

Construction of the ground state of Matrix Theory: Near the Origin

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A unique $Spin(9) \times SU(2)$ singlet state, involving only the fermionic degrees of freedom, corresponding to the supermembrane/M-theory matrix-model, is constructed. Any ground state of the theory evaluated at the origin must be proportional to the state.

*European Physical Society Europhysics Conference on High Energy Physics
July 16-22, 2009
Krakow, Poland*

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In early 60' Dirac considered a possibility that the electron could be described by a conducting membrane with a spherical topology [1]. His motivation was to explain the existence of the muon and its mass in terms of the excitation of the ground state identified with the electron. Ironically, although Dirac's findings did not match the experiential data, the modifications of the membrane-action he considered are nowadays one of the key tools in analyzing the D-brane dynamics.

As found by Goldstone and Hoppe [2] the quantum theory of the Dirac's membrane can be formulated by taking the large N limit of a certain matrix model of $N \times N$ matrices. The supersymmetric version of that model [3] has an intriguing property: not only does it describe the supermembrane but also it is nicely related to M-theory by the BFSS conjecture [4]. This relation makes the supermembrane matrix theory very attractive also in string theory.

The supermembrane matrix theory is a quantum mechanical model with $\mathcal{N} = 16$ supersymmetries, $SU(N)$ gauge invariance and $Spin(9)$ symmetry. The theory involves real bosonic variables x_{sA} (coordinates) and real fermionic ones $\theta_{\alpha A}$ (Majorana spinors) with $s = 1, \dots, 9$, $\alpha = 1, \dots, 16$ and $A = 1, \dots, N^2 - 1$ - spatial, spinor and color indices respectively. The corresponding supercharges and the Hamiltonian of the model are

$$\begin{aligned} \mathcal{Q}_\alpha &= \left(-i\partial_{sA}\gamma_{\alpha\beta}^s + \frac{1}{2}f_{ABC}x_{sB}x_{tC}\gamma_{\alpha\beta}^{st} \right) \theta_{\beta A}, \quad \gamma^{st} = \frac{1}{2}[\gamma^s, \gamma^t], \\ H &= -\partial_{sA}\partial_{sA} + \frac{1}{2}(f_{ABC}x_{sB}x_{tC})^2 + if_{ABC}\gamma_{\alpha\beta}^s\theta_{\alpha A}\theta_{\beta B}x_{sC}, \\ \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} &= \delta_{\alpha\beta}H + 2\gamma_{\alpha\beta}^s x_{sA} J_A, \quad J_A = -if_{ABC} \left(x_{sB}\partial_{sC} + \frac{1}{2}\theta_{\alpha B}\theta_{\alpha C} \right). \end{aligned} \quad (1)$$

Here γ^s are 16×16 dimensional, real matrices s.t. $\{\gamma^s, \gamma^t\} = 2\delta^{st}\mathbf{1}_{16 \times 16}$, $\theta_{\alpha A}$ are Grassmann numbers s.t. $\{\theta_{\alpha A}, \theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}$, and f_{ABC} are $SU(N)$ structure constants (real, antisymmetric). The physical (gauge invariant) Hilbert space consists of states $|\psi\rangle$ satisfying $J_A|\psi\rangle = 0$.

An important question in matrix theory is related to the construction of the ground state. While there are serious indications that such state exists [5] and a number of results concerning its asymptotic form [6], the state has never been constructed analytically. In this talk we report on some progress towards this direction [7].

Using elliptic regularity (see e.g. [8]), one can write any solution to the equation $H\Psi = 0$ in terms of a power series in the coordinates,

$$\Psi(x) = \psi^{(0)} + x_{tA}\psi_{tA}^{(1)} + \frac{1}{2}x_{tA}x_{uB}\psi_{tA,uB}^{(2)} + \dots,$$

with $\psi_{t_1 A_1 \dots t_k A_k}^{(k)}$ depending entirely on the fermionic variables. Inserting this into the equation $\mathcal{Q}_\beta \Psi = 0$ we find a class of equations satisfied by $\psi_{t_1 A_1 \dots t_k A_k}^{(k)}$, e.g.

$$\begin{aligned} \gamma_{\beta\hat{\alpha}}^t \theta_{\hat{\alpha}A} \psi_{tA}^{(1)} &= 0, \quad \gamma_{\beta\hat{\alpha}}^t \theta_{\hat{\alpha}A} \psi_{tA,uB}^{(2)} = 0, \\ -i\gamma_{\beta\hat{\alpha}}^t \theta_{\hat{\alpha}A} \psi_{tA,uB,vC}^{(3)} + \varepsilon_{ABC} \gamma_{\beta\hat{\alpha}}^{\nu} \theta_{\hat{\alpha}A} \psi^{(0)} &= 0, \end{aligned}$$

etc. for all $\hat{\beta}$. Solving the above equations would be an important result in understanding the behavior of the ground state around the origin. In the following part of this talk we will concentrate on the evaluation of $\psi^{(0)}$.

The only constraints which $\psi^{(0)}$ satisfies follow from the requirement that $\psi^{(0)}$ has to be both the $SU(N)$ and the $Spin(9)$ singlet. The corresponding equations are

$$\underbrace{f_{ABC}\theta_{\alpha A}\theta_{\alpha B}}_{SU(N)}\psi^{(0)} = 0, \quad \underbrace{\gamma_{\alpha\beta}^{st}\theta_{\alpha A}\theta_{\alpha A}}_{SO(9)}\psi^{(0)} = 0.$$

The existence and the uniqueness of such state for $SU(N=2)$ matrix theory was found earlier in [9] using symbolic programme. Here we present a group-theoretic construction of that result.

Out of the 16 real Majorana fermions one can build 8 creation and 8 annihilation fermionic operators and correspondingly construct the whole 256-dimensional fermionic Fock space \mathcal{H}_{256} . The state $\psi^{(0)}$ lies in the tensor product of three such spaces $\mathcal{H}_{256} \otimes \mathcal{H}_{256} \otimes \mathcal{H}_{256}$ each one corresponding to one color $A = 1, 2, 3$. Under $Spin(9)$ the representation **256** splits into **44** + **84** + **128**. Let us denote by $|st\rangle$, $|stu\rangle$ and $|s\alpha\rangle$ vectors which span the **44**, **84** and **128** respectively (for their explicit construction see the Appendix in [7]). Clearly, any product of a type

$$|su\rangle_1|tu\rangle_2|st\rangle_3, \quad |s\alpha\rangle_1|t\alpha\rangle_2|st\rangle_3, \quad |suv\rangle_1|tuv\rangle_2|st\rangle_3$$

(i.e. with no free indices) is a $Spin(9)$ singlet and hence $\psi^{(0)}$ could be some linear combination of the above examples. However it turns out that the additional $SU(2)$ invariance picks up a unique combination, namely

$$\psi^{(0)} \propto \begin{array}{|c|} \hline 1 \\ \hline 44 \\ \hline \end{array} + \frac{13}{36} \begin{array}{|c|} \hline 1 \\ \hline 844 \\ \hline \end{array}$$

where

$$\begin{array}{|c|} \hline 1 \\ \hline 44 \\ \hline \end{array} := |su\rangle_1|tu\rangle_2|st\rangle_3,$$

$$\begin{array}{|c|} \hline 1 \\ \hline 844 \\ \hline \end{array} := |suv\rangle_1|tuv\rangle_2|st\rangle_3 + |tuv\rangle_1|st\rangle_2|suv\rangle_3 + |st\rangle_1|suv\rangle_2|tuv\rangle_3.$$

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