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Holography and Anomalies *

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Abstract

The calculation of trace anomalies for conformal field theories theories which have a holographic dual is described. The method does not involve the solution of the equations of motion of the dual, gravitational theory. This holds for both the bulk trace anomalies and the surface(Graham-Witten) anomalies. A new proof of the universal formula for the type A trace anomaly is presented.

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1. Introduction

In even dimensions trace anomalies give important information about conformal field theories . Coupling the theory to an external space time metric the information about all the energy momentum tensor correlators is encoded in the generating functional. The conformal invariance of the theory ,i.e. the Ward identities following from the tracelessness of the energy momentum tensor is equivalent to the requirement that the generating functional is invariant under a local rescaling of the metric(Weyl invariance).

The trace anomalies are a violation of this requirement by local functionals which cannot be removed by the variation of a local counterterm . The possible terms which can appear were classified and belong to two classes:

a)type A given by the Euler characteristic in the respective , even dimension

b)type B whose number increases with the dimension and which are local Weyl invariant functionals

The above , "bulk" trace anomalies were generalized to other (surface) observables , in [1] .These new , Graham-Witten anomalies exist whenever the embedded submanifold has even dimension.

The calculation of trace anomalies in an interacting conformal theory is a nontrivial task . In superconformal theories the calculation is made easier by the relation of the trace anomalies to various chiral anomalies.

With the advent of holography [2] a new method for the calculation of trace anomalies became available through a classical calculation in the dual (super)gravity theory . In [3] such a calculation was performed giving impressive agreement with the known values of the anomalies in the conformal field theory . The calculation involved a systematic expansion of the solution of the equation of motion .

In the present contribution , based on [4] we review a new method to calculate trace anomalies in a holographic setup . The calculation does not require the solution of the equations of motion. The anomaly is given by a boundary term . This allows the calculation of the anomalies for arbitrary actions . In particular we obtain a universal formula for the type A anomaly for a general gravitational action in an arbitrary even dimension . The method is generalized to include also Graham-Witten anomalies.

The contribution is organized as follows:

-in Section 2 we describe the new method and apply it to the calculation of bulk trace anomalies and Graham-Witten anomalies

-in Section 3 we discuss the possible back reaction of the terms producing the Graham-Witten anomalies on the calculation of bulk trace anomalies

-in Section 4 we give a new proof for the universal formula describing the type A anomaly for arbitrary actions

2. Holographic anomalies

In this section we present a general scheme to compute holographic conformal anomalies. It is very much like the computation of the $SU(4)$ $\mathcal{R}$-current anomaly presented in [5]. The anomaly is the boundary term generated by a suitably chosen local symmetry transformation. In the case of the $\mathcal{R}$-current this is a $SU(4)$ gauge transformation and the boundary term, which is the $\mathcal{R}$-current anomaly of the CFT on the boundary ($\mathcal{N} = 4$ SYM) is due to the $SU(4)$ Chern-Simons term present in the 5d gauged supergravity that arises when one compactifies type IIB string theory on S^5. For the conformal anomalies the appropriate transformations are the so-called PBH-transformations, a subgroup of five dimensional diffeomorphisms introduced in [6] and reviewed in the following. This treatment of the conformal anomalies does not require the solution of the equations of motion and it does not depend on the introduction of a cutoff. This gives us confidence on the generality of their structure when we compare it with the results in the field theory.

In the first part of this section we deal with those anomalies origination from the bulk gravitational action. In the second part we extend the discussion to the trace anomalies originating from the Dirac-Nambu-Goto ("DNG" in the following) piece of the action ("Graham-Witten anomalies").

2.1. Anomalies from the bulk

We start with trace anomalies in the bulk. Besides giving a general illustration of the new way to calculate trace anomalies the explicit results will be used in the following for an alternative holographic representation of the anomalous pieces in the EE.

Consider a generic gravitational bulk action

$$S = \int_M \sqrt{G} f(R)\, d^{d+1}X \tag{2.1}$$

where f is an arbitrary *scalar* function of the curvature and its derivatives. We require that (2.1) admits AdS_{d+1} as a solution to the equations of motion: this imposes a mild inequality on the coefficients in $f(R)$.

We choose coordinates $X^\mu = (x^i, \rho)$ such that $\rho = 0$ is the boundary of AdS_{d+1} where the dual CFT lives. It is coupled to a metric (source for its energy momentum tensor) $g^{(0)}_{ij}(x)$. For the bulk metric we choose the Fefferman-Graham (FG) gauge [7][3]

$$ds^2 = G_{\mu\nu}\, dX^\mu dX^\nu = \left(\frac{d\rho}{2\rho}\right)^2 + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j \tag{2.2}$$

with $g_{ij}(x,0) = g^{(0)}_{ij}(x)$. For $g^{(0)}_{ij} = \eta_{ij}$ (2.2) is the metric of AdS_{d+1}, whose curvature radius we have set to one.

PBH (Penrose-Brown-Henneaux) transformations are those diffeomorphisms ξ^μ which preserve the FG-gauge [6], i.e. for which $\mathcal{L}_\xi G_{\rho\rho} = \mathcal{L}_\xi G_{\rho i} = 0$. The solution is parametrized by an arbitrary function $\sigma(x)$:[1]

$$\xi^\rho = -2\rho\sigma(x)\,, \qquad \xi^i = a^i(x,\rho) = \frac{1}{2}\int_0^\rho d\rho' g^{ij}(x,\rho')\partial_j\sigma(x) \tag{2.3}$$

In particular, $\delta_\sigma g^{(0)}_{ij} = 2\sigma g^{(0)}_{ij}$, i.e. $\sigma(x)$ is the parameter of Weyl rescalings of the boundary metric.

The group property for PBH transformations can be shown to be

$$\xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu + \delta_2 \xi_1^\mu - \delta_1 \xi_2^\mu = 0 \tag{2.4}$$

The last two terms are due to the dependence of the transformation parameters on $g_{ij}(x,\rho)$. The essential property of the PBH transformations is that on the boundary they coincide with the action of the Weyl group. Therefore in holography the Weyl group becomes embedded in the $d+1$ dimensional diffeomorphisms and the study of Weyl anomalies is reduced to an analysis of how diffeomorphisms act.

Under a bulk diffeomorphism the action (2.1) is invariant up to a boundary term

$$\delta_\xi(\sqrt{G} f) = \partial_\mu(\sqrt{G}\xi^\mu f) \tag{2.5}$$

$$\delta_\xi S = \int_{\partial M} d^d x \sqrt{G} f(R)\, \xi^\rho|_{\rho=0} = -2\int_{\partial M} d^d x\sqrt{G} f(R)\, \rho\, \sigma|_{\rho=0} \tag{2.6}$$

where in the second line we have restricted the diffeomorphism to a PBH transformation. The finite piece of this boundary term is the holographic Weyl anomaly.

[1] The choice of lower limit in the ρ' integral means that we do not consider diffeomorphisms of the boundary. They are of no interest here.

A comment is in order here: we consider passive diffeomorphism transformations which act on the fields rather than the coordinates. The reason for doing is that we want to keep the boundary fixed.

Following [7](see also [8]) we expand the metric as

$$g_{ij}(x,\rho)=\sum_{n=0}^{\infty}\overset{(n)}{g}_{ij}(x)\rho^n+\dots \tag{2.7}$$

The ... denote logarithmic terms ($\sim\log\rho$) which are present for even d. They do not play a role in our analysis. The integrand in (2.1) has likewise an expansion of the form [6]

$$\sqrt{G}f(R)=\sqrt{g^{(0)}}\rho^{-\frac{d}{2}-1}b(x,\rho)=\sqrt{g^{(0)}}\rho^{-\frac{d}{2}-1}\sum_{n=0}^{\infty}b_n(x)\rho^n \tag{2.8}$$

As was shown in [6], b and thus each b_n, satisfies the Wess-Zumino consistency condition

$$\int d^dx\sqrt{g^{(0)}}(\sigma_1\delta_{\sigma_2}-\sigma_2\delta_{\sigma_1})b=0 \tag{2.9}$$

A simple way to see this is as follows (cf. the Appendix). For $\mathcal{O}=\sqrt{G}f(R)$ one derives $\delta_{\sigma_1}\mathcal{O}=\partial_\mu(\xi_1^\mu\mathcal{O})$ and $[\delta_{\sigma_2},\delta_{\sigma_1}]\,\mathcal{O}=0$ by virtue of the group property (2.4).

On-shell b_n is a local, covariant expression constructed from $g^{(0)}_{ij}$. For $d=2n$ it is the coefficient of the boundary term at $\mathcal{O}(\rho^{-1})$ and represents the Weyl anomaly of the dual $2n$-dimensional CFT. The bulk gravitational action thus plays the same role for the Weyl anomaly of the CFT as does the CS term for the $\mathcal{R}$-current anomaly.

For general $d=2n$, the on-shell b_n depends also on some of the derivatives of g_{ij} at $\rho=0$ and not only on the boundary value $g^{(0)}_{ij}$. These higher derivatives need some information contained in the equation of motion. However, for $d=4$, b_2 can be computed without the need to solve the equations of motion. As we will show momentarily, in $d=4$, besides $g^{(0)}$ only the coefficient $g^{(1)}$ of the FG expansion of the bulk metric appears. This second coefficient is universal because it is uniquely determined by its behavior under PBH transformations and locality [6]:

$$\overset{(1)}{g}_{ij}=\frac{1}{(d-2)}\Big(R_{ij}-\frac{1}{2(d-1)}g_{ij}R\Big) \tag{2.10}$$

where R is the curvature of $g^{(0)}$ and $g_{ij}\equiv g^{(0)}_{ij}$. The universality of $g^{(1)}$ will be spoiled if we take back reaction into account, as we will do in section 5.

On dimensional grounds b_n can at most be linear in $g^{(n)}$, as both carry length-dimension $-2n$ ($\rho \sim \text{length}^2$). By assumption, $f(R)$ is such that Anti-de-Sitter space is a solution of the equations of motion. Expand the action around this solution. In this expansion the term linear in the fluctuations around the AdS-metric can only be a total derivative (or vanish altogether). Consider the terms $\nabla^\mu \nabla^\nu \delta G_{\mu\nu}$ and $\Box \mathrm{tr}\delta G$. For fluctuations $\delta G_{ij} = \rho^{n-1} g^{(n)}_{ij}$ the possibly dangerous terms, i.e. those which might contribute to b_n, are of the type $\rho^n \mathrm{tr}\, g^{(n)}$. It is straightforward to show that their coefficient is zero for $d = 2n$. Higher derivative terms in the variation of the action will involve coefficients $g^{(m)}$ for $m < n$. We stress that the above argument showing that in $d = 2n$ $g^{(n)}$ does not appear in the b_n term in the expansion of the action does not prevent the participation of $g^{(n)}$ in the equation of motion in the usual way [3] of calculating the anomalies.
To summarize, to find the Weyl anomaly of the $d = 2n$-dimensional dual CFT all we have to do is to extract the coefficient of $1/\rho$ is the expansion of the gravitational action. In $d = 4$ this only involves $g^{(0)}$ and $g^{(1)}$ and is thus completely fixed. On general grounds this can always be written as a linear combination $a\, E_4 - c\, C^2 + e\, \Box R$ where C^2 is the square of the Weyl tensor, E_4 the Euler density (i.e. $\int_M \sqrt{g} E_4 \propto \chi(M)$). In the Appendix we will rederive the general expression for a, already found, by different means, in [6].

2.2. Graham-Witten anomalies

In this subsection we will study the trace anomalies for submanifolds ("Graham-Witten" anomalies) . We will follow the method used in the previous subsection for bulk anomalies which does not depend on the equations of motion and does not need a cutoff. This will enable us to discuss the general structure of the Graham-Witten anomalies and the anomalies produced by more general submanifold actions having the same symmetries as DNG. For the DNG action our method reproduces the result of [1].
We start with a classification of the possible Graham-Witten anomalies for the case when the submanifold has dimension 2 embedded in a manifold of dimension d.
Candidates for the Graham-Witten anomaly are solutions to the Wess-Zumino consistency condition satisfying the following conditions: they should be local expressions constructed from the second fundamental form and from curvatures, linear in the Weyl parameter σ; they should have two derivatives (appropriate for the two dimensional case considered here); they should be cohomologically non-trivial. Among those we distinguish between type A which satisfy the WZ condition non-trivially and type B which satisfy them trivially having expressions which are Weyl invariant.

To find the candidates for the anomaly, we will need, besides well-known expressions for the Weyl-transformation of the curvature tensors, the transformation of the second fundamental form and of its trace:

$$\delta_\sigma K^i_{ab} = -h_{ab} P^{ij} \partial_j \sigma \,, \qquad \delta_\sigma K^i \equiv \delta_\sigma (h^{ab} K^i_{ab}) = -2\sigma K^i - k\, P^{ij} \partial_j \sigma \tag{2.11}$$

where $P^{ij} = g^{ij} - h^{ij} = g^{ij} - h^{ab} \partial_a X^i \partial_b X^j$ projects to the normal space of the hypersurface. It is then straightforward to show that the following list exhausts all possible Weyl invariant expressions:

$$\sqrt{h} h^{ac} h^{bd} C_{abcd}, \qquad \sqrt{h}(\mathrm{tr}(K^i K^j) - \frac{1}{2} K^i K^j) g_{ij}, \quad \sqrt{h}(K^i K^j g_{ij} - 4h^{ab} \overset{(1)}{g}_{ab} + 2R^{(2)}) \tag{2.12}$$

where $R^{(2)}$ is the curvature scalar of the induced metric, C_{abcd} the pull-back of the bulk Weyl tensor and $\overset{(1)}{g}_{ab} = \partial_a \overset{(0)}{X}{}^i \partial_b \overset{(0)}{X}{}^j \overset{(1)}{g}_{ij}$ is the pull-back of (2.10). However, with the help of the Gauss-Codazzi equation one shows that

$$h^{ac} h^{bd} C_{abcd} = R^{(2)} - 2h^{ab} \overset{(1)}{g}_{ab} + \tfrac{1}{2} K^i K^j g_{ij} - (\mathrm{tr}(K^i K^j) - \tfrac{1}{2} K^i K^j) g_{ij} \tag{2.13}$$

i.e. the above Weyl invariant expressions are not all independent. We will choose the first two as a basis.

Candidates for the type A anomaly are

$$\sqrt{h} R^{(2)} \sigma, \quad \sqrt{h} K^i \partial_i \sigma, \quad \sqrt{h} \Box \sigma \tag{2.14}$$

where these expressions are restricted to the submanifold. The first is the well-known trace anomaly in $d = 2$. The second one, on the other hand, is trivial as it can be written as the Weyl variation of a local term:

$$\delta_\sigma (K^i K^j g_{ij}) = -4 K^i \partial_i \sigma \tag{2.15}$$

where one uses (2.11) and $K_i h^{ij} = 0$. The third one is again trivial being the variation of R the scalar bulk curvature restricted to the submanifold.

We thus arrive at the following basis of GW anomalies when the submanifold is two dimensional:

$$\begin{aligned} &\text{type A:} \quad \sqrt{h} R^{(2)} \sigma \\ &\text{type B:} \quad \sqrt{h} h^{ac} h^{bd} C_{abcd}\, \sigma, \quad \sqrt{h} g_{ij} (\mathrm{tr}(K^i K^j) - \tfrac{1}{2} K^i K^j)\, \sigma \end{aligned} \tag{2.16}$$

In terms of this basis the anomaly found by Graham and Witten, who considered the case where the hypersurface degrees of freedom in the CFT have their holographic description in terms of the DNG action, is

$$\mathcal{A}_{GW} = \frac{1}{4}\int_{\partial\Sigma} d^2x\,\sqrt{h}\Big(h^{ac}h^{bd}C_{abcd} - g_{ij}(\mathrm{tr}(K^iK^j) - \tfrac{1}{2}K^iK^j) - R^{(2)}\Big)\sigma \tag{2.17}$$

We proceed now to an analysis of the Graham-Witten anomalies in a holographic setup. We will leave the dimensions of space-time d and of the submanifold k general and at the end of the discussion we will go back to the specific $k=2$ case.

In the holographic realization we have to consider a $(k+1)$-dimensional submanifold Σ embedded into the $(d+1)$-dimensional bulk M such that it ends on a k-dimensional submanifold $\partial\Sigma$ on the d-dimensional boundary. Denote, as before, the bulk coordinates by $X^\mu = (x^i, \rho)$ and the world-volume coordinates by $\tau^\alpha = (x^a, \tau)$ with $i = 1, \ldots, d$ and $a = 1, \ldots, k$. The embedding is $X^\mu : \Sigma \mapsto M$, i.e. $X^\mu = X^\mu(\tau^\alpha)$.

We assume that the action contains in addition to the usual bulk component (2.1) another component defined on the $k+1$ submanifold. The additional piece is invariant both under usual bulk diffeomorphisms and under reparametrizations of the world volume.

We want first to generalize the PBH transformations (2.3) to this new situation where we have two linked gauge invariances.

We first fix the gauge. For the bulk we go to FG gauge (2.2) as before. The reparametrizations of Σ are fixed by imposing

$$\tau = \rho \qquad \text{and} \qquad h_{a\tau} = 0 \tag{2.18}$$

Under a reparametrization of Σ, parametrized by $\tilde{\xi}^\alpha$, X^μ transforms as a scalar, i.e. $\delta_{\tilde{\xi}} X^\mu = \tilde{\xi}^\alpha \partial_\alpha X^\mu$. In particular $\delta_{\tilde{\xi}}\rho = \tilde{\xi}^\alpha\partial_\alpha\rho = \tilde{\xi}^\tau = 0$ after fixing the $\tau = \rho$ gauge. Also, if we require that $\delta_{\tilde{\xi}} h_{a\tau} = 0$, we find that $\tilde{\xi}^a$ must be independent of τ. This means that all world-volume reparametrizations of Σ are fixed except the ones acting on $\partial\Sigma$.

We perform now a target space PBH transformations $\delta\rho = -2\rho\sigma$, $\delta x^i = a^i$ (cf. (2.3)). To stay in the $\tau = \rho$ gauge we must make a compensating world-volume diffeomorphism

$$\tilde{\xi}^\tau = -2\tau\sigma \tag{2.19}$$

The resulting change of the induced metric must be compensated in order to keep $h_{a\tau} = 0$:

$$\delta h_{a\tau} = \partial_a\tilde{\xi}^\tau h_{\tau\tau} + \partial_\tau\tilde{\xi}^b h_{ab} = 0 \tag{2.20}$$

With $\tilde{\xi}^\tau = -2\tau\sigma$ this can be integrated to

$$\tilde{\xi}^a = 2\int_0^\tau d\tau' \; \tau' h_{\tau\tau} h^{ab} \partial_b \sigma \tag{2.21}$$

where all functions in the integrand depend on τ' (through $X^i(x^a, \tau)$). Here

$$h_{\tau\tau} = \partial_\tau X^\mu \partial_\tau X^\nu G_{\mu\nu} = \frac{1}{4\tau^2} + \frac{1}{\tau} \partial_\tau X^i \partial_\tau X^j g_{ij}(X, \tau) \tag{2.22}$$

Expand g_{ij} in powers of ρ (cf. (2.7)) and X^i in powers of τ (with $\tau = \rho$)

$$X^i(\tau, x^a) = \overset{(0)}{X}{}^i(x^a) + \tau \, \overset{(1)}{X}{}^i(x^a) + \tau^2 \, \overset{(2)}{X}{}^i(x^a) + \dots \tag{2.23}$$

With the definition

$$h_{ab} = \frac{1}{\rho} \partial_a X^i \partial_b X^j g_{ij}(X) = \frac{1}{\rho} \partial_a \overset{(0)}{X}{}^i \, \partial_b \overset{(0)}{X}{}^j \, \overset{(0)}{g}{}_{ij}(\overset{(0)}{X}) + \mathcal{O}(1) \equiv \frac{1}{\rho} \, \overset{(0)}{h}{}_{ab}(X) + \mathcal{O}(1) \tag{2.24}$$

we obtain from (2.21)

$$\tilde{\xi}^a = \frac{1}{2} \tau \, \overset{(0)}{h}{}^{ab} \partial_b \sigma + \mathcal{O}(\tau^2) \tag{2.25}$$

We can now determine how X^i changes under PBH. It transforms as

$$\delta X^i = \tilde{\xi}^\alpha \partial_\alpha X^i - a^i \tag{2.26}$$

with a^i from (2.3). This implies

$$\begin{aligned} &\delta \, \overset{(0)}{X}{}^i = 0 \\ &\delta \, \overset{(1)}{X}{}^i = -2\sigma \, \overset{(1)}{X}{}^i + \frac{1}{2} \, \overset{(0)}{h}{}^{ab} \partial_a \overset{(0)}{X}{}^i \, \partial_b \sigma - \frac{1}{2} \, \overset{(0)}{g}{}^{ij} \partial_i \sigma \end{aligned} \tag{2.27}$$

which is solved by

$$\overset{(1)}{X}{}^i = \frac{1}{2k} K^i \tag{2.28}$$

where

$$K^i = \overset{(0)}{h}{}^{ab} K^i_{ab} = \overset{(0)}{h}{}^{ab} \Big(\partial_a \partial_b \overset{(0)}{X}{}^i - \overset{(0)}{\Gamma}{}^c_{ab} \partial_c \overset{(0)}{X}{}^i + \overset{(0)}{\Gamma}{}^i_{jk} \partial_a \overset{(0)}{X}{}^j \partial_b \overset{(0)}{X}{}^k \Big) \tag{2.29}$$

is the trace of the second fundamental form, i.e. the extrinsic curvature, of the embedded submanifold $\partial\Sigma$.

We remark that the universality of $\overset{(1)}{X}{}^i$ is analogous to the universality of $g^{(1)}$, c.f. (2.10). The higher $X^{(n)}$, just like the higher $g^{(n)}$, are not universal, the reason being that their behavior under PBH transformations admits homogeneously transforming terms [2].

We succeeded therefore to put also this more general situation, with the action having two components, into a framework similar to the one we had for the bulk action alone. The action of the Weyl transformations on the boundary is embedded into bulk diffeomorphisms and world volume reparametrizations (2.19), (2.21). Moreover, besides the $g^{(1)}$ component of the bulk metric also the $X^{(1)}$ component of the embedding have a universal form determined by the PBH transformations.

Using these results we are now prepared to analyze the Graham-Witten anomalies, i.e. the transformation properties of the additional piece of the action when the metric $g^{(0)}$ is Weyl transformed.

Following [1] we consider the case where the dynamics of the submanifold is governed by the DNG action:

$$S = \int_\Sigma \sqrt{h} \tag{2.30}$$

The generalization to arbitrary world-volume actions is straightforward. A particular case will be considered at the end of this section. The DNG action of Σ is invariant under passive world-volume diffeomorphisms up to a boundary term. The finite part of this boundary term (at $\tau = 0$) is the Graham-Witten anomaly

$$\mathcal{A} = \int_{\partial\Sigma} \sqrt{\det h}\, \tilde{\xi}^\tau|_{\mathrm{finite}} \tag{2.31}$$

Given that the τ-expansion of $X^{(1)}$ is universal only up to the first non-trivial order, we will be able to compute the anomaly, without further input from the equations of motion, only for $k = 2$. This is also the relevant dimension for the discussion of the EE in a four dimensional CFT.

We now evaluate (2.31). We need

$$\tilde{\xi}^\tau = -2\tau\sigma(X) = -2\tau\sigma(\overset{(0)}{X}) - 2\tau^2 \partial_i\sigma(\overset{(0)}{X})\, \overset{(1)}{X}{}^i + \mathcal{O}(\tau^3) \tag{2.32}$$

[2] For $g^{(2)}_{ij}$ this is e.g. $g^{(0)}_{ij} C^2$ and for $X^{(1)}$ any one of the terms in (2.12) (without the $\sqrt{h}$ factor), multiplied by $X^{(0)}$.

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and $\det(h) = h_{\tau\tau}\det(h_{ab})$ with

$$
\begin{aligned}
h_{\tau\tau} &= \frac{1}{4\tau^2} + \frac{1}{\tau}\overset{(1)}{X}{}^i \overset{(1)}{X}{}^j \overset{(0)}{g}_{ij} + \ldots = \frac{1}{4\tau^2}(1 + 4\tau \overset{(1)}{X}{}^i \overset{(1)}{X}{}^j \overset{(0)}{g}_{ij}) + \ldots \\
\det h_{ab} &= \frac{1}{\tau^k}\det(\overset{(0)}{h}_{ab})(1 + \tau \overset{(0)}{h}{}^{ab} \overset{(1)}{h}_{ab}) + \ldots
\end{aligned}
\tag{2.33}
$$

where

$$
\begin{aligned}
\overset{(1)}{h}_{ab} &= \partial_a \overset{(1)}{X}{}^i \partial_b \overset{(0)}{X}{}^j \overset{(0)}{g}_{ij} + \partial_a \overset{(0)}{X}{}^i \partial_b \overset{(1)}{X}{}^j \overset{(0)}{g}_{ij} + \partial_a \overset{(0)}{X}{}^i \partial_b \overset{(0)}{X}{}^j \overset{(1)}{g}_{ij} + \partial_a \overset{(0)}{X}{}^i \partial_b \overset{(0)}{X}{}^j \partial_k \overset{(0)}{g}_{ij} \overset{(1)}{X}{}^k \\
&= \overset{(1)}{g}_{ab} - \frac{1}{k} K^i K^j_{ab} \overset{(0)}{g}_{ij}
\end{aligned}
\tag{2.34}
$$

With the help of these expressions we finally find, for $k = 2$,

$$
\mathcal{A}_{GW} = \frac{1}{8}\int_{\partial\Sigma} d^2x \sqrt{\det h}\Big((g_{ij}K^iK^j - 4h^{ab} \overset{(1)}{g}_{ab})\sigma - 2K^i\partial_i\sigma\Big) \tag{2.35}
$$

where h and g now denote the boundary metrics. Eq.(2.35) is in agreement with [1]. As remarked above, the last term is cohomologically trivial. The rest can be written in terms of the basis (2.16). The result was already given in (2.17).

In analogy to allowing general bulk actions, as we did in Section 3.1, the dynamics of the hypersurface might be given by generalizations of the DNG action

$$
S = \int_\Sigma \sqrt{h} f(R^\Sigma, K, X, \ldots) \tag{2.36}
$$

where f is a scalar function. In this case the GW anomaly will also change:

$$
\mathcal{A} = \int_{\partial\Sigma} \sqrt{\det h} f \tilde{\xi}^\tau|_{\text{finite}} \tag{2.37}
$$

The fact that the anomaly satisfies the WZ condition is again a consequence of the group property of the PBH transformations.

As a particular example we consider the action

$$
S = \int_\Sigma \sqrt{h} f(R^{(\Sigma)}) \tag{2.38}
$$

where $R^{(\Sigma)}$ is the Ricci scalar computed with $h_{\alpha\beta}$ with the expansion

$$
R^{(\Sigma)} = 6 + \left(R^{(2)} - 2 \overset{(0)}{h}{}^{ab} \overset{(1)}{g}_{ab} + \tfrac{1}{2} \overset{(0)}{g}_{ij} K^i K^j\right)\tau + \ldots \tag{2.39}
$$

For instance, if we choose $f(R^{(\Sigma)}) = 1 - \frac{1}{2}R^{(\Sigma)}$ the GW-anomaly is purely type A. Alternatively we can choose an action for which the $R^{(2)}$ anomaly vanishes .

3. The back reaction and the Graham-Witten anomalies

The holographic realization as used above involved smooth bulk metrics. We would like to examine if in the holographic realization a singular metric due to the back reaction of the DNG could appear and if they may have an influence on the anomaly calaculation. Of course the boundary value of the metric $g^{(0)}$ is smooth but the solution in the bulk can acquire singular components if the back reaction of the DNG component of the action is taken into account.

In the previous section we have treated the dynamics of the bulk independently producing a solution $g_{ij}(x,\rho)$. The embedded surface evolved in this bulk background following the dynamics prescribed by the DNG action. In this section we will take back reaction of the hypersurface on the bulk into account, solve the coupled equations of motion and evaluate the $\mathcal{O}(\rho^{-1})$ term of the on-shell action. According to [3] this computes the Weyl anomaly, in addition to the contribution coming from the DNG piece.

The total action is

$$S=\int_{\mathcal{M}} d\rho\, d^dx\sqrt{G}(R-2\Lambda)+\int_{\Sigma} d\tau\, d^kx\sqrt{h_{\alpha\beta}} \tag{3.1}$$

The equation of motion for the metric can be cast in the form

$$\sqrt{G}\left(\frac{1}{2}(R-2\Lambda)G^{\mu\nu}-R^{\mu\nu}\right)=\Delta^{\mu\nu} \tag{3.2}$$

with

$$\Delta^{\mu\nu}=-\frac{1}{2}\int_{\Sigma}\sqrt{h}h^{\mu\nu}\delta^{(d)}(x-X(\tau))\,\delta(\rho-\tau) \tag{3.3}$$

Inserting its trace into the action results in

$$S=2d\int_{\mathcal{M}}\sqrt{G}+\frac{d-k-2}{d-1}\int_{\Sigma}\sqrt{h} \tag{3.4}$$

where the second term vanishes if $\mathrm{codim}(\Sigma)=2$, which is the case of interest where $d=4$ and $k=2$. However, the DNG piece of the action will feed back, through the equations of motion, into the coefficients $\overset{(n)}{g}_{ij}$.

Using the FG expansion (2.7) one finds the following expression for the on-shell action at $\mathcal{O}(\rho^{-1})$ [3]

$$\frac{1}{4}\sqrt{\det g^{(0)}}\left(\mathrm{tr}\ \overset{(2)}{g}-\frac{1}{2}\mathrm{tr}(\overset{(1)}{g}{}^2)+\frac{1}{4}(\mathrm{tr}\ \overset{(1)}{g})^2\right) \tag{3.5}$$

For $g^{(1)}$ one finds, by solving the (ij)-component of (3.2) at leading non-trivial order in its ρ-expansion

$$\overset{(1)}{g}_{ij} = \frac{1}{2}\Big(\overset{(0)}{R}_{ij} - \frac{1}{6} \overset{(0)}{R} \overset{(0)}{g}_{ij} \Big) + \delta \overset{(1)}{g}_{ij} \tag{3.6}$$

where

$$\delta \overset{(1)}{g}_{ij} = -\frac{1}{4\sqrt{g^{(0)}}} \int d\tau \sqrt{\overset{(0)}{h}} \Big(\overset{(0)}{h}_{ij} - \frac{2}{3} \overset{(0)}{g}_{ij} \Big) \delta^{(d)}(x - \overset{(0)}{X}) \tag{3.7}$$

and

$$\overset{(0)}{h}_{ij} = \overset{(0)}{g}_{ik} \overset{(0)}{g}_{jl} \overset{(0)}{h}{}^{ab} \partial_a \overset{(0)}{X}{}^{k} \partial_b \overset{(0)}{X}{}^{l} \tag{3.8}$$

Note that $\delta g^{(1)}$ is Weyl invariant and $g^{(1)}$ is no longer universal. Consistency with PBH and dimensional arguments restrict the most general nonuniversal addition to $g^{(1)}$, which would result for general bulk and hypersurface action to the above form, but with arbitrary coefficients for $h^{(0)}$ and $g^{(0)}$ in (3.7).
To find $\mathrm{tr}(g^{(2)})$ it suffices to solve the $(\rho\rho)$-component of (3.2) at lowest non-trivial order:

$$\mathrm{tr}(\overset{(2)}{g}) = \frac{1}{4}\mathrm{tr}(\overset{(1)}{g}{}^{2}) - \frac{1}{32}\frac{1}{\sqrt{g^{(0)}}} \int d\tau \sqrt{\overset{(0)}{h}} K^i K^j \overset{(0)}{g}_{ij} \delta^{(4)}(x - \overset{(0)}{X}) \tag{3.9}$$

The expressions (3.6) and (3.9) represent singular contributions to the bulk metric solution. Using the singular contributions to linear order in the δ-function we will find the contributions to the Graham-Witten anomaly. Quadratic and higher order terms in the δ-functions require a regularization producing local counterterms which do not influence the anomalies.
We find for (3.5) for the case $d = 4, k = 2$

$$\frac{\sqrt{g^{(0)}}}{8}\Big((\mathrm{tr}\, \overset{(1)}{g})^2 - \mathrm{tr}(\overset{(1)}{g}{}^{2}) \Big)\Big|_{\mathrm{universal}} + \frac{1}{64} \int d\tau \sqrt{\overset{(0)}{h}} (K^i K^j \overset{(0)}{g}_{ij} - 4 \overset{(0)}{h}{}^{ab} \overset{(1)}{g}_{ab})\, \delta^{(d)}(x - \overset{(0)}{X}) \tag{3.10}$$

where the universal $g^{(1)}$ was given in (2.10). Again there was a crucial cancellation, related to the one observed above, for $\mathrm{codim}(\Sigma) = 2$.
Compare (3.10) to (2.35): we have shown that for the simplest bulk and hypersurface actions, taking into account the back-reaction leads to the same GW anomaly for the total action (3.1).
The above result has a simple explanation which will allow us to generalize the result for arbitrary bulk actions. The contributions to the bulk metric specified above once inserted in the bulk action to linear order in the δ-function produce a term localized on the submanifold. Moreover this term has the same symmetries as the DNG action.

Therefore we can use the procedure discussed in Section 3.2. The additional Graham - Witten anomaly is given by an expression analogous to (2.31) the DNG integrand being replaced by the term of the bulk action specified above. We will need therefore just the expansion to order τ^2 (order ρ^2 in our gauge) of the integrand in the first term in (3.1). Remembering that $g^{(2)}$ does not appear in the expansion we get the following terms (the curvatures are computed with $g^{(0)}$):

$$\mathrm{tr}(\overset{(1)}{g}{}^2) - (\mathrm{tr}\ \overset{(1)}{g})^2 - \overset{(1)}{g}_{ij}\, R^{ij} + \frac{1}{2} R\, \mathrm{tr}\ \overset{(1)}{g} \tag{3.11}$$

In the expression (3.11) we left out terms in which derivatives act on $g^{(1)}$. We will discuss them in the general setting.

Now, using (3.6) in (3.11) it is easy to verify that all the terms linear in $\delta g^{(1)}$ vanish without any need to specify the exact coefficients in $\delta g^{(1)}$. What is the reason for this vanishing? As we discussed in Section 3.2 an expression obtained by (2.31) satisfies automatically the Wess-Zumino condition. Independently of the exact form of the bulk action for dimensional reasons the only expressions which could appear in (3.11) linear in $\delta g^{(1)}$ are R_{ab} – the pullback of the Ricci curvature or R – the bulk scalar curvature restricted to the submanifold. Indeed they do appear in individual terms in (3.11). However once they are multiplied with the Weyl parameter σ it is easy to check that they do not fulfill the Wess-Zumino condition and therefore they must cancel in the full expression.

Finally we return to the derivative terms left out above. Again by a dimensional argument verified explicitly for the aforementioned terms these contributions have the form $\Box\sigma$ or $K^i\partial_i\sigma$, restricted to the submanifold. These expressions do satisfy the Wess-Zumino condition but they are cohomologically trivial being the variations of local expressions as we discussed in Section 3.2.

In conclusion, for an arbitrary bulk action in $d = 5$ and an arbitrary three dimensional DNG action the Graham-Witten anomalies remain unchanged after the back reaction on the bulk metric is included. This is a consequence of the fact that the Graham-Witten anomalies classified in section 3.2 cannot originate from the $g^{(1)}$ back reaction term the only one available in $d = 5$.

4. Derivation of the universal type A anomaly coefficient

In [6] it was shown that for any gravitational action (2.1) with has AdS_{2n+1} as a solution to the equations of motion, the coefficient a of the unique type A anomaly of dual CFT is

$$a_n = \frac{b_0}{2^{2n}(n!)^2} \tag{4.1}$$

where $b_0 = f(AdS)$. In [6] this was derived by looking at a conformally flat metric $g^{(0)}$ and solving the PBH-transformation equation for b_n. Here we will present an alternative derivation which uses the ideas of [9]. There it was observed that while the type B Weyl anomalies have a trivial descent, the unique (in any even dimensions) type A anomaly has a non-trivial descent.[3] These features might, in fact, serve as the defining distinction between the two classes of anomalies, which can also be applied to the hypersurface anomalies discussed in sect. 3.

We begin with a review of the results of [9]. Define

$$\mathcal{O}^{j_1\ldots j_p}_{12\ldots p+1} = \frac{4^p\, n!}{2^n(n-p)!}\sqrt{g}g_{i_1k_1}\ldots g_{i_pk_p}\epsilon^{i_1j_1\ldots i_nj_n}\epsilon^{k_1l_1\ldots k_nl_n} \times R_{i_{p+1}j_{p+1}k_{p+1}l_{p+1}}\cdots R_{i_nj_nk_nl_n}\sigma_{[1}\partial_{l_1}\sigma_2\ldots\partial_{l_p}\sigma_{p+1]} \tag{4.2}$$

$$\mathcal{O}_1 = \sigma_1\sqrt{g}\,E_{2n} \tag{4.3}$$

with

$$E_{2n} = \frac{1}{2^n}\epsilon^{i_1j_1\ldots i_nj_n}\epsilon^{k_1l_1\ldots k_nl_n}R_{i_1j_1k_1l_1}\cdots R_{i_nj_nk_nl_n} \tag{4.4}$$

the d-dimensional Euler density. The normalization is such that $E_{2n} = R^n + \ldots$. $\mathcal{O}_1$ is at the top of the descent which is

$$\delta_{[p+1}\mathcal{O}^{j_1\ldots j_{p-1}}_{1\ldots p]} = \partial_p\ \mathcal{O}^{j_1\ldots j_p}_{1\ldots p+1} \tag{4.5}$$

and

$$\mathcal{O}^{j_1\ldots j_n}_{12\ldots n+1} = 2^n(n!)^2\sqrt{g}\,\sigma_{[1}\nabla^{j_1}\sigma_2\ldots\nabla^{j_n}\sigma_{n+1]} \tag{4.6}$$

is at the bottom. In deriving (A.5) we need the Weyl variation of the Riemann tensor

$$\delta R_{ijkl} = 2\sigma R_{ijkl} + g_{ik}\nabla_j\nabla_l\sigma + g_{jl}\nabla_i\nabla_k\sigma - g_{il}\nabla_j\nabla_k\sigma - g_{jk}\nabla_i\nabla_l\sigma \tag{4.7}$$

[3] The descent of cohomologically trivial contributions stops after the second step.

The holographic version of the descent starts with the $d+1$ dimensi $\mathcal{O} = \sqrt{G}f(R)$. Under PBH

$$\delta_1 \mathcal{O} = \partial_\mu(\xi_1^\mu \mathcal{O}) \equiv \partial_\mu \mathcal{O}_1^\mu \tag{4.8}$$

If we define

$$\mathcal{O}_{1\ldots p}^{\mu_1\ldots\mu_p} = \xi_{[1}^{\mu_1} \cdots \xi_{p]}^{\mu_p} \mathcal{O} \tag{4.9}$$

we can show, using the group property (2.4)

$$\delta_{p+1} \mathcal{O}_{1\ldots p}^{\mu_1\ldots\mu_p} = \partial_{\mu_{p+1}} \mathcal{O}_{1\ldots p+1}^{\mu_1\ldots\mu_{p+1}} \tag{4.10}$$

Using (2.8) for the ρ-expansion of $\mathcal{O}$ and $\xi^\rho = 2\sigma\rho\,, \xi^i = \frac{1}{2}\rho g_{(0)}^{ij}\partial_j\sigma + \mathcal{O}(\rho^2)$ we find

$$\mathcal{O}_{1\ldots n+1}^{\rho j_1\ldots j_n} = \frac{1}{2^n}\sqrt{g}\, b_0\, \sigma_{[1} \nabla^{j_1}\sigma_2 \cdots \nabla\sigma_{n+1]}^{j_n} \tag{4.11}$$

Comparing this with (A.6) we conclude that the holographic type A Weyl anomaly in $d = 2n$ dimensions is $a_n E_{2n}$ with a_n as in (A.1).

References

[1] C. R. Graham and E. Witten, "Conformal anomaly of submanifold observables in AdS/CFT correspondence," Nucl. Phys. B **546**, 52 (1999) [arXiv:hep-th/9901021].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000), hep-th/9905111.

[3] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].

[4] A. Schwimmer and S. Theisen, "Entanglement Entropy ,Trace Anomalies and Holography," Nucl. Phys. B **801**, 1 (2008); [arxiv:hep-th /0802.1017]

[5] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[6] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, "Diffeomorphisms and holographic anomalies," Class. Quant. Grav. **17**, 1129 (2000) [arXiv:hep-th/9910267].

[7] C. Fefferman and R. Graham, "Conformal Invariants," Astèrisque, hors série, 1995, p.95.

[8] A. D. Rendall, "Asymptotics of solutions of the Einstein equations with positive cosmological constant," Annales Henri Poincare **5**, 1041 (2004) [arXiv:gr-qc/0312020].

[9] N. Boulanger, "General solutions of the Wess-Zumino consistency condition for the Weyl anomalies," JHEP **0707**, 069 (2007) [arXiv:0704.2472 [hep-th]]; "Algebraic Classification of Weyl Anomalies in Arbitrary Dimensions," Phys. Rev. Lett. **98**, 261302 (2007) [arXiv:0706.0340 [hep-th]].

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