e. the NLO terms are not small compared to the LO terms. If we were to compute even one order higher, i.e. at $\mathscr{O}(\alpha_s^2)$ we would find corrections of the form $\alpha_s^2/\beta^2$ which cause the same problem. On top of this, we would also find terms of the form $\alpha_s^2 \log\beta$. While these terms are not quite as bad they can still be dangerous in the sense that for small $\beta$ the large logarithm $\log\beta$ can compensate the small coupling $\alpha_s$. In Eq. (2.2) the corresponding term is harmless as it appears with a factor $\beta^2$ but beyond NLO potentially large logarithms are present. To summarize, a strict order by order in $\alpha_s$ approach is fine if we are (far) above threshold, i.e. $\beta \simeq 1$, but in the threshold region such an approach completely fails.

To get a meaningful theoretical result in the threshold region we have to reorganize the perturbative series and perform a double expansion. That is we do not only expand in the small coupling $\alpha_s$ but also in the small $\beta$ and for the purpose of counting the order we set $\alpha_s \simeq \beta \ll 1$. The perturbative series then looks like

$$R = \beta \sum_n \left(\frac{\alpha_s}{\beta}\right)^n \times \left(\underbrace{1}_{\mathrm{LO}} + [\sim \underbrace{\alpha_s,\ \beta}_{\mathrm{NLO}}] + [\sim \underbrace{\alpha_s^2,\ \beta^2,\ \alpha_s\beta}_{\mathrm{NNLO}}] + \ldots\right) \tag{2.3}$$

In this context the LO result takes into account the $\mathscr{O}(\alpha_s^0\beta^1)$ term of Eq. (2.1), the $\mathscr{O}(\alpha_s^1\beta^0)$ term of Eq. (2.2) as well as all terms of the form $\mathscr{O}(\alpha_s^n\beta^{1-n})$. NLO terms are suppressed w.r.t. the LO result by one small factor, either $\alpha_s$ or $\beta$. Correspondingly, NNLO terms are suppressed by any combination of two small factors and so on and from now on we always mean this when referring to the order of the perturbative expansion.

For a LO computation in the threshold region we have to sum infinitely many terms. At each order in $\alpha_s$ we have to take the leading term in $1/\beta$. To do this in a systematic way we have to use an effective theory approach. A detailed discussion of such an approach is well beyond the scope of this lecture (see e.g. Ref. [3]) but I will try to give an idea how this works in what follows. The main idea is to exploit the hierarchy of scales. In the threshold region, the top quarks are non-relativistic, i.e. their kinetic energy $E$ is much smaller than their mass. Thus we have $E \sim m\beta^2 \ll \vec{p} \sim m\beta \ll m$, where $\vec{p}$ is the momentum of the top. Let us now consider how this can be used to study the interaction of two heavy non-relativistic quarks.

From a quantum field theory point of view the quarks interact by exchanging gluons. Using Feynman rules for the corresponding diagram at $\mathscr{O}(\alpha_s)$ shown in panel $(a)$ of Figure 1 we obtain

$$-i\,\mathscr{M}^{(0)} = 4\pi\alpha_s\,\mathscr{C}\,\bar{u}(p-q)\gamma^\mu u(p)\otimes\bar{v}(p')\gamma^\nu v(p'+q)\;\frac{-i}{q^2}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right) \tag{2.4}$$

where $\xi$ is the gauge parameter, $q$ is the exchanged momentum and $\mathscr{C}$ is a colour factor. If the quarks are in a colour singlet state (e.g. created from a photon) we have $\mathscr{C} = C_F = 4/3$ whereas in an colour octet state we have $\mathscr{C} = C_F - N_c/2 = -1/6$. In the centre-of-mass frame of the quark pair the momenta of the quarks are given by $p = (m+E,\vec{p})$ and $p' = (m+E',-\vec{p})$ with $\vec{p}\sim\vec{p}\,'\sim m\beta$ and $E\sim E'\sim m\beta^2$. To ensure that the quarks are still non-relativistic after the exchange of the gluon with momentum $q$ we need $q^0\sim m\beta^2$ and $\vec{q}\sim m\beta$. A field with momentum that scales in such a way is called a potential field. Using all the information we have we can simplify the expression given in Eq. (2.4). First we note that the $\xi$-dependent terms vanish due to the Dirac equation. Secondly we use $q^2\sim-\vec{q}^{\,2}$ because $q^0\ll\vec{q}$. Finally we note that $\bar{u}\gamma^0 u\sim 1+\mathscr{O}(1/m^2)$ whereas $\bar{u}\vec{\gamma}u\sim\vec{p}/m\sim\beta$ (and similarly for $v$ spinors). Thus the leading part of the interaction does not know anything about the spin of the quarks. This is familiar from the hydrogen atom where the leading interaction is spin independent. Putting everything together Eq. (2.4) simplifies to

$$\mathscr{M}^{(0)} = 1\otimes 1\;\frac{-4\pi\alpha_s\,\mathscr{C}}{\vec{q}^{\,2}} \tag{2.5}$$

Thus in the colour singlet case the leading term from the exchange of a single potential gluon results in an interaction term $-4\pi\alpha_s C_F/\vec{q}^{\,2}$ whose Fourier transform is $-\alpha_s C_F/r$ with $r=|\vec{r}|$. This is the QCD version of the Coulomb potential (hence the name potential gluon).

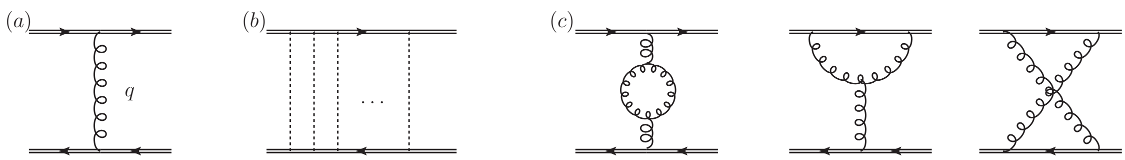


**Figure 1:** (a) exchange of a single gluon between non-relativistic quarks; (b) ladder diagram with multiple exchange of potential gluons, corresponding to LO potential of non-relativistic quarks; (c) some of the diagrams contributing to the NLO correction to the potential of non-relativistic quarks.

To understand what happens if several potential gluons are exchanged it is convenient to go to coordinate space. In fact we can now write down the Scrödinger equation

$$\left(-\frac{\vec{\nabla}^2}{m}-\frac{C_F\,\alpha_s}{r}-E-i\Gamma\right)G_c(\vec{r};E) = \delta^{(3)}(\vec{r}) \tag{2.6}$$

for the Coulomb Green function $G_c$ which includes the Coulomb potential to all orders. We also take into account the width of the top quark. Using our counting rules we note that all terms are of the same order: $\vec{\nabla}^2/m\sim\vec{p}^{\,2}/m\sim\beta^2$, $\alpha_s/r\sim\alpha_s\vec{p}\sim\alpha_s\beta$, $E\sim\beta^2$ and counting the electroweak coupling[2] as $\alpha\sim\alpha_s^2$ we also have $\Gamma\sim\alpha m\sim\beta^2$. Thus we cannot treat the exchange of potential

[2]This counting is motivated simply from a numerical point of view.

gluons in a perturbative way. We have to solve Eq. (2.6) exactly, taking into account multiple potential gluon exchange. In fact this resums the terms of order $\mathscr{O}(\alpha_s^n\beta^{1-n})$ mentioned before and corresponds to the sum of diagrams shown in panel $(b)$ of Figure 1.

The solution to Eq. (2.6) is essentially the well-known solution for the hydrogen atom and can for example be written as

$$G_c(\vec{r};E) = \frac{m\sqrt{-m\bar{E}}}{2\pi} e^{-r\sqrt{-m\bar{E}}} \int_0^\infty d\tau\, e^{-2r\sqrt{-m\bar{E}}\,\tau} \left(\frac{1+\tau}{\tau}\right)^\lambda \tag{2.7}$$

with $\lambda \equiv C_F\alpha_s/(2\sqrt{-\bar{E}/m})$ and $\bar{E} \equiv E + i\Gamma$. However, we need $G_c$ for $\vec{r} = 0$, since the imaginary part of $G_c(0;E)$ via the optical theorem is related to the total cross section. But at $\vec{r} = 0$ the Coulomb Green function has an ultraviolet singularity. Indeed, after setting $\vec{r} \to 0$ in Eq. (2.7) the $\tau$ integration results in a divergence. In order to deal with this we have to work in momentum space. Fortunately the ultraviolet singularity is restricted to the first two terms of the expansion of $G_c$ in $\alpha_s$. Therefore we can compute these terms in momentum space and combine the result with the well-known result for $G_c$ for all higher orders in $\alpha_s$. The result is

$$G_c(0;E) = -\frac{\alpha_s C_F\, m^2}{4\pi}\left(-\frac{1}{4\bar{\varepsilon}} + \frac{1}{2\lambda} + \frac{1}{2}\log\frac{-4m\bar{E}}{\mu^2} - \frac{1}{2} + \gamma_E + \psi(1-\lambda)\right) \tag{2.8}$$

where $1/\bar{\varepsilon} = 1/\varepsilon + \log(4\pi) - \gamma_E$ is the dimensionally regulated ultraviolet singularity with the corresponding scale $\mu$. The cross section is proportional to the imaginary part of $G_c$

$$R = \frac{6\pi\, e_t^2\, N_c}{m^2}\, \mathrm{Im}\, G_c(0,E) \tag{2.9}$$

and thus is not affected by the ultraviolet singularity. However it contains terms to all orders in $\alpha_s$ with $\alpha_s$ always appearing in the combination $\lambda \sim \alpha_s/\beta$. The terms $\mathscr{O}(\alpha_s^n\beta^{1-n})$ have been resummed. Note also that even strictly at threshold, $E = 0$ the result is well defined. This is due to the width of the top, but would actually even hold if the width was negligible as is the case for the bottom quark.

If we were to plot Eq. (2.9) as a function of $E$ we would obtain a result similar to the LO curves in Figure 2. Below threshold the cross section rapidly vanishes. Around threshold there is a peak due to the fact that the two top quarks try to form a bound state (similar to the $\Upsilon$ in the bottom case). However, due to the large width the tops decay before they can form a bound state. If we decrease $\Gamma$, the peak becomes higher and thinner and in the limit $\Gamma \to 0$ we would find delta peaks below threshold, corresponding to bound states. This observation leads directly to a simple method for a precise determination of $m$ and $\Gamma$, roughly speaking by measuring the position and the width of the peak. Of course, in a real experiment the curves would look rather different due to bremsstrahlung and other effects. However, a detailed analysis shows that a precise measurement of $m$ with an error of $\delta m \sim 20 - 50$ MeV and $\Gamma$ with an error of $\delta\Gamma \sim 30$ MeV is still possible [4].

With such a high precision in $m$ great care has to be taken to use a suitable definition of the mass. In particular, to achieve $\delta m \lesssim \Lambda_{\mathrm{QCD}} \simeq 200$ MeV we have to abandon the pole mass. The reason is that the pole mass has an ambiguity of order $\Lambda_{\mathrm{QCD}}$ in perturbation theory. This can be understood by considering a quark-antiquark meson. The mass of the meson is given by twice the pole mass of the quark minus the binding energy. Being a physical quantity, the mass of the meson is well defined. However, due to non-perturbative effects, the binding energy has an ambiguity $\sim \Lambda_{\rm QCD}$ in perturbation theory. This entails that the pole mass must have the same ambiguity such that the effect cancels for the meson mass. Therefore we have to use a so-called short-distance threshold mass. The word short-distance refers to the absence of non-perturbative ambiguities and there are several possible definitions [5].

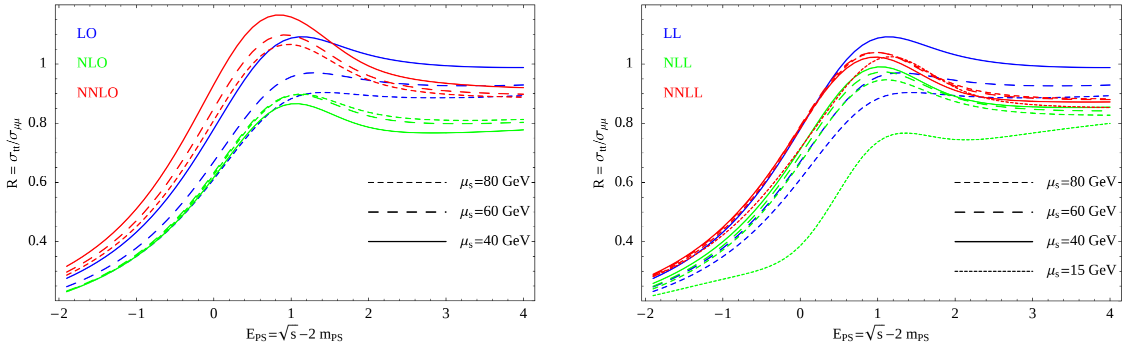


**Figure 2:** Threshold scan for top pair production near threshold up to NNLO, without (left panel) and with (right panel) resummation of $\log\beta$ terms. The results and plots are taken from Ref. [7].

The result Eq. (2.8) is of course not the full story but corresponds only to the LO result according to Eq. (2.3). To go beyond we have to take into account higher-order corrections to the Coulomb potential as well as corrections higher order in $\beta$. The latter will contain relativistic corrections and spin dependent corrections familiar from the hydrogen atom. The potentials can be computed in momentum space, in principle to any order, by computing Feynman diagrams. Some diagrams contributing at NLO are shown in panel $(c)$ of Figure 1. In general divergences are encountered that are regulated using dimensional regularization. Once the potential is known, standard quantum mechanics perturbation theory can be used to compute higher-order corrections to $G_c$. This program has been carried out up to NNNLO [8]. In the left panel of Figure 2 we show the results at LO, NLO and NNLO for various scale choices. The variation of the result with the scale $\mu$ is an indication on how reliable the theoretical prediction is. The scale dependence is still rather large at NNLO, indicating large corrections and a rather poor convergence. This poor convergence can partly be understood by recalling the presence of potentially large $\log\beta$ terms. In fact these terms can be resummed as well [6, 7] and this improves the convergence substantially, as shown in the right panel of Figure 2. On the other hand there are also large non-logarithmic corrections and a full exploitation of future experimental data would require additional theoretical work. In this context it should also be mentioned that the electroweak decay of the top is not properly taken into account in the current calculations.

## 3. Top-Higgs Yukawa coupling

If the mass of fermions is generated through the Higgs mechanism there is a close relation between the mass of a fermion and the strength of its coupling to the Higgs. Because the top has a Yukawa coupling that is very close to its natural value $y_t \sim 1$, the top is the prime candidate for this test. Ideally we would measure the top-Higgs Yukawa coupling with the same precision as the mass. Unfortunately such a precision is well beyond what can be achieved, but there are (at least) two ways to measure $y_t$ at a LC.

The first uses once more the threshold scan. The potential between the top quarks also receives a contribution from the exchange of a Higgs boson $\delta V = -y_t^2 e^{-m_H r}/((4\pi)^2 r)$. This potential is much smaller than the usual QCD potential, but since the latter has been taken into account to a very high precision, a very precise experimental determination of the cross section in the threshold region could be used to determine $y_t$. However, this measurement hinges on a careful assessment of other (small) contributions of similar size and theoretical uncertainties. At this moment it is difficult to estimate to what precision a measurement could be made, but in the literature a precision of 35% for $y_t$ has been mentioned [4].

Another option is to measure $y_t$ more directly by considering $t\bar{t}H$ production. This is similar to what is done at a hadron collider. The cross section is obviously proportional to $y_t^2$ and if it can be measured precisely enough it leads to a determination of $y_t$. The advantage of using a LC rather than the LHC is that potentially a better precision can be obtained. On the other hand the process requires a large centre-of-mass energy $\sqrt{s}$ of the collider which is not much of an issue at the LHC. The precision with which $y_t$ can be measured at a LC crucially depends on the Higgs mass and $\sqrt{s}$. As an example, if the Higgs mass is in the expected range 120 GeV $< m_H <$ 170 GeV at a $\sqrt{s} = 800$ GeV linear collider $y_t$ can be measured with a precision better than 10% [9].

## 4. Top pairs at LHC

At the LHC the main production mechanism for top pair production is through the process $gg \to t\bar{t}$ with $q\bar{q} \to t\bar{t}$ playing a smaller role. The top quark pairs a produced either in a colour singlet or in a colour octet state. Computing $\hat{\sigma}^{(1)}_{t\bar{t}}$, the total partonic cross section to NLO in $\alpha_s$ we schematically find [10]

$$\hat{\sigma}^{(1)}_{tt} = \hat{\sigma}^{(0)}_{tt}\left(1+\alpha_s\left[\sim\frac{1}{\beta} + \sim\log^2\beta + \sim\log\beta \ + \mathrm{cst}\right]\right) \tag{4.1}$$

which has very similar features as Eq. (2.2). The potentially large logarithms have been resummed quite a while ago [11] but, somewhat surprisingly, the $\alpha_s/\beta$ terms have been resummed only recently [12]. The results are shown in Figure 3. For the colour singlet component, the result is very similar to the results obtained with $e^+e^-$ in the initial state. The colour octet contribution however behaves differently. Because the colour factor has a different sign, the colour octet potential is not attractive and therefore cannot give rise to a bound state. In the top case this means there is no bump from a would-be bound state. Very close to threshold the colour singlet is dominant, but soon the octet contribution becomes more important.

At a hadronic collider the partonic cross section $\sigma_{t\bar{t}}$ has to be folded with parton distribution functions. Furthermore it is very difficult to measure the invariant mass of the top pairs to a high precision. As a result it is very unlikely that the top width can be measured, despite the nice peak in the total partonic cross section. However, the top quark mass can of course be measuread at the LHC and in fact has been measured at the Tevatron. The main method to do this is by considering the invariant mass of the decay products of the top. Another option, more similar to what would be done at a LC, is to measure the cross section very precisely and use its dependence on $m$ for

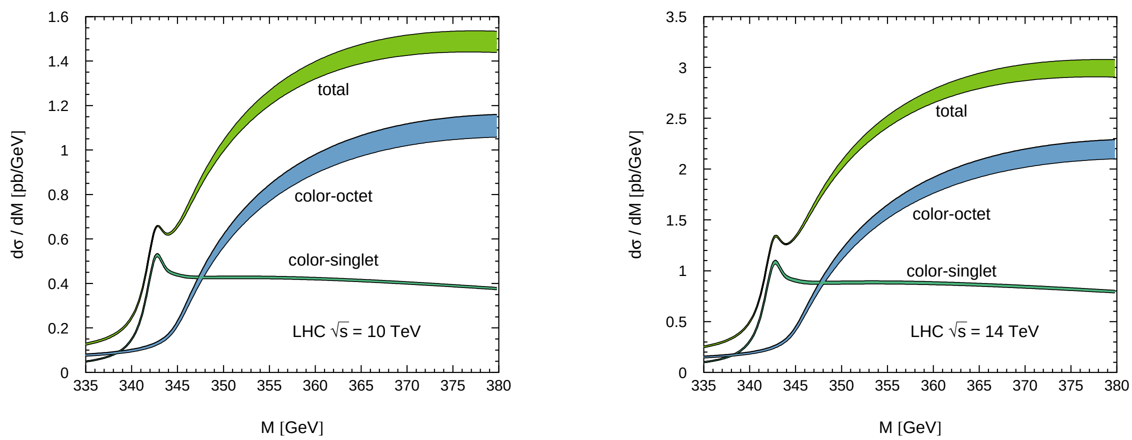


**Figure 3:** Invariant mass distribution for the colour singlet and octet partonic cross section at the LHC with $\sqrt{s} = 10$ TeV (left panel) and $\sqrt{s} = 14$ TeV (right panel). The results and plots are taken from Ref. [12].

a determination of the mass. However, both methods have – by the standards of a linear collider measurement – relatively large inherent uncertainties and it is very difficult to see how at a hadron collider a measurement of $m$ with a precision better than $\delta m \sim \Gamma \sim 1.5$ GeV can be made.

## References


[1] V. S. Fadin and V. A. Khoze, JETP Lett. **46**, 525 (1987) [Pisma Zh. Eksp. Teor. Fiz. **46**, 417 (1987)]; V. S. Fadin and V. A. Khoze, Sov. J. Nucl. Phys. **48**, 309 (1988) [Yad. Fiz. **48**, 487 (1988)].

[2] A. O. G. Kallen and A. Sabry, Kong. Dan. Vid. Sel. Mat. Fys. Med. **29N17** (1955) 1.

[3] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. **77** (2005) 1423 [arXiv:hep-ph/0410047].

[4] M. Martinez and R. Miquel, Eur. Phys. J. C **27** (2003) 49 [arXiv:hep-ph/0207315].

[5] I. I. Y. Bigi, M. A. Shifman and N. Uraltsev, Ann. Rev. Nucl. Part. Sci. **47**, 591 (1997) [arXiv:hep-ph/9703290];
M. Beneke, Phys. Lett. B **434**, 115 (1998) [arXiv:hep-ph/9804241];
A. H. Hoang, Z. Ligeti and A. V. Manohar, Phys. Rev. Lett. **82**, 277 (1999) [arXiv:hep-ph/9809423].

[6] A. H. Hoang, A. V. Manohar, I. W. Stewart and T. Teubner, Phys. Rev. Lett. **86**, 1951 (2001) [arXiv:hep-ph/0011254];

[7] A. Pineda and A. Signer, Nucl. Phys. B **762** (2007) 67 [arXiv:hep-ph/0607239].

[8] M. Beneke, Y. Kiyo and K. Schuller, PoS **RADCOR2007** (2007) 051 [arXiv:0801.3464 [hep-ph]].

[9] A. Gay, Eur. Phys. J. C **49** (2007) 489 [arXiv:hep-ph/0604034].

[10] P. Nason, S. Dawson and R. K. Ellis, Nucl. Phys. B **303** (1988) 607.

[11] R. Bonciani, S. Catani, M. L. Mangano and P. Nason, Nucl. Phys. B **529** (1998) 424 [Erratum-ibid. B **803** (2008) 234] [arXiv:hep-ph/9801375].

[12] Y. Kiyo, J. H. Kuhn, S. Moch, M. Steinhauser and P. Uwer, Eur. Phys. J. C **60** (2009) 375 [arXiv:0812.0919 [hep-ph]].