PROCEEDINGS OF SCIENCE

Fermions coupled to emergent noncommutative gravity

Daniela Klammer*
University of Vienna, Austria
E-mail: daniela.klammer@univie.ac.at

Harold Steinacker
University of Vienna, Austria
E-mail: harold.steinacker@univie.ac.at

We study the coupling of fermions to Yang-Mills matrix models in the framework of emergent noncommutative gravity. It is shown that the matrix model action provides an appropriate coupling for fermions to gravity, albeit with a non-standard spin-connection. Integrating out the fermions in a nontrivial geometrical background induces indeed the Einstein-Hilbert action for on-shell geometries plus a dilaton-like term. This result explains UV/IR mixing as a gravity effect. It also illuminates why UV/IR mixing remains even in supersymmetric models, except in the $N = 4$ case.

Black Holes in General Relativity and String Theory
24 - 30 August, 2008
Veli Lošinj, Croatia

*Speaker.

PoS(BHs, GR and Strings)037

Contents

1. Introduction 2

2. Matrix models and effective geometry 3

3. Quantization and induced gravity 5

4. Relation with gauge theory on $\mathbb{R}^4_\theta$ 8

5. Comparison with UV/IR mixing 10

6. Discussion and outlook 13

1. Introduction

Noncommmutativity & gravity. Heisenberg's uncertainty principle together with Einsteins' general theory of relativity lead to the conclusion that the classical concept of spacetime loses its meaning in the small. When measuring a spacetime coordinate with great accuracy a, there is an uncertainty in momentum of the order $1/a$. That is to say measuring small distances requires high energies, which will curve locally the region of spacetime you want to measure. When the gravitational field becomes so strong as to prevent any signal from escaping that region the operational meaning of this localization gets lost. The process of measuring a spacetime coordinate to infinite accuracy is thus as a matter of principle not possible.

It has been shown in a fundamental paper by Doplicher, Fredenhagen and Roberts [1] that the above argument leads to uncertainty relations for the spacetime coordinates which can be derived from *noncommuting spacetime coordinates*, such as

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}. \tag{1.1}$$

One can then study so called called "noncommutative (NC) quantum field theories" on spaces with such noncommuting coordinates; for basic reviews see e.g. [2, 3]. In NC field theories quantum fluctuations of spacetime coordinates occur naturally. Thus it is believed that these theories could play an important rôle on the way towards a quantum theory of gravity. Recently, a specific realization of this idea was published under the name of "emergent noncommutative gravity," see [4] and [5–8]. There, matrix models of noncommutative gauge theory describe dynamical noncommutative spaces. The main lesson learned is that gravity is already contained in noncommutative gauge theories. There is no need to add new concepts. Here we discuss specific results of this approach: We study the successful coupling of fermions to the framework of emergent noncommutative gravity.

2. Matrix models and effective geometry

Consider the matrix model action

$$S_{YM} = -Tr[Y^a,Y^b][Y^{a'},Y^{b'}]g_{aa'}g_{bb'} \tag{2.1}$$

for

$$g_{aa'} = \delta_{aa'} \quad \text{or} \quad g_{aa'} = \eta_{aa'} \tag{2.2}$$

in the Euclidean resp. Minkowski case. $g_{aa'}$ should not be interpreted as a fixed, physical background metric, but rather as a prescription to fix the signature. Here the "covariant coordinates" Y^a for $a = 1,2,3$, are hermitian matrices or operators acting on some Hilbert space $\mathscr{H}$. We will denote the commutator of two matrices as

$$[Y^a,Y^b] = i\theta^{ab} \tag{2.3}$$

so that $\theta^{ab} \in L(\mathscr{H})$ is an antihermitian matrix, which is *not* assumed to be constant here. We study configurations Y^a (not necessarily solutions of the equation of motion) which can be interpreted as quantizations of a Poisson manifold $(\mathscr{M},\theta^{ab}(y))$ with general Poisson structure $\theta^{ab}(y)$. This defines the geometrical background under consideration, and essentially any (local) Poisson manifold is a possible background Y^a. In particular, we assume that θ^{ab} is small and well approximated by the classical Poisson structure $\theta^{ab}(y)$ in a semi-classical expansion. More formally, this means that there is an isomorphism of vector spaces

$$\mathscr{C}(\mathscr{M}) \to \mathscr{A} \subset L(\mathscr{H}) \tag{2.4}$$

where $\mathscr{C}(\mathscr{M})$ denotes the space of functions on $\mathscr{M}$, and $\mathscr{A}$ is the algebra generated by Y^a, interpreted as quantized algebra of functions. In particular, Y^a corresponds to a classical coordinate function[1] y^a on $\mathscr{M}$. This can be used to define a star product on $\mathscr{C}(\mathscr{M})$. Moreover, Y^a defines a derivation on $\mathscr{A}$ via

$$[Y^a, f] \sim \mathrm{i}\theta^{ab}(y)\partial_b f(y). \tag{2.5}$$

In this paper, we restrict ourselves to the "irreducible" case, i.e. we assume that $\mathscr{A}$ is in some sense dense in $L(\mathscr{H})$. Then any matrix ("function") in $L(\mathscr{H})$ can be expressed as a function of Y^a resp. y^a. From the gauge theory point of view in Section 4, it means that we restrict ourselves to the $U(1)$ case. This is most interesting for us since the UV/IR mixing (see Sect. 5) happens in the trace-$U(1)$ sector. For the general case see [4,8].

Scalars. To begin with, we consider the case of scalar fields i.e. hermitian matrices Φ coupled to the matrix model (2.1). There it is seen most easily how the effective metric appears. The only possibility to write down kinetic terms for matter fields is through commutators[2] $[Y^a,\Phi] \sim$

[1]The coordinates y^a are preferred ones since in their frame $g_{aa'}$ equals $\delta_{aa'}$ resp. $\eta_{aa'}$. In other frames $g_{aa'}$ will not be constant.

[2]Throughout this paper, $\sim$ indicates the leading contribution in a semi-classical expansion in powers of θ^{ab}.

PoS(BHS, GR and Strings)037

$i\theta^{ab}(y)\frac{\partial}{\partial y^b}\Phi$, and one is lead to the action

$$S[\Phi] = (2\pi)^2\,\mathrm{Tr}\,g_{aa'}[Y^a,\Phi][Y^{a'},\Phi] \sim \int d^4y\,\rho(y)\,G^{ab}(y)\,\frac{\partial}{\partial y^a}\Phi(y)\frac{\partial}{\partial y^b}\Phi(y). \tag{2.6}$$

Here

$$G^{ab}(y) = \theta^{ac}(y)\theta^{bd}(y)g_{cd} \tag{2.7}$$

is the effective metric for the scalar field Φ. The Poisson manifold naturally acquires a metric structure $(\mathscr{M},\theta^{ab}(y),G^{ab}(y))$ determined by the Poisson structure. The metric is thus no fundamental building block of the theory. We also used $\mathrm{Tr} \sim \int \mathrm{d}^4y\rho(y)$, where

$$\rho(y) = (\det\theta^{ab}(y))^{-1/2} = |G_{ab}(y)|^{1/4} \equiv e^{-\sigma} \quad (\equiv \Lambda^4_{NC}(y)) \tag{2.8}$$

is the symplectic measure on $(\mathscr{M},\theta^{ab}(y))$ which can be naturally interpreted as non-commutative scale Λ^4_{NC}. After appropriate rescaling of $G^{ab}(y)$, this can be rewritten in covariant form

$$S[\Phi] = \int d^4y\,\tilde{G}^{ab}(y)\partial_{y^a}\Phi\partial_{y^b}\Phi \tag{2.9}$$

with the effective metric

$$\tilde{G}^{ab} = |G_{ab}|^{1/4}\,G^{ab} = \rho(y)\,G^{ab}, \qquad |\tilde{G}^{ab}| = 1 \tag{2.10}$$

being unimodular in the preferred y^a coordinates.

Fermions. The most obvious action for a spinor which can be written down in the matrix model framework[3] is

$$S = (2\pi)^2\,\mathrm{Tr}\overline{\Psi}\gamma_a[Y^a,\Psi] \;\sim\; \int d^4y\,\rho(y)\,\overline{\Psi}i\gamma_a\theta^{ab}(y)\partial_b\Psi \tag{2.11}$$

ignoring possible nonabelian gauge fields here to simplify the notation. This is written for the case of Minkowski signature, the Euclidean version involves the obvious replacement $\bar{\Psi} \to \Psi^\dagger$. This defines the (matrix) Dirac operator

$$\not{D}\Psi = \gamma_a[Y^a,\Psi] \;\sim\; i\gamma_a\theta^{ab}(y)\partial_b\Psi. \tag{2.12}$$

We can compare this with the standard covariant derivative for spinors

$$\not{D}_{\mathrm{comm}}\Psi = i\gamma^a\,e^\mu_a\left(\partial_\mu + \Sigma_{ab}\,\omega^{ab}_\mu\right)\Psi \tag{2.13}$$

where

$$\omega^{ab}_\mu = i\frac{1}{2}\,e^{a\nu}\left(\nabla_\mu e^b_\nu\right) \tag{2.14}$$

is the spin connection, and $\Sigma_{ab} = \mathrm{i}/4[\gamma_a,\gamma_b]$ is the representation of the Lorentz algebra. Comparing

[3]In particular, fermions should also be in the adjoint, otherwise they cannot acquire a kinetic term. This does not rule out its applicability in particle physics, see e.g. [13].

(2.12) with (2.13), we observe again that in the geometry defined by (2.7),

$$e^{\mu}_{b}(y) := \theta^{\mu c}(y) g_{cb} \tag{2.15}$$

plays the rôle of a preferred vielbein. However this must be used with great care, because the distinction between the coordinate index μ and the Lorentz index a is lost in the special "gauge" inherent in (2.15).

One notices that the spin connection does not appear in (2.11), which seems very strange at first. In spite of this strange feature, the action (2.11) is a good action for a fermion propagating in the geometry defined by $\tilde{G}_{ab}$. To see this, recall that the spin connection determines how the spinors are rotated under parallel transport along a trajectory. However, the spin-connection ω_{μ}^{ab} can always be eliminated (via parallel-transport resp. a suitable gauge choice) along an open trajectory. Then the conventional kinetic term (2.13) boils down to (2.11). Therefore in the point-particle limit, the trajectory of a fermion with action (2.11) will follow properly the geodesics of the metric[4] $\tilde{G}_{ab}$, albeit with a different rotation of the spin. Furthermore, the induced gravitational action obtained by integrating out the fermion in (2.11) indeed induces the expected Einstein-Hilbert term $\int d^4y R[\tilde{G}]\Lambda^2$ at least for "on-shell geometries", albeit with an unusual numerical coefficient and an extra term depending on σ. All this shows that (2.11) defines a reasonable action for fermions in the background defined by $\tilde{G}_{ab}$.

Equations of motion. So far we considered arbitrary background configurations Y^a as long as they admit a geometric interpretation. The equations of motion derived from the action (2.1)

$$[Y^a,[Y^{a'},Y^b]]\,g_{aa'} = 0 \quad \overset{\text{semi-cl. limit}}{\rightarrow} \quad \theta^{ma}\partial_m\theta^{nb}g_{ab} = 0 \tag{2.16}$$

select on-shell geometries among all possible backgrounds, such as the Moyal-Weyl quantum plane (4.2). However since we are interested in the quantization here, we will need general off-shell configurations below.

3. Quantization and induced gravity

Next we study the quantization of our matrix model coupled to fermions. In principle, the quantization is defined in terms of a ("path") integral over all matrices Y^a and Ψ. In 4 dimensions, we can only perform perturbative computations for the "gauge sector" Y^a, while the fermions can be integrated out formally in terms of a determinant. Let us focus here on the effective action at one loop[5] obtained by integrating out the fermionic fields,

$$e^{-\Gamma_\Psi} = \int \mathrm{d}\Psi \mathrm{d}\overline{\Psi} e^{-S[\Psi]} \quad \text{with} \quad \Gamma_\Psi = -\frac{1}{2}\mathrm{Tr}\log \not{D}^2 \,. \tag{3.1}$$

for a non-interacting fermionic field with action Eq. (2.11).

[4]for massless particles, the geodesics of $\tilde{G}_{ab}$ coincide with those of G_{ab}. Masses should be generated spontaneously, which is not considered here.

[5]For the sake of rigor we work in Euclidean case now.

PoS(BHS, GR and Strings)037

Square of the Dirac operator and induced action. The square of the Dirac operator takes the following form

$$\not{D}^2\Psi = \gamma_a\gamma_b[Y^a,[Y^b,\Psi]] \sim -\gamma_a\gamma_b\theta^{ac}\partial_c(\theta^{bd}\partial_d\Psi) = -G^{cd}\partial_c\partial_d\Psi - a^d\partial_d\Psi, \tag{3.2}$$

with

$$a^d \;=\; \gamma_a\gamma_b\theta^{ma}\partial_m\theta^{db} = -2i\Sigma_{ab}\theta^{ac}\partial_c\theta^{bd} + g_{ab}\theta^{ac}\partial_c\theta^{bd}. \tag{3.3}$$

$\not{D}^2$ defines the quadratic form

$$S_{\text{square}} = (2\pi)^2\,\text{Tr}\Psi^\dagger\,\not{D}^2\Psi \sim \int d^4y\,\rho(y)\Psi^\dagger\,\not{D}^2\Psi = \int d^4y\,|G_{ab}|^{1/4}\Psi^\dagger\,\not{D}^2\Psi, \tag{3.4}$$

which is very similar to the scalar action. In terms of the unimodular metric $\tilde{G}_{ab}$ Eq. (2.10), S_{square} can be written in standard covariant form

$$S_{\text{square}} = \int d^4y\sqrt{|\tilde{G}|}\;\overline{\Psi}\widetilde{\not{D}^2}\Psi \quad \text{with} \quad \widetilde{\not{D}^2}\Psi = -\left(\tilde{G}^{cd}\partial_c\partial_d\Psi + e^{-\sigma}\,a^d\,\partial_d\Psi\right). \tag{3.5}$$

We now compute the effective action using

$$\frac{1}{2}\text{Tr}(\log\widetilde{\not{D}^2} - \log\widetilde{\not{D}^2_0}) \;=\; -\frac{1}{2}\text{Tr}\int_0^\infty d\alpha\frac{1}{\alpha}\left(e^{-\alpha\widetilde{\not{D}^2}} - e^{-\alpha\widetilde{\not{D}^2_0}}\right)e^{-\frac{1}{2\alpha\tilde{\Lambda}^2}}, \tag{3.6}$$

where $\widetilde{\Lambda}^2$ denotes the cutoff for $\widetilde{\not{D}^2}$ regularizing the divergence for small α. Now we can apply the heat kernel expansion

$$\text{Tr}\,e^{-\alpha\widetilde{\not{D}^2}} = \sum_{n\geq 0}\left(\frac{\alpha}{2}\right)^{\frac{n-4}{2}}\int_{\mathscr{M}} d^4y\;a_n\left(y,\widetilde{\not{D}^2}\right) \tag{3.7}$$

where the Seeley-de Witt coefficients $a_n(y,\widetilde{\not{D}^2})$ are given by [14]

$$\begin{aligned}
a_0(y) \;&=\; \frac{1}{16\pi^2}\,\text{tr}\,\mathbb{1},\\
a_2(y) \;&=\; \frac{1}{16\pi^2}\text{tr}\left(\frac{R[\widetilde{G}]}{6}\;\mathbb{1}+\mathscr{E}\right),\\
\mathscr{E} \;&=\; -\widetilde{G}^{mn}\left(\partial_m\Omega_n + \Omega_m\Omega_n - \widetilde{\Gamma}^k_{mn}\Omega_k\right), && (3.8)\\
\Omega_m \;&=\; \frac{1}{2}\widetilde{G}_{mn}\left(e^{-\sigma}a^n + \widetilde{\Gamma}^n\right), && (3.9)
\end{aligned}$$

where tr denotes the trace over the spinorial matrices. The effective action is therefore

$$\Gamma_\Psi \;=\; \frac{1}{16\pi^2}\int d^4y\left(2\,\text{tr}(\mathbb{1})\,\widetilde{\Lambda}^4 + \text{tr}\left(\frac{R[\tilde{G}]}{6}\;\mathbb{1}+\mathscr{E}\right)\widetilde{\Lambda}^2 + O(\log\tilde{\Lambda})\right), \tag{3.10}$$

where $\text{tr}(\mathbb{1}) = 4$ assuming Dirac fermions. Everything is expressed in terms of the unimodular metric $\tilde{G}_{ab}$, which can be written in terms of G_{ab} using

$$\begin{aligned} R[\widetilde{G}] &= \rho(y)\left(R[G]+3\Delta_G\sigma-\frac{3}{2}G^{ab}\partial_a\sigma\partial_b\sigma\right), \\ \Delta_G\sigma &= G^{ab}\partial_a\partial_b\sigma-\Gamma^c\partial_c\sigma, \\ \Gamma^a &= G^{bc}\Gamma^a_{bc}, \\ e^{-\sigma(y)} &= \rho(y)=(\det G_{ab})^{1/4}, \\ \widetilde{\Gamma}^a &= \widetilde{G}^{cd}\widetilde{\Gamma}^a_{cd} = e^{-\sigma}\Gamma^a-e^{-\sigma}(\partial_b\sigma)G^{ba}. \end{aligned} \tag{3.11}$$

Note the relative minus sign of the various terms in the effective action Γ_Ψ compared with the induced action due to a scalar field [5],

$$\Gamma_\Phi=\frac{1}{16\pi^2}\int d^4y\left(-2\widetilde{\Lambda}^4-\frac{1}{6}R[\widetilde{G}]\widetilde{\Lambda}^2+O(\log\widetilde{\Lambda})\right). \tag{3.12}$$

hence

$$\Gamma_\Psi+4\Gamma_\Phi = \frac{1}{16\pi^2}\int d^4y\,\mathrm{tr}\,\mathscr{E}\,\widetilde{\Lambda}^2. \tag{3.13}$$

This expresses the cancellation of the induced actions due to fermions and bosons, apart from the $\mathscr{E}$ term. For the standard coupling of Dirac fermions to gravity on commutative spaces, one has [15]

$$\mathrm{tr}\,\mathscr{E}_{\mathrm{comm}}=-R \tag{3.14}$$

which originates from an additional constant term $-\frac{1}{4}R$ in $\not{D}^2_{\mathrm{comm}}$ (Lichnerowicz's formula). In our case, $\mathscr{E}$ turns out to be somewhat modified due to the missing spin connection, nevertheless it contains the appropriate curvature scalar plus an additional term, see Eq. (3.24).

Ricci scalar in terms of θ^{mn}. The curvature is given as usual by

$$R_{abc}{}^d=\partial_b\Gamma^d_{ac}-\partial_a\Gamma^d_{bc}+\Gamma^e_{ac}\Gamma^d_{eb}-\Gamma^e_{bc}\Gamma^d_{ea}. \tag{3.15}$$

The Ricci scalar is then

$$R=G^{ac}R^b_{abc}=G^{ac}\left(\partial_b\Gamma^b_{ac}-\partial_a\Gamma^b_{bc}+\Gamma^e_{ac}\Gamma^b_{eb}-\Gamma^e_{bc}\Gamma^b_{ea}\right). \tag{3.16}$$

By plugging in the explicit formula for metric tensor,

$$G^{mn}(y)=\theta^{ma}(y)\theta^{nb}(y)g_{ab} \tag{3.17}$$

one can express the Ricci scalar R in terms of θ. By making use of the Jacobi identity,

$$\partial_a\theta^{-1}_{bc}+\partial_c\theta^{-1}_{ab}+\partial_b\theta^{-1}_{ca}=0 \tag{3.18}$$

$$\partial_a\theta^{pq}=-\left(\partial_c\theta^{-1}_{am}\right)\left(\theta^{mp}\theta^{cq}-\theta^{mq}\theta^{cp}\right), \tag{3.19}$$

and by exploiting relations coming from partial integration several terms appearing in the compu-

PoS(BHs, GR and Strings)037

tation of $\mathrm{tr}\mathscr{E}$ and $R[\widetilde{G}]$ are equivalent[6]. After a rather lengthy computation, which can be found in [6], one yields the following compact form for the Ricci scalar in terms of the unimodular metric $\widetilde{G}_{ab}$

$$\begin{aligned}\int d^4y R[\widetilde{G}]\widetilde{\Lambda}^2 \;=\; & e^{-\sigma}\Big\{\frac{1}{2}G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab} - \frac{1}{2}G^{mn}G^{pq}\left(\partial_p\theta^{-1}_{ma}\right)\left(\partial_q\theta^{-1}_{nb}\right)g^{ab} \\ & -\frac{1}{2}\left(\partial_p\theta^{pa}\right)G^{qk}\left(\partial_k\theta^{-1}_{qa}\right) + \frac{1}{2}G^{mn}\left(\partial_m\sigma\right)\left(\partial_n\sigma\right)\Big\}\widetilde{\Lambda}^2. \end{aligned} \tag{3.20}$$

Evaluation of tr$\mathscr{E}$. We also need to evaluate

$$\mathrm{tr}\mathscr{E} \;=\; -\mathrm{tr}\widetilde{G}^{ab}\left(\partial_a\Omega_b + \Omega_a\Omega_b - \widetilde{\Gamma}^r_{ab}\Omega_r\right), \tag{3.21}$$

where

$$\begin{aligned}\Omega_m \;&=\; \frac{1}{2}\widetilde{G}_{mn}\left(\widetilde{a}^n + \tilde{\Gamma}^n\right) \\ &=\; \frac{1}{2}\left(G_{mn}\gamma_a\gamma_b\theta^{pa}\left(\partial_p\theta^{nb}\right) - G_{mn}\left(\partial_p G^{pn}\right) + \partial_m\sigma\right)\end{aligned} \tag{3.22}$$

and $\tilde{a}^n = e^{-\sigma}a^n$. For the computation of $\mathrm{tr}\mathscr{E}$ we use again the Jacobi identity (3.18) and relations from partial integration and we find:

$$\mathrm{tr}\mathscr{E} \;=\; e^{-\sigma}\Big\{G^{kl}G^{mn}\left(\partial_k\theta^{-1}_{ma}\right)\left(\partial_l\theta^{-1}_{nb}\right)g^{ab} - G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab}\Big\}. \tag{3.23}$$

Comparing with (3.20) for $\widetilde{\Lambda}^2$ regarded as constant cutoff of $\Delta_{\widetilde{G}}$, we can write this as

$$\begin{aligned}\mathrm{tr}\mathscr{E} \;&=\; -2R[\tilde{G}] - \left(\partial_p\theta^{pa}\right)G^{qk}\left(\partial_k\theta^{-1}_{qa}\right) + G^{mn}\left(\partial_m\sigma\right)\left(\partial_n\sigma\right) \\ &\overset{\mathrm{eom}}{=}\; -2R[\tilde{G}] + G^{mn}\partial_m\sigma\partial_n\sigma,\end{aligned} \tag{3.24}$$

assuming on-shell geometries (2.16) in the last line. This formula applies for Dirac fermions, and with an additional factor $\frac{1}{2}$ for Weyl fermions. It is remarkable that $\mathrm{tr}\mathscr{E}$ is essentially given by the appropriate curvature scalar $R[\tilde{G}]$, and up to a contribution from the dilaton-like scaling factor $\rho = e^{-\sigma}$. This is a very reasonable modification of the standard result (3.14), as desired and tells us that Einstein-Hilbert action also emerges for fermions at one-loop.

4. Relation with gauge theory on $\mathbb{R}^4_\theta$

One motivation to study NC field theories comes from the fact that NC spacetime coordinates in the small tend to cure the UV divergencies. However, the supposedly removed divergencies reappear in the infrared limit $p \to 0$. This effect is the notorious *UV/IR mixing* which spoils renormalizability [17]. In the framework of emergent noncommutative gravity the UV/IR mixing problem of noncommutative gauge theories is understood in terms of an induced gravity action. In order to show this we want to interpret the fermionic action (2.11) as action for a Dirac fermion on the Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$ coupled to a $U(1)$ gauge field in the adjoint. This point of view is

[6]By means of these relations one can also check that the action (3.4) is indeed hermitian.

PoS(BHS, GR and Strings)037

obtained by writing the general covariant coordinate resp. matrix Y^a as

$$Y^a = X^a + \mathscr{A}^a \,. \tag{4.1}$$

Here X^a are generators of the Moyal-Weyl quantum plane, which satisfy

$$[X^a, X^b] = i\bar{\theta}^{ab} \,, \tag{4.2}$$

where $\bar{\theta}^{ab}$ is a *constant* antisymmetric tensor. These are particular solutions of the equations of motion (2.16). The effective geometry for the Moyal-Weyl plane is flat, given by

$$\begin{aligned} \bar{g}^{ab} &= \bar{\theta}^{ac}\,\bar{\theta}^{bd} g_{cd} \\ \tilde{g}^{ab} &= \bar{\rho}\,\bar{g}^{ab}, \qquad \det\tilde{g}^{ab} = 1 \\ \bar{\rho} &= (\det\bar{\theta}^{ab})^{-1/2} = |\bar{g}_{ab}|^{1/4} \equiv \Lambda_{NC}^4 \,. \end{aligned} \tag{4.3}$$

Consider now the change of variables

$$\mathscr{A}^a(x) = -\bar{\theta}^{ab} A_b(x) \tag{4.4}$$

where A_a are hermitian matrices interpreted as smooth functions on $\mathbb{R}^4_\theta$. Thus we can write

$$[Y^a, f] = [X^a + \mathscr{A}^a, f] = i\bar{\theta}^{ab}\left(\frac{\partial}{\partial x^b} f + i\,[A_b, f]\right) \equiv i\bar{\theta}^{ab} D_b f \,, \tag{4.5}$$

giving for the quadratic form (3.4)

$$\begin{aligned} S_{square} &= (2\pi)^2\,\mathrm{Tr}\,\Psi^\dagger \gamma_a \gamma_b \left[Y^a, \left[Y^b, \Psi\right]\right] \\ &= -\int d^4x\,\bar{\rho}\,\Psi^\dagger\,\gamma_a\gamma_b\bar{\theta}^{am}\bar{\theta}^{bn} D_m D_n \Psi \\ &= \int d^4x\,\Psi^\dagger\,\widetilde{\not{D}^2}_A \Psi \,. \end{aligned} \tag{4.6}$$

This is an exact expression on $\mathbb{R}^4_\theta$, where

$$\widetilde{\not{D}^2}_A = -\bar{\rho}\,\gamma_a\gamma_b\bar{\theta}^{am}\bar{\theta}^{bn} D_m D_n = -\,\tilde{\gamma}^m\tilde{\gamma}^n\,D_m D_n \,, \tag{4.7}$$

and

$$\tilde{\gamma}^a = (\det\bar{g}_{ab})^{\frac{1}{8}}\,\gamma_b\,\bar{\theta}^{ba}, \qquad \{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\tilde{g}^{ab} \,. \tag{4.8}$$

We now want to rewrite the geometrical results of Section 3 in terms of gauge theory on $\mathbb{R}^4_\theta$ in x-coordinates. To do this, note that most formulas of Section 3 are not generally covariant, but only valid in the preferred y-coordinates defined by the matrix models where $g_{ab} = \delta_{ab}$ resp. $g_{ab} = \eta_{ab}$. Eq. (4.1) defines the leading-order relation between y and x coordinates,

$$y^a = x^a - \bar{\theta}^{ab}\bar{A}_b + O(\theta^2) \,. \tag{4.9}$$

See [6] for details of this change of variables. Let us moreover denote $\bar{\partial}_a = \partial/\partial x^a$. The Poisson tensor can be written in terms of the $\mathfrak{u}(1)$ field strength as

$$i\theta^{ab}(y) = \left[Y^a, Y^b\right] = i\bar{\theta}^{ab} - i\bar{\theta}^{ac}\bar{\theta}^{bd}\bar{F}_{cd}, \tag{4.10}$$

where $\bar{F}_{cd}$ is a rank two tensor in x coordinates on $\mathbb{R}^4_\theta$. We also need the effective metric (2.7) in x-coordinates,

$$G^{ab} = \left(\bar{\theta}^{ac} - \bar{\theta}^{ai}\bar{\theta}^{cj}\bar{F}_{ij}\right)\left(\bar{\theta}^{bd} - \bar{\theta}^{be}\bar{\theta}^{df}\bar{F}_{ef}\right)g_{cd}. \tag{4.11}$$

We find for the one-loop induced action

$$\begin{aligned}\Gamma_\Psi &= \int d^4y\left(a_0\widetilde{\Lambda}^4 + a_2\widetilde{\Lambda}^2 + O\left(\log\widetilde{\Lambda}\right)\right)\\ &= -4\Gamma_\Phi - \frac{1}{16\pi^2}\int d^4y\frac{\rho(y)}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}\,\widetilde{\Lambda}^2.\end{aligned} \tag{4.12}$$

Finally, there is a nontrivial relation between the cutoff $\widetilde{\Lambda}$ of the geometrical action and the cutoff Λ of the $\mathfrak{u}(1)$ gauge theory, which follows from the identity

$$S_{\text{square}} = \operatorname{Tr}\Psi^\dagger\gamma_a\gamma_b\left[Y^a,\left[Y^b,\Psi\right]\right] = \int d^4y\,\Psi^\dagger\widetilde{\not{D}^2}_{\widetilde{G}}\Psi = \int d^4y\frac{\rho(y)}{\bar{\rho}}\Psi^\dagger\widetilde{\not{D}^2}_A\Psi. \tag{4.13}$$

For the Lapacians this means

$$\widetilde{\not{D}^2}_{\widetilde{G}} = \frac{\rho(y)}{\bar{\rho}}\widetilde{\not{D}^2}_A. \tag{4.14}$$

Since we implement the cutoffs using Schwinger parameterization they are related as follows

$$\widetilde{\Lambda}^2 = \frac{\rho(y)}{\bar{\rho}}\Lambda^2. \tag{4.15}$$

This makes sense provided $\rho(y)/\bar{\rho}$ varies only on large scales respectively small momenta $p \ll \Lambda$, which is our working assumption. We obtain as a final result for the geometric one-loop effective action expressed in terms of gauge theory on $\mathbb{R}^4_\theta$

$$\begin{aligned}\Gamma_\Psi &= -4\Gamma_\Phi - \int d^4x\bar{\rho}\frac{\Lambda^2}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}\\ &= -4\Gamma_\Phi + \int\frac{d^4p}{(2\pi)^4}\widetilde{g}^{ac}\widetilde{g}^{bd}\bar{F}_{ab}(p)\bar{F}_{cd}(-p)\frac{p^2}{\Lambda_{NC}^4}\frac{\Lambda^2}{2}\end{aligned} \tag{4.16}$$

where $p^2 = p_ip_jg^{ij}$. This agrees precisely with the one-loop computation in the gauge theory point of view obtained below. Note that the last term corresponds to $\operatorname{tr}\mathscr{E}$ in (3.13).

5. Comparison with UV/IR mixing

In this section, we compare the geometrical form of the one-loop effective action obtained in the previous section with the one-loop effective action obtained from the gauge theory point of view. The strategy is to apply first the concept of covariant coordinates to obtain a noncommutative gauge theory coupled to fermions and compute thereafter the one-loop effective action. The result is of course the same, which provides not only a nontrivial check for our geometrical interpretation, but also sheds new light on the conditions to which extent the semi-classical analysis of the previous section is valid. This generalizes the results of [5] to the fermionic case. We find as expected that the UV/IR mixing terms obtained by integrating out the fermions are given by the induced geometrical resp. gravitational action (3.10), in a suitable IR regime. In particular, we need an explicit, physical momentum cutoff Λ.

Using the variables and conventions of the previous section, the action (2.11) can be exactly rewritten as $U(1)$ gauge theory on $\mathbb{R}^4_\theta$, which in the Euclidean case takes the form

$$\begin{aligned} S[\Psi] &= (2\pi)^2\, \mathrm{Tr} \Psi^\dagger \gamma_a [Y^a, \Psi] \\ &= \int d^4x\, \tilde{\Psi}^\dagger i \tilde{\gamma}^a (\bar{\partial}_a \tilde{\Psi} + ig[A_a, \tilde{\Psi}]) \end{aligned} \tag{5.1}$$

We introduce an explict coupling constant g, and define a rescaled fermionic field

$$\tilde{\Psi} = |\bar{g}_{ab}|^{\frac{1}{16}}\, \Psi \tag{5.2}$$

in order to obtain the properly normalized effective metric $\tilde{g}^{ab}$; we will omit the tilde on Ψ henceforth. Recall also that only $U(1)$ gauge fields are considered here, because only those correspond to the nontrivial geometry considered in the previous section.

We need the $O(A^2)$ contribution to the one-loop effective action obtained by integrating out the fermionic field Ψ. While this computation has been discussed several times in the literature [16–20], the known results are not accurate enough for our purpose, i.e. in the regime $p^2, \Lambda^2 < \Lambda^2_{NC}$ where the semiclassical geometry is expected to make sense. We need to analyze carefully the IR regime of the well-known effective cutoff $\Lambda_{eff}(p)$ (5.7) for non-planar graphs as $p \to 0$, keeping Λ fixed. In this regime the non-planar diagrams almost coincide with the planar diagrams, and the leading IR corrections due to the nonplanar diagrams correspond to the induced geometrical terms in (3.10). This has not been considered in previous attempts to explain UV/IR mixing, e.g. in terms of exchange of closed string modes [21, 22].

To proceed we use the fermionic Feynman rules and consider the Feynman diagram in Figure 1 corresponding to

$$\begin{aligned} \Gamma_\Psi &= -\frac{1}{2}\mathrm{Tr}\log\Delta_0 - \frac{g^2}{2}\left\langle \int d^4x\, \bar{\rho}\, \bar{\Psi} \tilde{\gamma}^a [A_a, \Psi] \int d^4y\, \bar{\rho}\, \bar{\Psi} \tilde{\gamma}^b [A_b, \Psi] \right\rangle \\ &= -\frac{1}{2}\mathrm{Tr}\log\Delta_0 + \Gamma_\Psi(A). \end{aligned} \tag{5.3}$$

The minus sign in front is due to the fermionic loop. This integral looks explicitly as follows

$$\begin{aligned} \Gamma_\Psi &= -4g^2 \int \frac{d^4p}{(2\pi)^4} A_{a'}(p) A_{b'}(-p)\, \tilde{g}^{a'a} \tilde{g}^{b'b} \int \frac{d^4k}{(2\pi)^4} \frac{2k_a k_b + k_a p_b + p_a k_b - \tilde{g}_{ab} k(k+p)}{(k\cdot k)((k+p)\cdot(k+p))} \\ &\quad \times \left(e^{-ik_i\theta^{ij}p_j} - 1\right) \end{aligned} \tag{5.4}$$

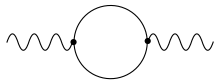

Figure 1: Fermionic one-loop diagram.

which is quite close to the bosonic case, using the notation

$$k\cdot k \equiv k_i k_j \tilde{g}^{ij} \qquad k^2 \equiv k_i k_j g^{ij}\,. \tag{5.5}$$

An evaluation of the integral gives

$$\begin{aligned}\Gamma_\Psi \;=\; & -4\Gamma_\Phi - g^2 n_f \int \frac{d^4p}{(2\pi)^4} A_{a'}(p) A_{b'}(-p)\, \tilde{g}^{a'a}\tilde{g}^{b'b}\,(p_a p_b - \tilde{g}_{ab}\, p\cdot p) \\ & \frac{1}{8\pi^2}\int_0^1 dz \Big(K_0\big(2\sqrt{\frac{z(1-z)p\cdot p}{\Lambda^2}}\big) - K_0\big(2\sqrt{\frac{z(1-z)p\cdot p}{\Lambda_{eff}^2}}\big)\Big)\,,\end{aligned} \tag{5.6}$$

for Dirac fermions, where

$$\Lambda_{eff}^2 \;=\; \frac{1}{1/\Lambda^2 + \frac{1}{4}\frac{p^2}{\Lambda_{NC}^4}} = \Lambda_{eff}^2(p) \tag{5.7}$$

is the "effective" cutoff for non-planar graphs, and Λ_{NC} is defined in (4.3). To proceed we consider the IR regime

$$\frac{p^2\Lambda^2}{\Lambda_{NC}^4} < 1\,. \tag{5.8}$$

Then both Λ and Λ_{eff} are large, and we can use an asymptotic expansions for the Bessel function

$$K_0\Big(2\sqrt{\frac{m^2}{\Lambda^2}}\Big) \;=\; -\left(\gamma + \log\big(\sqrt{\frac{m^2}{\Lambda^2}}\big)\right) + O\Big(\frac{m^2}{\Lambda^2}\log\big(\frac{\Lambda}{m}\big)\Big). \tag{5.9}$$

Moreover, in the valid regime $p\Lambda < \Lambda_{NC}^2$ one is allowed to expand the effective cutoff

$$\Lambda_{eff}^2 = \Lambda^2 - p^2\frac{\Lambda^4}{4\Lambda_{NC}^4} + \ldots, \qquad \Lambda_{eff}^4 = \Lambda^4 - p^2\frac{\Lambda^6}{2\Lambda_{NC}^4} + \ldots\,. \tag{5.10}$$

We obtain our final result

$$\begin{aligned}\Gamma_\Psi + 4\Gamma_\Phi \;\sim\; & \frac{1}{4}\frac{g^2}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\, \tilde{g}^{a'a}\tilde{g}^{b'b}\, \bar{F}_{ab}(p)\bar{F}_{a'b'}(-p)\, \frac{p^2\Lambda^2}{\Lambda_{NC}^4}\,, \\ =\; & \frac{1}{4}\frac{g^2}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\, \bar{\rho}^2\Lambda^2 p^2\, \bar{g}^{a'a}\bar{g}^{b'b}\, \bar{F}_{ab}(p)\bar{F}_{a'b'}(-p)\,,\end{aligned} \tag{5.11}$$

where $p^2 = p_a p_b g^{ab}$. There are obvious modifications due to the appropriate expansion of Λ_{eff}^2 if one approaches the border of the IR regime (5.8).

To compare this with the geometrical results, we must take into account the different regularizations used in the heat-kernel expansion (3.6) and in the above one-loop computation. It was shown in [5] that these regularizations agree if we replace Λ^2 with $2\Lambda^2$ in the one-loop computation above[7]. We then find complete agreement with the result (4.16) obtained using the geometrical point of view. Notice in particular that the induced gravitational action is nontrivial even in the case of e.g. $N=1$ supersymmetry. This is now understood in terms of induced gravity, and full cancellation is obtained only in the case of $N=4$ supersymmetry. This will be discussed below.

Cancellations and supersymmetry It is very interesting to compare the fermionic and the bosonic contribution to the gravitational action. As is well-known [16, 19], we note that the fermionic contribution to the one-loop effective action in NC gauge theory does not quite cancel the scalar contribution, due to (5.11). This means that even in supersymmetric cases some UV/IR mixing may remain. From the geometrical point of view, this terms corresponds to a gravitational action $\mathrm{tr}\,\mathscr{E}\,\tilde\Lambda^2 = -2R[\tilde G]\,\tilde\Lambda^2 + \dots$, so that the cutoff $\tilde\Lambda^2$ should be interpreted as effective gravitational constant $\frac{1}{G}$. This is completely analogous to the commutative case, where the gravitational term (3.14) is induced. The remaining UV/IR mixing term cancels only in the case of $N=4$ supersymmetry. We can therefore identify $\tilde\Lambda$ as the scale of $N=4$ SUSY breaking (assuming such a model), above which the model is finite. These observations strongly suggest that for the model to be well-defined at the quantum level, $N=4$ SUSY is required above the gravity scale i.e. the Planck scale. This is realized by the IKKT model [10] on a NC background.

6. Discussion and outlook

In this paper, fermions are studied in the framework of emergent noncommutative gravity, as realized through matrix models of Yang-Mills type. The matrix model strongly suggests a particular fermionic term in the action, corresponding to a specific coupling to a background geometry with nontrivial metric $\tilde G_{\mu\nu}$. This coupling is similar to the standard coupling of fermions to a gravitational background, except that the spin connection vanishes in the preferred coordinates associated with the matrix model.

The main result of this paper is that in spite of this unusual feature, the resulting fermionic action is very reasonable, and properly describes fermions coupled to emergent gravity. In the point particle limit, fermions propagate along the appropriate trajectories, albeit with a different rotation of the spin. At the quantum level, we find an induced gravitational action which includes the expected Einstein-Hilbert term with a modified coefficient, as well as an additional term for a scalar density reminiscent of a dilaton. There are further terms which vanish for on-shell geometries. We conclude that the framework of emergent gravity does extend to fermions in a reasonable manner, and might well provide - in a suitable extension - a physically viable theory of gravity.

In a second part of the paper, we compare this induced gravitational action with the well-known UV/IR mixing in NC gauge theory due to fermions. Generalizing the results in [5] for scalar fields, we find as expected that the UV/IR mixing can be explained precisely by the gravitational point of view. This also provides a nice understanding for the fact that some UV/IR mixing

[7]while this was strictly speaking established only for the bosonic case, the argument should extend to the fermionic case without difficulties.

PoS(BHs, GR and Strings)037

remains in supersymmetric cases, and only disappears for $N = 4$ supersymmetry. The reason is that a gravitational action is induced even in supersymmetric cases, except in $N = 4$ SUSY. This in turn leads to the conjecture that the gravitational constant should be related to the scale of $N = 4$ SUSY breaking, which is quite reasonable. All of these findings suggest that the IKKT model on a noncommutative background [9–12] should be the most promising candidate for a realistic version of emergent gravity. These issues will be discussed in more detail elsewhere.

Acknowledgments The work of D.K. was supported by the FWF project P20017, and the work of H.S. was supported in part by the FWF project P18657 and in part by the FWF project P20017.

References

[1] S. Doplicher, K. Fredenhagen, and J. Roberts, *The Quantum structure of space-time at the Planck scale and quantum fields*, *Commun. Math. Phys.* **172** (1995) 187 [arXiv:hep-th/0303037].

[2] R. Szabo, *Quantum Field Theory on Noncommutative Spaces*, *Phys. Rept.* **378** (2003) 207, [arXiv:hep-th/0109162].

[3] M. Douglas, N. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977, [arXiv:hep-th/0106048].

[4] H. Steinacker, *Emergent Gravity from Noncommutative Gauge Theory*, *JHEP* **12**, (2007) 049; [arXiv:0708.2426 [hep-th]].

[5] H. Grosse, H. Steinacker and W. Wohlgenannt, *Emergent Gravity, Matrix Models and UV/IR Mixing*, [arXiv:hep-th/0802.0973].

[6] D. Klammer and H. Steinacker, *Fermions and Emergent Noncommutative Gravity*, *JHEP* **0808** (2008) 07 [arXiv:0805.1157[hep-th]].

[7] H. Steinacker, *Emergent Gravity and Noncommutative Branes from Yang-Mills Matrix Models*, *Nucl. Phys. B* **810** (2009) 1, [arXiv:0806.2032[hep-th]].

[8] H. Steinacker, *Covariant Field Equations, Gauge Fields and Conservation Laws from Yang-Mills Matrix Models*, [arXiv:0812.3761[hep-th]].

[9] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, *Noncommutative Yang-Mills in IIB matrix model*, *Nucl. Phys. B* **565** (2000) 176 [arXiv:hep-th/9908141].

[10] N. Ishibashi, H. Kawai, Y. Kitayawa, A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys. B* **498** (1997) 467 [arXiv:hep-th/9612115]

[11] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *String scale in noncommutative Yang-Mills*, *Nucl. Phys. B* **583** (2000) 159 [arXiv:hep-th/0004038].

[12] Y. Kitazawa and S. Nagaoka, *Graviton propagators on fuzzy G/H*, *JHEP* **0602** (2006) 001 [arXiv:hep-th/0512204]. Y. Kitazawa and S. Nagaoka, *Graviton propagators in supergravity and noncommutative gauge theory*, *Phys. Rev. D* **75** (2007) 046007 [arXiv:hep-th/0611056].

[13] H. Steinacker and G. Zoupanos, *Fermions on spontaneously generated spherical extra dimensions*, *JHEP* **0709** (2007) 017 [arXiv:0706.0398 [hep-th]].

[14] P. B. Gilkey, *Invariance theory, the heat equation and the Atiyah-Singer index theorem*, Wilmington, Publish or Perish, 1984.

[15] D.V. Vassilevich, *Heat kernel expansion: User's manual*, *Phys. Rept.* **388** (2003) 279 [arXiv:hep-th/0306138].

[16] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, *JHEP* **0012** (2000) 002 [arXiv:hep-th/0002075].

[17] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **0002** (2000) 020 [arXiv:hep-th/9912072].

[18] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on* $\mathbb{R}^4$, [arXiv:hep-th/9912167].

[19] V. V. Khoze and G. Travaglini, *Wilsonian effective actions and the IR/UV mixing in noncommutative gauge theories*, *JHEP* **0101**, 026 (2001) [arXiv:hep-th/0011218].

[20] M. Van Raamsdonk, *The meaning of infrared singularities in noncommutative gauge theories*, *JHEP* **0111** (2001) 006 [arXiv:hep-th/0110093].

[21] A. Armoni and E. Lopez, *UV/IR mixing via closed strings and tachyonic instabilities*, *Nucl. Phys. B* **632** (2002) 240 [arXiv:hep-th/0110113]; A. Armoni, E. Lopez and A. M. Uranga, *Closed strings tachyons and non-commutative instabilities*, *JHEP* **0302** (2003) 020 [arXiv:hep-th/0301099].

[22] S. Sarkar and B. Sathiapalan, *Aspects of open-closed duality in a background B-field*, *JHEP* **0505** (2005) 062 [arXiv:hep-th/0503009]; S. Sarkar and B. Sathiapalan, *Aspects of open-closed duality in a background B-field. II*, *JHEP* **0511** (2005) 002 [arXiv:hep-th/0508004]; S. Sarkar, *Closed string exchanges on C**2/Z(2) in a background B-field*, *JHEP* **0605** (2006) 020 [arXiv:hep-th/0602147]; S. Sarkar, *UV / IR mixing in noncommutative field theories and open closed string duality*, *Int. J. Mod. Phys. A* **21** (2006) 4763 [arXiv:hep-th/0606002].

PoS(BHs, GR and Strings)037