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Past horizons in Robinson-Trautman spacetimes

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We exhibit properties of past quasi-local horizons in vacuum Robinson-Trautman spacetimes. Trapped surfaces have to intersect the surface $r = 2m$, which cannot be null unless g is the Schwarzschild metric. The only Robinson-Trautman metric admitting a past nonexpanding horizon is the Schwarzschild metric and the C-metric.

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1. Introduction

The Robinson-Trautman (RT) metrics [1]

$$g = 2\mathrm{d}u(H\mathrm{d}u + \mathrm{d}r) - 2\frac{r^2}{P^2}\mathrm{d}\xi\mathrm{d}\bar{\xi} \tag{1.1}$$

were proposed in order to describe gravitational radiation from bounded sources. Here $u, r, \xi, \bar{\xi}$ are coordinates and $P = P(u, \xi, \bar{\xi})$ is an unknown function. For RT metrics the vacuum Einstein equations reduce to the definition of H in terms of P

$$H = P^2(\ln P)_{,\xi\bar{\xi}} - r(\ln P)_{,u} - \frac{m}{r} \tag{1.2}$$

and the RT equation for the function P

$$K_{,\xi\bar{\xi}} - 3m(P^{-2})_{,u} = 0\ . \tag{1.3}$$

Here $m =$const and

$$K = 2P^2(\ln P)_{,\xi\bar{\xi}}$$

is the Gauss curvature of the surfaces $u =$ const, $r = 1$.

Global structure, trapped surfaces and asymptotic behaviour of RT spacetimes were successfully studied by Penrose [2], Foster and Newman [3], Lukacs, Perjes, Porter and Sebestyen [4], Schmidt [5], Rendall [6], Tod [7], Singleton [8], Chruściel [9, 10], Chruściel and Singleton [11], Chow and Lun [12] and others. In this communication we summarize our results [13] on trapped surfaces and quasi-local horizons in RT spacetimes. These geometrical objects play an important role in modern theory of black (or white) holes (see e.g. [14] and references therein).

A nontrivial solution of the RT equation is given by

$$P = 1 + \frac{1}{2}\xi\bar{\xi}\ .$$

It defines the Schwarzschild metric in the Eddington-Finkelstein coordinates

$$g = \mathrm{d}u\big((1 - \frac{2m}{r})\mathrm{d}u + 2\mathrm{d}r\big) - r^2\frac{2\mathrm{d}\xi\mathrm{d}\bar{\xi}}{(1+\frac{1}{2}\xi\xi)^2}\ ,$$

ξ being the complex stereographic coordinate of the 2-dimensional sphere S_2. These coordinates cover the shaded half of the Penrose conformal diagram (Fig.1).

Let us consider RT metrics with sections $u =$ const, $r =$ const diffeomorphic to S_2. Then

$$\hat{P} = \frac{P}{1+\frac{1}{2}\xi\xi}$$

is a smooth and positive function on S_2. By virtue of the RT equation we can set

$$i\int_{S_2} P^{-2}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi} = 4\pi\ . \tag{1.4}$$

Then the surface area of the sections is $4\pi r^2$. For $m > 0$ these metrics can be developed from an initial surface $u = u_0$ to cover a manifold shown in Fig.2 [10].

When $u \to \infty$ they tend to the Schwarzschild metric. Thus, the future event horizon is given by $u = \infty$, $r = 2m$ and is similar to that in the Schwarzschild spacetime.

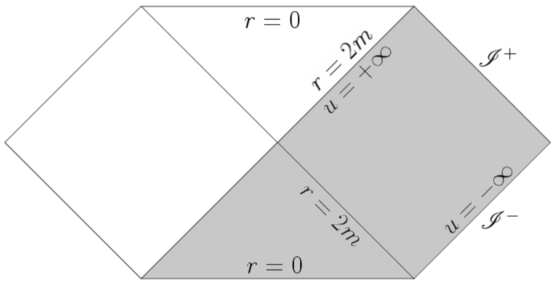

Figure 1: Conformal diagram of the Schwarzschild metric

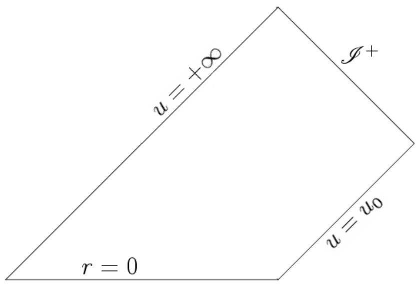

Figure 2: Conformal diagram of the RT metric with $m > 0$ [10]

2. Past trapped surfaces and horizons

The question arises whether crossections of $r = 2m$ with $u = \text{const} < \infty$ form the past event horizon as in the Schwarzschild spacetime. In general, the answer is 'no' since solving condition $H = 0$ together with the RT equation shows that (see [13] for details)

- *Surface $r = 2m$ is null for $u < \infty \Leftrightarrow g$ is Schwarzschild.*

Let $\mathscr{S}$ be a 2-dimensional spacelike surface given by

$$u = \text{const}, \quad r = R(\xi, \bar{\xi}) \ . \tag{2.1}$$

The ingoing and outgoing null vectors normal to $\mathscr{S}$ read

$$k = \partial_r$$
$$l = \partial_u - \left(H - \frac{P^2}{r^2} |R_{,\xi}|^2 \right) \partial_r + \frac{P^2}{r^2} (R_{,\bar{\xi}} \partial_\xi + R_{,\xi} \partial_{\bar{\xi}}) \ .$$

Expansion of k and l on $\mathscr{S}$ are, respectively,

$$\theta_{(k)} = -R^{-1}$$

and

$$\theta_{(l)} = \frac{1}{R} \left(-P^2 (\ln R)_{,\xi\bar{\xi}} + \frac{K}{2} - \frac{m}{R} \right) \ .$$

Hence, the surface $\mathscr{S}$ is trapped iff

$$-P^2(\ln R)_{,\xi\bar{\xi}}+\frac{K}{2}-\frac{m}{R}=0\ . \tag{2.2}$$

According to Tod [7] equation (2.2) admits unique (for each u) solution which defines an outermost marginally trapped surface $\mathscr{S}$.

Integrating equation (2.2) over $\mathscr{S}$ and using the Gauss-Bonnet theorem yields

$$\int_{S_2}\left(\frac{2m}{R}-1\right)\mathrm{d}\sigma=0\ ,$$

where $\mathrm{d}\sigma=iP^{-2}\mathrm{d}\xi\wedge\mathrm{d}\bar{\xi}$ is the surface 2-form. Hence, we obtain the following property

- *The trapped surface $\mathscr{S}$ intersects the surface $r=2m$.*

By varing u in (2.1) and (2.2) one can define a hypersurface $\mathscr{H}$ foliated by the marginally trapped surfaces $\mathscr{S}$. Chow and Lun [12] showed that $\mathscr{H}$ is a non-timelike surface (dynamical horizon). From the point of view of a theory of black holes it is important to know whether $\mathscr{H}$ can be null (nonexpanding horizon). Note that if $\mathscr{H}$ is null then the expansion-free null vector l is tangent to $\mathscr{H}$. It is also shear-free due to the Raychaudhuri equation. Independly of topological assumptions on intersections of $\mathscr{H}$ with u=const we obtain the following result (see [13] for a proof)

- *The only vacuum Robinson-Trautman metrics admitting a past nonexpanding horizon is the Schwarzschild solution and the C-metric.*

In the case of the C-metric coordinates u and ξ can be chosen in such a way that

$$K=K(x+u)\ ,\ P^2=\frac{12m}{K_{,x}}\ ,$$

where $x=\mathrm{Re}\xi$. The function K undergoes the equation

$$6mK_{,x}=-\frac{1}{3}K^3+bK+c$$

and the nonexpanding horizon $\mathscr{H}$ is given by

$$r=6m(K+a)^{-1}\ .$$

Here a, b and c are constants constrained by the condition

$$\frac{a^3}{3}-ab+c=0\ .$$

Note that in this case $\mathscr{H}$ does not admit regular spherical sections.

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