PROCEEDINGS OF SCIENCE

A recursive approach to the reduction of tensor Feynman integrals

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We describe a new, convenient, recursive tensor integral reduction scheme for one-loop n-point Feynman integrals. The reduction is based on the algebraic Davydychev-Tarasov formalism where the tensors are represented by scalars with shifted dimensions and indices, and then expressed by conventional scalars with generalized recurrence relations. The scheme is worked out explicitly for up to $n = 6$ external legs and for tensor ranks $R \leq n$. The tensors are represented by scalar one- to four-point functions in d dimensions. For the evaluation of them, the Fortran code for the tensor reductions has to be linked with a package like QCDloop or LoopTools/FF. Typical numerical results are presented.

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1. Introduction

n-point integrals I_n^R in loop momentum space of tensor rank R appear in any realistic evaluation of Feynman diagrams. There are several ways to calculate them, and one approach expresses them by a small set of scalar integrals. The first systematic treatment, in a Standard Model calculation, is known as the Passarino-Veltman reduction [1] and expresses four-point tensor integrals (and simpler ones) algebraically by scalar one- to four-point functions. Two of us made use of this scheme in the early days [2, 3, 4]. Nowadays tensor reductions became again a topic of research because at LEP2, LHC and ILC the interesting final states typically consist of more than two particles, some of them being massive. Several dedicated tensor reduction packages have been developed for the calculation of five- and six-point functions. As open-source packages we like to mention the Fortran packages LoopTools/FF [5, 6] (covering I_n^R with $n \leq 5, R \leq 4$) and Golem95 [7] (covering I_n^R with $n \leq 6$ and massless propagators) and the Mathematica package hexagon.m [8, 9] (covering I_n^R with $(n,R) \leq (6,4),(5,3)$).

Here, we describe a recursive implementation for tensor functions I_n^R with $n \leq 6, R \leq n$ for arbitrary internal masses. We use the Davydychev-Tarasov approach where the tensor integrals are first expressed by scalar integrals with higher dimensions and indices [10]. In a second step, the scalar integrals may be expressed algebraically by scalar one- to four-point functions [11] quite similar to the Passarino-Veltman reduction. In fact, when using the same basis, the approaches are equivalent. A difference, though, may arise in the algorithmic realization, and as a consequence in the numerical stability and speed of an implementation. For more comments on the differences of tensor reduction schemes we refer to the literature quoted. In a recent paper [12], we introduced a convenient and easy-to-program version of the reduction a la Davydychev-Tarasov, which allows a recursive determination of a chain of tensors. Here we will describe that scheme and present some numerical results. The integrals to be evaluated are:

$$I_n^{\mu_1\cdots\mu_R} = C(\varepsilon) \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}}, \tag{1.1}$$

where the denominators c_j have *indices* ν_j and *chords* q_j:

$$c_j = (k-q_j)^2 - m_j^2 + i\varepsilon. \tag{1.2}$$

The normalization $C(\varepsilon)$ plays a role for divergent integrals only and is conventional:

$$C(\varepsilon) = (\mu)^{2\varepsilon} \frac{\Gamma(1-2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}. \tag{1.3}$$

Here, we use $d = 4-2\varepsilon$ and $\mu = 1$. For the evaluation of the scalar functions we will rely on either LoopTools or QCDloops/FF [13, 6], and the latter one uses also $C(\varepsilon)$ as defined here.

2. Recursions

Our recursions begin with six-point functions where the well known formula [14, 15, 16, 9] may be used:

$$I_6^{\mu_1\ldots\mu_{R-1}\rho} = -\sum_{s=1}^{6} I_5^{\mu_1\ldots\mu_{R-1},s} \bar{Q}_s^{\rho}. \tag{2.1}$$

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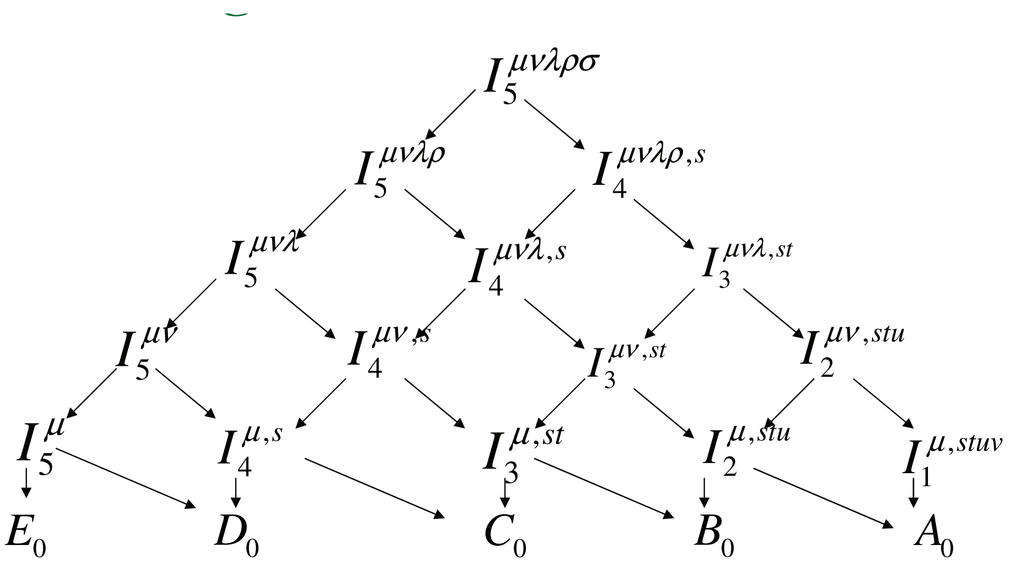

Figure 1: The recursion triangle.

The auxiliary vectors $\bar{Q}_s^\rho$ read:

$$\bar{Q}_s^\rho = \sum_{i=1}^{6} q_i^\rho \frac{\binom{0s}{0i}_6}{\binom{0}{0}_6} \quad , \quad s = 1 \ldots 6. \tag{2.2}$$

The $I_{n-1}^{\{\mu_1,\cdots\},s}$ is obtained from $I_n^{\{\mu_1,\cdots\}}$ by shrinking line s, and the $\binom{i,j,\cdots}{k,l,\cdots}_n$ are signed minors of the modified Cayley determinant $\left(\right)_n$ [17]. For further details of notations we refer to [12].

The further calculational chain may be read off from figure 1. Its basic idea is to represent an n-point tensor I_n^R by an n-point tensor of lower rank I_n^{R-1} and by all the $(n-1)$-point tensors of lower rank I_{n-1}^{R-1}. [1] For 5-point functions, we derived in [12]:

$$I_5^{\mu_1\ldots\mu_{R-1}\mu} = I_5^{\mu_1\ldots\mu_{R-1}} Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1\ldots\mu_{R-1},s} Q_s^\mu , \tag{2.3}$$

$$Q_s^\mu = \sum_{i=1}^{n} q_i^\mu \frac{\binom{s}{i}_n}{\left(\right)_n}, \quad s = 0, \ldots, n. \tag{2.4}$$

The formula is the analogue to (2.1). For n-point functions with $n < 5$, the corresponding representations contain additional terms because the number of independent chords is then less than four so that the chords don't form a complete basis for $d = 4$. There are several modifications to be applied, and we like to reproduce only one example with auxiliary terms:

$$I_4^{\mu\nu\lambda\rho} = I_4^{\mu\nu\lambda} Q_0^\rho - \sum_{t=1}^{4} I_3^{\mu\nu\lambda,t} Q_t^\rho - G^{\mu\rho} T^{\nu\lambda} - G^{\nu\rho} T^{\mu\lambda} - G^{\lambda\rho} T^{\mu\nu}, \tag{2.5}$$

[1] Similar recursive realizations of the Passarino-Veltman reduction may be found in [16], see there figures 2 and 3.

p_1	0.5	0.0	0.0	0.5
p_2	0.5	0.0	0.0	– 0.5
p_3	– 0.19178191	– 0.12741180	– 0.08262477	– 0.11713105
p_4	– 0.33662712	0.06648281	0.31893785	0.08471424
p_5	– 0.21604814	0.20363139	– 0.04415762	– 0.05710657
$p_6 = -(p_1+p_2+p_3+p_4+p_5)$				

Table 1: Phase space point of massless six-point functions taken from [7].

with the additional tensor and vector components:

$$T^{\mu\nu} = I_4^{\mu,[d+]} Q_0^{\nu} - \sum_{t=1}^{4} I_3^{\mu,[d+],t}\, Q_t^{\nu} - G^{\mu\nu} I_4^{[d+]^2}, \tag{2.6}$$

$$G^{\mu\lambda} = \frac{1}{2} g^{\mu\lambda} - \sum_{i,j=1}^{4} q_i^{\mu} q_j^{\lambda} \frac{\binom{i}{j}_4}{\left(\right)_4}, \tag{2.7}$$

$$I_4^{\mu,[d+]} = I_4^{[d+]} Q_0^{\mu} - \sum_{t=1}^{4} I_3^{[d+],t} Q_t^{\mu}, \tag{2.8}$$

$$I_3^{\mu,[d+],t} = I_3^{[d+],t} Q_0^{t,\mu} - \sum_{u=1}^{4} I_2^{[d+],tu} Q_u^{t,\mu}, \tag{2.9}$$

$$Q_u^{t,\mu} = \sum_{i=1}^{4} q_i^{\mu} \frac{\binom{ut}{it}_4}{\binom{t}{t}_4}, \quad u = 0, \ldots, 4. \tag{2.10}$$

They may be, finally, represented by the scalar integrals in d dimensions $I_4, I_3^t, I_2^{tu}, I_1^{tuw}$, where the indices t, u, w indicate truncations of corresponding lines:

$$I_4^{[d+]^2} = \left[\frac{\binom{0}{0}_4}{\left(\right)_4} I_4^{[d+]} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{\left(\right)_4} I_3^{[d+],t} \right] \frac{1}{d-1}, \tag{2.11}$$

$$I_4^{[d+]} = \frac{\binom{0}{0}_4}{\left(\right)_4} I_4 - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{\left(\right)_4} I_3^t, \tag{2.12}$$

$$I_3^{[d+],t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^t - \sum_{u=1}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{tu} \right] \frac{1}{d-2}, \tag{2.13}$$

$$I_2^{[d+],tu} = \left[\frac{\binom{0tu}{0tu}_4}{\binom{tu}{tu}_4} I_2^{tu} - \sum_{w=1}^{4} \frac{\binom{0tu}{wtu}_4}{\binom{tu}{tu}_4} I_1^{tuw} \right] \frac{1}{d-1}. \tag{2.14}$$

The representations for the simpler tensors have been given in [12].

3. Numerical results

An example of a completely massless tensor reduction uses the momenta given in table 1. We combined our tensor reduction with the scalar master integrals from QCDloop [13]. Table 2 contains sample tensor components of a six-point function with rank $R = 5$. It shows an agreement of eight digits between the results of our package Hexagon.F [18] and those of Golem95 [7] for the constant terms of the tensor components.

	Hexagon.F		Golem95	
F^{03121}	0.158428987E+0,	0.41670698E–1	0.158428981E+0 ,	0.41670700E–1
F^{11020}	– 0.143913860E+1,	– 0.16464705E+0	– 0.143913853E+1 ,	– 0.16464708E+0
F^{20200}	0.242928780E+2,	0.55504184E+2	0.242928776E+2 ,	0.55504182E+2
F^{22130}	0.225563941E+0,	0.23192857E+0	0.225563949E+0 ,	0.23192851E+0
F^{33333}	0.244568135E+0,	0.74014604E+0	0.244568138E+0 ,	0.74014610E+0

Table 2: Real and imaginary parts of selected tensor components of rank $R = 5$ massless hexagon integrals; comparison of the packages Hexagon.F and Golem95.

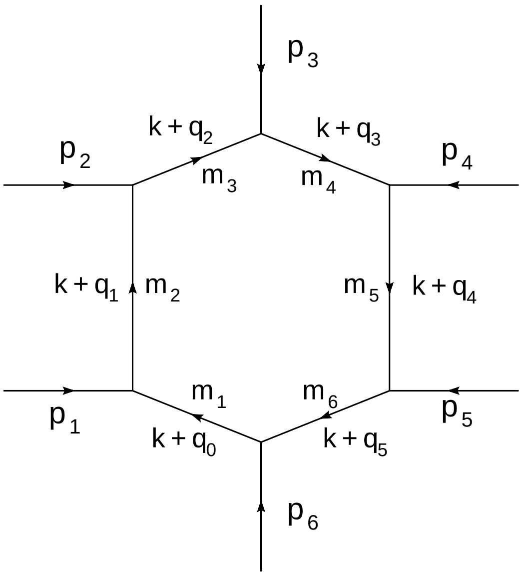

Figure 2: Momenta flow for the massive six-point topology.

As a second example, we reproduce components of the massive tensor integral $I_6^{\alpha\beta\gamma\delta\varepsilon}$ in table 4. The kinematics is defined by figure 2 (with $q_0 = 0$) and table 3. All tensor components are finite. The numbers could not be checked by another open source program, but we had the opportunity to compare them with an unpublished numerical package [19].

Further numerical results may be found in the transparencies of the talk [20].

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p_1	0.21774554 E+03	0.0	0.0	0.21774554 E+03
p_2	0.21774554 E+03	0.0	0.0	– 0.21774554 E+03
p_3	– 0.20369415 E+03	– 0.47579512 E+02	0.42126823 E+02	0.84097181 E+02
p_4	– 0.20907237 E+03	0.55215961 E+02	– 0.46692034 E+02	– 0.90010087 E+02
p_5	– 0.68463308 E+01	0.53063195 E+01	0.29698267 E+01	– 0.31456871 E+01
p_6	– 0.15878244 E+02	– 0.12942769 E+02	0.15953850 E+01	0.90585932 E+01
$m_1 = m_2 = m_3 = m_5 = m_6 = 110.0, \quad m_4 = 140.0$				

Table 3: Randomly chosen phase space point of six-point functions with massive particles.

	Hexagon.F	
F^{03121}	0.29834730E–09,	– 0.68229122E–10
F^{11020}	0.42830755E–09,	0.42574811E–09
F^{20200}	– 0.71172947E–08,	0.10102923E–07
F^{22130}	– 0.29200434E–09,	0.78553811E–10
F^{33333}	0.17451484E–07,	– 0.30914316E–07

Table 4: Selected tensor components of rank $R = 5$ massive hexagon integrals;

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