

\mathcal{R}^4 terms in $d = 4$ from string theory

Filipe Moura^{*†}

*Security and Quantum Information Group, Instituto de Telecomunicações,
Instituto Superior Técnico, Departamento de Matemática, Av. Rovisco Pais, 1049-001 Lisboa,
and*

*Centro de Matemática da Universidade do Minho, Escola de Ciências, Campus de Gualtar,
4710-057 Braga, Portugal.*

E-mail: fmoura@math.ist.utl.pt

We analyze the reduction to four dimensions of the \mathcal{R}^4 terms which are part of the ten-dimensional string effective actions, both at tree level and one loop. We show that there are two independent combinations of \mathcal{R}^4 present, at one loop, in the type IIA four dimensional effective action, which means they both have their origin in M-theory. The $d = 4$ heterotic effective action also has such terms. This contradicts the common belief that there is only one \mathcal{R}^4 term in four-dimensional supergravity theories, given by the square of the Bel-Robinson tensor. In pure $\mathcal{N} = 1$ supergravity this new \mathcal{R}^4 combination cannot be directly supersymmetrized, but we show that, when coupled to a scalar chiral multiplet (violating the U(1) R -symmetry), it emerges in the action after elimination of the auxiliary fields. We then move to the extended ($\mathcal{N} = 8$) supersymmetrization of this term, where no other coupling can be taken. We show that such supersymmetrization cannot be achieved at the linearized level. This is in conflict with the theory one gets after toroidal compactification of type II superstrings being $\mathcal{N} = 8$ supersymmetric. We interpret this result in face of the recent claim that perturbative supergravity cannot be decoupled from string theory in $d \geq 4$, and $\mathcal{N} = 8$, $d = 4$ supergravity is in the swampland.

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1. \mathcal{R}^4 terms in $d = 10$

Type II superstring α'^3 effective actions contain two independent bosonic terms I_X, I_Z containing four powers of the Riemann tensor, from which two separate superinvariants are built [1, 2]. These terms are given, at linear order in the NS-NS gauge field B_{mn} , by:

$$I_X = t_8 t_8 \mathcal{R}^4 + \frac{1}{2} \varepsilon_{10} t_8 B \mathcal{R}^4, \quad I_Z = -\varepsilon_{10} \varepsilon_{10} \mathcal{R}^4 + 4 \varepsilon_{10} t_8 B \mathcal{R}^4. \quad (1.1)$$

For the heterotic string another two independent \mathcal{R}^4 terms Y_1 and Y_2 appear at order α'^3 [1, 2, 3]. Their parts which involve only the Weyl tensor are given respectively by

$$Y_1 := t_8 (\text{tr} \mathcal{W}^2)^2, \quad Y_2 := t_8 \text{tr} \mathcal{W}^4. \quad (1.2)$$

Each t_8 tensor has eight free spacetime indices. It acts in four two-index antisymmetric tensors, as defined in [4, 5]. In our case,

$$\begin{aligned} t_8 t_8 \mathcal{R}^4 &= t^{mnpqrstu} t^{m'n'p'q'r's't'u'} \mathcal{R}_{mnm'n'} \mathcal{R}_{ppq'q'} \mathcal{R}_{rsr's'} \mathcal{R}_{tut'u'}, \\ \varepsilon_{10} t_8 B \mathcal{R}^4 &= t^{mnpqrstu} \varepsilon^{vwm'n'p'q'r's't'u'} B_{vw} \mathcal{R}_{mnm'n'} \mathcal{R}_{ppq'q'} \mathcal{R}_{rsr's'} \mathcal{R}_{tut'u'}, \\ \varepsilon_{10} \varepsilon_{10} \mathcal{R}^4 &= \varepsilon_{vw}^{mnpqrstu} \varepsilon^{vwm'n'p'q'r's't'u'} \mathcal{R}_{mnm'n'} \mathcal{R}_{ppq'q'} \mathcal{R}_{rsr's'} \mathcal{R}_{tut'u'}. \end{aligned} \quad (1.3)$$

The effective action of type IIB theory must be written, because of its well known $SL(2, \mathbb{Z})$ invariance, as a product of a single linear combination of order α'^3 invariants and an overall function of the complexified coupling constant $\Omega = C^0 + ie^{-\phi}$, C^0 being the axion. The order α'^3 part of this effective action which involves only the Weyl tensor is given in the string frame by

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{\text{IIB}} \Big|_{\alpha'^3} = -e^{-2\phi} \alpha'^3 \frac{\zeta(3)}{3 \times 2^{10}} \left(I_X - \frac{1}{8} I_Z \right) - \alpha'^3 \frac{1}{3 \times 2^{16} \pi^5} \left(I_X - \frac{1}{8} I_Z \right). \quad (1.4)$$

The corresponding part of the action of type IIA superstrings has a relative "-" sign flip in the one loop term [6]. This sign difference is because of the different chirality properties of type IIA and type IIB theories, which reflects on the relative GSO projection between the left and right movers:

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{\text{IIA}} \Big|_{\alpha'^3} = -e^{-2\phi} \alpha'^3 \frac{\zeta(3)}{3 \times 2^{10}} \left(I_X - \frac{1}{8} I_Z \right) - \alpha'^3 \frac{1}{3 \times 2^{16} \pi^5} \left(I_X + \frac{1}{8} I_Z \right). \quad (1.5)$$

Heterotic string theories in $d = 10$ have $\mathcal{N} = 1$ supersymmetry, which allows corrections already at order α' , including \mathcal{R}^2 corrections. These corrections come both from three and four graviton scattering amplitudes and anomaly cancellation terms (the Green-Schwarz mechanism). Up to order α'^3 , the terms from this effective action which involve only the Weyl tensor are given by

$$\begin{aligned} \frac{1}{\sqrt{-g}} \mathcal{L}_{\text{heterotic}} \Big|_{\alpha' + \alpha'^3} &= e^{-2\phi} \left[\frac{1}{16} \alpha' \text{tr} \mathcal{R}^2 + \frac{1}{2^9} \alpha'^3 Y_1 - \frac{\zeta(3)}{3 \times 2^{10}} \alpha'^3 \left(I_X - \frac{1}{8} I_Z \right) \right] \\ &\quad - \alpha'^3 \frac{1}{3 \times 2^{14} \pi^5} (Y_1 + 4Y_2). \end{aligned} \quad (1.6)$$

In order to consider these terms in the context of supergravity, one should write them in the Einstein frame. To pass from the string to the Einstein frame, we redefine the metric in d (noncompact) dimensions through a conformal transformation involving the dilaton, given by

$$g_{\mu\nu} \rightarrow \exp\left(\frac{4}{d-2}\phi\right) g_{\mu\nu}, \quad \mathcal{R}_{\mu\nu}{}^{\rho\sigma} \rightarrow \exp\left(-\frac{4}{d-2}\phi\right) \tilde{\mathcal{R}}_{\mu\nu}{}^{\rho\sigma}, \quad (1.7)$$

with $\tilde{\mathcal{R}}_{\mu\nu}{}^{\rho\sigma} = \mathcal{R}_{\mu\nu}{}^{\rho\sigma} - \delta_{[\mu}{}^{[\rho} \nabla_{\nu]} \nabla^{\sigma]} \phi$.

Let $I_i(\mathcal{R}, \mathcal{M})$ be an arbitrary term in the string frame lagrangian. $I_i(\mathcal{R}, \mathcal{M})$ is a function, with conformal weight w_i , of any given order in α' , of the Riemann tensor \mathcal{R} and any other fields - gauge fields, scalars, and also fermions - which we generically designate by \mathcal{M} . The transformation above takes $I_i(\mathcal{R}, \mathcal{M})$ to $e^{\frac{4}{d-2}w_i\phi} I_i(\tilde{\mathcal{R}}, \mathcal{M})$. After considering all the dilaton couplings and the effect of the conformal transformation on the metric determinant factor $\sqrt{-g}$, the string frame lagrangian

$$\frac{1}{2}\sqrt{-g} e^{-2\phi} \left(-\mathcal{R} + 4(\partial^\mu \phi) \partial_\mu \phi + \sum_i I_i(\mathcal{R}, \mathcal{M}) \right) \quad (1.8)$$

is converted into the Einstein frame lagrangian

$$\frac{1}{2}\sqrt{-g} \left(-\mathcal{R} - \frac{4}{d-2}(\partial^\mu \phi) \partial_\mu \phi + \sum_i e^{\frac{4}{d-2}(1+w_i)\phi} I_i(\tilde{\mathcal{R}}, \mathcal{M}) \right). \quad (1.9)$$

Next we will take the terms we wrote above, but reduced to four dimensions, in the Einstein frame.

2. \mathcal{R}^4 terms in $d = 4$

In four dimensions, the Weyl tensor can be decomposed in self-dual and antiself-dual parts ¹:

$$\mathcal{W}_{\mu\nu\rho\sigma} = \mathcal{W}_{\mu\nu\rho\sigma}^+ + \mathcal{W}_{\mu\nu\rho\sigma}^-, \quad \mathcal{W}_{\mu\nu\rho\sigma}^\mp := \frac{1}{2} \left(\mathcal{W}_{\mu\nu\rho\sigma} \pm \frac{i}{2} \varepsilon_{\mu\nu}{}^{\lambda\tau} \mathcal{W}_{\lambda\tau\rho\sigma} \right). \quad (2.1)$$

The totally symmetric Bel-Robinson tensor is given in four dimensions by $\mathcal{W}_{\mu\rho\nu\sigma}^+ \mathcal{W}_{\tau\lambda}^{-\rho\sigma}$. In the van der Warden notation, using spinorial indices [7], to $\mathcal{W}_{\mu\rho\nu\sigma}^+, \mathcal{W}_{\mu\rho\nu\sigma}^-$ correspond the totally symmetric $\mathcal{W}_{ABCD}, \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ being given by (in the notation of [8])

$$\mathcal{W}_{ABCD} := -\frac{1}{8} \mathcal{W}_{\mu\nu\rho\sigma}^+ \sigma_{\dot{A}\dot{B}}^{\mu\nu} \sigma_{\dot{C}\dot{D}}^{\rho\sigma}, \quad \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} := -\frac{1}{8} \mathcal{W}_{\mu\nu\rho\sigma}^- \sigma_{\dot{A}\dot{B}}^{\mu\nu} \sigma_{\dot{C}\dot{D}}^{\rho\sigma}.$$

The Bel-Robinson tensor is simply given by $\mathcal{W}_{ABCD} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}$; the decomposition (2.1) is written as

$$\mathcal{W}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = -2\varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} \mathcal{W}_{ABCD} - 2\varepsilon_{AB} \varepsilon_{CD} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}. \quad (2.2)$$

In four dimensions, there are only two independent real scalar polynomials made from four powers of the Weyl tensor [9], given by

$$\mathcal{W}_+^2 \mathcal{W}_-^2 = \mathcal{W}^{ABCD} \mathcal{W}_{ABCD} \mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \quad (2.3)$$

$$\mathcal{W}_+^4 + \mathcal{W}_-^4 = \left(\mathcal{W}^{ABCD} \mathcal{W}_{ABCD} \right)^2 + \left(\mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} \right)^2. \quad (2.4)$$

In particular, the Weyl-dependent parts of I_X, I_Z, Y_1, Y_2 , when computed directly in $d = 4$ (i.e. replacing the ten dimensional indices m, n, \dots by the four dimensional indices μ, ν, \dots), should be

¹We used latin letters - m, n, \dots - to represent ten dimensional spacetime indices. From now on we will be only working with four dimensional spacetime indices which, to avoid any confusion, we represent by greek letters μ, ν, \dots

expressed in terms of them. The resulting \mathcal{W}^4 terms are (see [10] for details):

$$I_X - \frac{1}{8}I_Z = 96\mathcal{W}_+^2\mathcal{W}_-^2, \quad (2.5)$$

$$I_X + \frac{1}{8}I_Z = 48(\mathcal{W}_+^4 + \mathcal{W}_-^4) + 672\mathcal{W}_+^2\mathcal{W}_-^2, \quad (2.6)$$

$$Y_1 = 8\mathcal{W}_+^2\mathcal{W}_-^2, \quad (2.7)$$

$$Y_1 + 4Y_2 = 80\mathcal{W}_+^2\mathcal{W}_-^2 + 4(\mathcal{W}_+^4 + \mathcal{W}_-^4). \quad (2.8)$$

$I_X - \frac{1}{8}I_Z$ is the only combination of I_X and I_Z which in $d = 4$ contains only the square of the Bel-Robinson tensor (2.3), but not (2.4). Interestingly, from (1.1) exactly this very same combination is the only one which does not depend on the ten dimensional B^{mn} field and, therefore, due to its gauge invariance, is the only one that can appear in string theory at arbitrary loop order.

We should consider another possibility: could there be any four-dimensional \mathcal{W}^4 terms coming from the original ten-dimensional $I_X + \frac{1}{8}I_Z$ term in (1.1), but this time including the (four dimensional) $B^{\mu\nu}$ field, as a scalar, after toroidal compactification and dualization [11]? Let's take

$$\partial^{[\mu} B^{\nu\rho]} = \varepsilon^{\mu\nu\rho\sigma} \partial_\sigma D. \quad (2.9)$$

$B^{\mu\nu}$ is a pseudo 2-form under parity; after dualization in $d = 4$, D is a true scalar. This way, from the $\varepsilon_{10} t_8 B \mathcal{R}^4$ term in $d = 10$ one gets in $d = 4$, among other terms, derivatives of scalars and at most an \mathcal{R}^2 factor. (One also gets simply derivatives of scalars, without any Riemann tensor.) An \mathcal{R}^4 factor would only come, after dualization, from a higher-order term, always multiplied by derivatives of scalars. Therefore we cannot get any \mathcal{R}^4 terms this way.

We then write the effective actions (1.4), (1.5), (1.6) in four dimensions, in the Einstein frame (considering only terms which are simply powers of the Weyl tensor, without any other fields except their couplings to the dilaton, and introducing the $d = 4$ gravitational coupling constant κ):

$$\frac{\kappa^2}{\sqrt{-g}} \mathcal{L}_{\text{IB}} \mathcal{R}^4 = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{2^{11} \pi^5} e^{-4\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2, \quad (2.10)$$

$$\frac{\kappa^2}{\sqrt{-g}} \mathcal{L}_{\text{IIA}} \mathcal{R}^4 = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{2^{12} \pi^5} e^{-4\phi} \alpha'^3 [(\mathcal{W}_+^4 + \mathcal{W}_-^4) + 224 \mathcal{W}_+^2 \mathcal{W}_-^2] \quad (2.11)$$

$$\begin{aligned} \frac{\kappa^2}{\sqrt{-g}} \mathcal{L}_{\text{het}} \mathcal{R}^2 + \mathcal{R}^4 &= -\frac{1}{16} e^{-2\phi} \alpha' (\mathcal{W}_+^2 + \mathcal{W}_-^2) + \frac{1}{64} (1 - 2\zeta(3)) e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 \\ &\quad - \frac{1}{3 \times 2^{12} \pi^5} e^{-4\phi} \alpha'^3 [(\mathcal{W}_+^4 + \mathcal{W}_-^4) + 20 \mathcal{W}_+^2 \mathcal{W}_-^2]. \end{aligned} \quad (2.12)$$

These are only the moduli-independent \mathcal{R}^4 terms. Strictly speaking not even these terms are moduli-independent, since they are all multiplied by the volume of the compactification manifold, a factor we omitted for simplicity. But they are always present, no matter which manifold is taken. The complete action, for every different compactification, includes many other moduli-dependent terms which we do not consider here: we are mostly interested in a \mathbb{T}^6 compactification.

3. \mathcal{R}^4 terms and $d = 4$ supersymmetry: some known results

Up to now, we have only been considering bosonic terms for the effective actions, but we are interested in their full supersymmetric completion in $d = 4$. In general each superinvariant consists

of a leading bosonic term and its supersymmetric completion, given by a series of terms with fermions. We are particularly focusing on \mathcal{R}^4 terms.

It has been known for a long time that the square of the Bel-Robinson tensor $\mathcal{W}_+^2 \mathcal{W}_-^2$ can be made supersymmetric, in simple [12, 13] and extended [14, 15, 16] four dimensional supergravity. For the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ there is a "no-go theorem", based on $\mathcal{N} = 1$ chirality arguments [17]: for a polynomial $I(\mathcal{W})$ of the Weyl tensor to be supersymmetrizable, each one of its terms must contain equal powers of $\mathcal{W}_{\mu\nu\rho\sigma}^+$ and $\mathcal{W}_{\mu\nu\rho\sigma}^-$. The whole polynomial must then vanish when either $\mathcal{W}_{\mu\nu\rho\sigma}^+$ or $\mathcal{W}_{\mu\nu\rho\sigma}^-$ do. The only exception is $\mathcal{W}^2 = \mathcal{W}_+^2 + \mathcal{W}_-^2$, which in $d = 4$ is part of the Gauss-Bonnet topological term and is automatically supersymmetric.

But the new term (2.4) is part of the heterotic and type IIA effective actions at one loop which must be supersymmetric, even after compactification to $d = 4$. One must then find out how this term can be made supersymmetric, circumventing the $\mathcal{N} = 1$ chirality argument from [17].

One must keep in mind the assumptions in which this argument was derived, namely the preservation by the supersymmetry transformations of R -symmetry which, for $\mathcal{N} = 1$, corresponds to $U(1)$ and is equivalent to chirality. That is true for pure $\mathcal{N} = 1$ supergravity, but to this theory and to all extended supergravity theories except $\mathcal{N} = 8$ one may add matter couplings and extra terms which violate $U(1)$ R -symmetry and yet can be made supersymmetric, inducing corrections to the supersymmetry transformation laws which do not preserve $U(1)$ R -symmetry. Since [17] only deals with the term (2.4) by itself, one can consider extra couplings to it and only then try to supersymmetrize. These couplings could eventually (but not necessarily) break $U(1)$ R -symmetry. This procedure is very natural, considering the scalar couplings that multiply (2.4) in (2.11), (2.12).

Considering couplings to other multiplets and breaking $U(1)$ may be possible in $\mathcal{N} = 4$ supergravity, for \mathbb{T}^6 compactifications of heterotic strings, but $\mathcal{N} = 1$ supergravity has the advantage of being much less restrictive than its extended counterparts. To our purposes, the simplest and most obvious choice of coupling is to $\mathcal{N} = 1$ chiral multiplets. That is what we do in the following.

4. $\mathcal{W}_+^4 + \mathcal{W}_-^4$ in $\mathcal{N} = 1$ matter-coupled supergravity

The $\mathcal{N} = 1$ supergravity multiplet is very simple. What also makes this theory easier is the existence of several different full off-shell formulations. We work in standard "old minimal" supergravity, having as auxiliary fields a vector $A_{A\dot{A}}$, a scalar M and a pseudoscalar N , given as $\theta = 0$ components of superfields $G_{A\dot{A}}, R, \bar{R}$:² $G_{A\dot{A}}| = \frac{1}{3}A_{A\dot{A}}, \bar{R}| = 4(M + iN), R| = 4(M - iN)$. Besides there is a chiral superfield W_{ABC} and its hermitian conjugate $\bar{W}_{\bar{A}\bar{B}\bar{C}}$, which together at $\theta = 0$ constitute the field strength of the gravitino. The Weyl tensor shows up as the first θ term: in the notation of (2.2), at the linearized level, $\nabla_{\underline{D}}W_{\underline{ABC}}| = \mathcal{W}_{\underline{ABCD}} + \dots$

$\mathcal{W}_+^4 + \mathcal{W}_-^4$ is proportional to the $\theta = 0$ term of $(\nabla^2 W^2)^2 + \text{h.c.}$, which cannot result from a superspace integration. This whole term itself is $U(1)$ R -symmetric, like $\nabla_{\underline{D}}W_{\underline{ABC}}$; indeed, the components of the Weyl tensor are $U(1)$ R -neutral, according to the weights [8] $\nabla_A \mapsto +1, R \mapsto +2, G_m \mapsto 0, W_{ABC} \mapsto -1$.

This way, as expected, one needs some extra coupling to (2.4) in order to break $U(1)$ R -symmetry. We can use the fact that there are many more matter fields with its origin in string theory

²The $\mathcal{N} = 1$ superspace conventions are exactly the same as in [8, 13].

and many different matter multiplets to which one can couple the $\mathcal{N} = 1$ supergravity multiplet in order to build superinvariants. This way we hope to find some coupling which breaks $U(1)$ R -symmetry and simultaneously supersymmetrizes (2.4), which could result from the elimination of the matter auxiliary fields.

Having this in mind, we consider a chiral multiplet, represented by a chiral superfield Φ (we could take several chiral multiplets Φ_i , but we restrict ourselves to one for simplicity), and containing a scalar field $\varphi = \Phi|$, a spin- $\frac{1}{2}$ field $\nabla_A \Phi|$, and an auxiliary field $F = -\frac{1}{2} \nabla^2 \Phi|$. This superfield and its hermitian conjugate couple to $\mathcal{N} = 1$ supergravity in its simplest version through a superpotential $P(\Phi) = d + a\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3$ and a Kähler potential $K(\Phi, \bar{\Phi}) = -\frac{3}{\kappa^2} \ln \left(-\frac{\Omega(\Phi, \bar{\Phi})}{3} \right)$, with $\Omega(\Phi, \bar{\Phi}) = -3 + \Phi\bar{\Phi} + c\Phi + \bar{c}\bar{\Phi}$.

In order to include the term (2.4), we take the following effective action:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{6\kappa^2} \int E \left[\Omega(\Phi, \bar{\Phi}) + \alpha'^3 \left(b\Phi (\nabla^2 W^2)^2 + \text{h.c.} \right) \right] d^4\theta - \frac{2}{\kappa^2} \left(\int \varepsilon P(\Phi) d^2\theta + \text{h.c.} \right) \\ &= \int \frac{\varepsilon}{4\kappa^2} \left[\left(\bar{\nabla}^2 + \frac{\bar{R}}{3} \right) \left(\Omega(\Phi, \bar{\Phi}) + \alpha'^3 \left(b\Phi (\nabla^2 W^2)^2 + \text{h.c.} \right) \right) - 8P(\Phi) \right] d^2\theta + \text{h.c.} \end{aligned} \quad (4.1)$$

E is the superdeterminant of the supervielbein; ε is the chiral density. The $\Omega(\Phi, \bar{\Phi})$ and $P(\Phi)$ terms represent the most general renormalizable coupling of a chiral multiplet to pure supergravity [18]; the extra terms represent higher-order corrections. Of course (4.1) is meant as an effective action and therefore does not need to be renormalizable. The component expansion of this action may be found using the explicit θ expansions for ε and $\nabla^2 W^2$ given in [8].

It is well known that an action of this type in pure supergravity (without the higher-order corrections) will give rise, in x -space, to a leading term given by $\frac{1}{6\kappa^2} e \Omega| \mathcal{R}$ instead of the usual $-\frac{1}{2\kappa^2} e \mathcal{R}$.³ In order to remove the extra $\varphi \mathcal{R}$ terms in $\frac{1}{6\kappa^2} e \Omega| \mathcal{R}$, one takes a $\varphi, \bar{\varphi}$ -dependent conformal transformation [18]; if one also wants to remove the higher order $\varphi \mathcal{R}$ terms, this conformal transformation must be α' -dependent. Here we are only interested in obtaining the supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$; therefore we will not be concerned with the Ricci terms of any order.

If one expands (4.1) in components, one does not directly get (2.4), but one should look at the auxiliary field sector. Because of the presence of the higher-derivative terms, the auxiliary field from the original conformal supermultiplet A_m also gets higher derivatives in its equation of motion, and therefore it cannot be simply eliminated [13, 15]. Here we only consider the much simpler terms which include the chiral multiplet auxiliary field F . Take the superfields

$$\tilde{\mathcal{C}} = c + \alpha'^3 b (\nabla^2 W^2)^2, \tilde{\Omega}(\Phi, \bar{\Phi}, \tilde{\mathcal{C}}, \bar{\tilde{\mathcal{C}}}) = -3 + \Phi\bar{\Phi} + \tilde{\mathcal{C}}\Phi + \bar{\tilde{\mathcal{C}}}\bar{\Phi}, \quad (4.2)$$

so that the action (4.1) becomes

$$\mathcal{L} = \frac{1}{4\kappa^2} \int \varepsilon \left[\left(\bar{\nabla}^2 + \frac{1}{3} \bar{R} \right) \tilde{\Omega}(\Phi, \bar{\Phi}, \tilde{\mathcal{C}}, \bar{\tilde{\mathcal{C}}}) - 8P(\Phi) \right] d^2\theta + \text{h.c.} \quad (4.3)$$

and all the α'^3 corrections considered in it become implicitly included in $\tilde{\Omega}(\Phi, \bar{\Phi}, \tilde{\mathcal{C}}, \bar{\tilde{\mathcal{C}}})$ through $\tilde{\mathcal{C}}, \bar{\tilde{\mathcal{C}}}$. We also define $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}|$ and the functional derivative $P_\Phi = \partial P / \partial \Phi$. From now on, we will

³As usual in supergravity theories we work with the vielbein and not with the metric. Therefore, here we write e , the determinant of the vielbein, instead of $\sqrt{-g}$.

work in x -space and assume there is no confusion between the superfield functionals $\tilde{\Omega}(\Phi, \bar{\Phi}, \tilde{C}, \bar{C})$, $P(\Phi)$, P_Φ and their corresponding x -space functionals $\tilde{\Omega}(\varphi, \bar{\varphi}, \tilde{C}, \bar{C})$, $P(\varphi)$, P_φ . The terms we are looking for are given by [18]

$$\kappa^2 \mathcal{L}_{F, \bar{F}} = \frac{e}{9} \tilde{\Omega}(\varphi, \bar{\varphi}, \tilde{C}, \bar{C}) \left| M - iN - \frac{3(\varphi + \bar{C})F}{\tilde{\Omega}(\varphi, \bar{\varphi}, \tilde{C}, \bar{C})} \right|^2 - \frac{e(3 + \tilde{C}\bar{C})F\bar{F}}{\tilde{\Omega}^2(\varphi, \bar{\varphi}, \tilde{C}, \bar{C})} + e\tilde{P}_\varphi F + e\bar{\tilde{P}}_{\bar{\varphi}} \bar{F}. \quad (4.4)$$

This equation would be exact, with $\tilde{P}_\varphi = P_\varphi$ and $\bar{\tilde{P}}_{\bar{\varphi}} = \bar{P}_{\bar{\varphi}}$, if we were only considering the $\theta = 0$ components of \tilde{C}, \bar{C} . But, of course (as it is clear from (4.1)), coupled to F we will have $\nabla_{\tilde{A}}(\nabla^2 W^2)^2$ and $\bar{\nabla}^2(\nabla^2 W^2)^2$ terms (and $\nabla_A(\bar{\nabla}^2 \bar{W}^2)^2$ and $\nabla^2(\bar{\nabla}^2 \bar{W}^2)^2$ terms coupled to \bar{F}). These terms will not play any role for our purpose (which is to show that there exists a supersymmetric lagrangian which contains (2.4), and not necessarily to compute it in full), and therefore we do not compute them explicitly. We write them in (4.4) because we include them in \tilde{P}_φ , through the definition (analogous for $\bar{\tilde{P}}_{\bar{\varphi}}$) $\tilde{P}_\varphi = P_\varphi + (\nabla_{\tilde{A}} \tilde{C} + \bar{\nabla}^2 \tilde{C} \text{ terms})$.

The first term in (4.4) contains the well known term $-\frac{1}{3}e(M^2 + N^2)$ from "old minimal" supergravity. Because the auxiliary fields M, N belong to the chiral compensating multiplet, their field equation should be algebraic, despite the higher derivative corrections [13, 15]. That calculation should still require some effort; plus, those M, N auxiliary fields should not generate *by themselves* terms which violate U(1) R -symmetry: these terms should only occur through the elimination of F, \bar{F} . This is why we will only be concerned with these auxiliary fields, which therefore can be easily eliminated through their field equation

$$\left(\frac{(\bar{\varphi} + \tilde{C})(\varphi + \bar{C})}{\tilde{\Omega}(\varphi, \bar{\varphi}, \tilde{C}, \bar{C})} - \frac{3 + \tilde{C}\bar{C}}{\tilde{\Omega}^2(\varphi, \bar{\varphi}, \tilde{C}, \bar{C})} \right) F = -\bar{\tilde{P}}_{\bar{\varphi}} - \frac{1}{3}(\bar{\varphi} + \tilde{C})(M - iN).$$

Replacing F, \bar{F} in $\mathcal{L}_{F, \bar{F}}$, one gets

$$\kappa^2 \mathcal{L}_{F, \bar{F}} = -e \frac{\tilde{P}_\varphi \bar{\tilde{P}}_{\bar{\varphi}} \tilde{\Omega}^2(\varphi, \bar{\varphi}, \tilde{C}, \bar{C})}{(\bar{\varphi} + \tilde{C})(\varphi + \bar{C}) \tilde{\Omega}(\varphi, \bar{\varphi}, \tilde{C}, \bar{C}) - (\tilde{C}\bar{C} + 3)} + M, N \text{ terms}. \quad (4.5)$$

This is a nonlocal, nonpolynomial action. Since we take it as an effective action, we can expand it in powers of the fields $\varphi, \bar{\varphi}$, but also in powers of \tilde{C}, \bar{C} . These last fields contain both the couplings of Φ to supergravity c and the string parameter α' ; expanding in these fields is equivalent to expanding in a certain combination of these parameters. Here one should notice that we are only considering up to α'^3 terms. If we wanted to consider higher (than α'^3) order corrections, together with these we should also have included *a priori* in (4.1) the leading higher order corrections, which should be independently supersymmetrized. Considering solely the higher than α'^3 order corrections coming directly from the elimination of (any of) the auxiliary fields from the α'^3 effective action (4.1) would be misleading. The correct expansion of (4.1) to take, in the first place, is in α'^3 . That is what we do in the following, after replacing \tilde{C}, \bar{C} by their explicit superfield expressions given by (4.2) and taking $\theta = 0$. We also exclude the M, N contributions and the higher θ terms from \tilde{C}, \bar{C}

in $\tilde{P}_\phi, \tilde{P}_{\bar{\phi}}$, for the reasons mentioned before: they are not significant for the term we are looking for. The resulting lagrangian we get (which we still call $\mathcal{L}_{F,\bar{F}}$ to keep its origin clear, although it is not anymore the complete lagrangian resulting from the elimination of F, \bar{F}) is

$$\begin{aligned} \kappa^2 \mathcal{L}_{F,\bar{F}} = & -e \frac{P_\phi \tilde{P}_{\bar{\phi}} \Omega^2(\phi, \bar{\phi})}{(\bar{\phi} + c)(\phi + \bar{c})\Omega(\phi, \bar{\phi}) - (c\bar{c} + 3)} \\ & + \alpha^3 \frac{e P_\phi \tilde{P}_{\bar{\phi}} \Omega(\phi, \bar{\phi})}{((\bar{\phi} + c)(\phi + \bar{c})\Omega(\phi, \bar{\phi}) - (c\bar{c} + 3))^2} \left[-2 \left(b\phi (\nabla^2 W^2)^2 \right) \right. \\ & + \left. \bar{b}\bar{\phi} \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \right] ((\bar{\phi} + c)(\phi + \bar{c})\Omega(\phi, \bar{\phi}) - (c\bar{c} + 3)) \\ & + \Omega(\phi, \bar{\phi}) \left(-b\bar{c}\phi (\nabla^2 W^2)^2 \Big| - \bar{b}c\bar{\phi} \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \Big| \right) \\ & + (\bar{\phi} + c)(\phi + \bar{c}) \left(b\phi (\nabla^2 W^2)^2 \Big| + \bar{b}\bar{\phi} \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \Big| \right) \\ & + \Omega(\phi, \bar{\phi}) \left(b(\bar{c} + \phi) (\nabla^2 W^2)^2 \Big| + \bar{b}(c + \bar{\phi}) \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \Big| \right) \Big] + \dots \end{aligned} \quad (4.6)$$

If we look at the last line of the previous equation, we can already identify the term we are looking for. This is still a nonlocal, nonpolynomial action, which we expand now in powers of the fields $\phi, \bar{\phi}$ coming from the denominators and the $P_\phi \tilde{P}_{\bar{\phi}}$ factors. We obtain

$$\begin{aligned} \kappa^2 \mathcal{L}_{F,\bar{F}} = & -15e \frac{(3 + c\bar{c})}{(3 + 4c\bar{c})^2} (m\bar{a}\phi + \bar{m}a\bar{\phi})(c\phi + \bar{c}\bar{\phi}) \\ & + e \frac{2c^3\bar{c}^3 + 60c^2\bar{c}^2 + 117c\bar{c} - 135}{(3 + 4c\bar{c})^3} a\bar{a}\phi\bar{\phi} - 36\alpha^3 e \left(b\bar{c} (\nabla^2 W^2)^2 \Big| \right. \\ & + \left. \bar{b}c \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \Big| \right) \frac{a\bar{a} + m\bar{a}\phi + \bar{m}a\bar{\phi} + g\bar{a}\phi^2 + \bar{g}a\bar{\phi}^2 + m\bar{m}\phi\bar{\phi}}{(3 + 4c\bar{c})^2} \\ & - 3\alpha^3 a\bar{a} \frac{74c^2\bar{c}^2 + 192c\bar{c} - 657}{(3 + 4c\bar{c})^4} \phi\bar{\phi} \left(b\bar{c} (\nabla^2 W^2)^2 \Big| + \bar{b}c \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \Big| \right) \\ & + 15\alpha^3 e \frac{a\bar{a} + m\bar{a}\phi + \bar{m}a\bar{\phi}}{(3 + 4c\bar{c})^3} \left[(\bar{c}^2 (21 + 4c\bar{c})\bar{\phi} + (-9 + 6c\bar{c})\phi) b (\nabla^2 W^2)^2 \Big| \right. \\ & + \left. (c^2 (21 + 4c\bar{c})\phi + (-9 + 6c\bar{c})\bar{\phi}) \bar{b} \left(\bar{\nabla}^2 \bar{W}^2 \right)^2 \Big| \right] + \dots \end{aligned} \quad (4.7)$$

This way we are able to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$, although we had to introduce a coupling to a chiral multiplet. These multiplets show up after $d = 4$ compactifications of superstring and heterotic theories and truncation to $\mathcal{N} = 1$ supergravity [19]. Since from $\nabla_D W_{ABC} \Big| = \mathcal{W}_{ABCD} + \dots$ we have $\nabla^2 W^2 \Big| = -2\mathcal{W}_+^2 + \dots$, the factor in front of \mathcal{W}_+^4 (resp. \mathcal{W}_-^4) in (4.7) is given by $\frac{72b\bar{c}a\bar{a}}{(3+4c\bar{c})^2}$ (resp. $\frac{72\bar{b}ca\bar{a}}{(3+4c\bar{c})^2}$). Therefore, for this supersymmetrization to be effective the factors a from $P(\phi)$ and c from $\Omega(\phi, \bar{\phi})$ (and of course b from (4.1)) must be nonzero.

The action (4.7) includes the $\mathcal{N} = 1$ supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$, but without any coupling to a scalar field or only with couplings to powers of the scalar field from the chiral multiplet, which may be seen as compactification moduli. But, as one can see from (2.11), (2.12), this term

should be coupled to powers of the dilaton. It is well known [19] that in $\mathcal{N} = 1$ supergravity the dilaton is part of a linear multiplet, together with an antisymmetric tensor field and a Majorana fermion. One must then work out the coupling to supergravity of the linear and chiral multiplets. As usual one starts from conformal supergravity and obtain Poincaré supergravity by coupling to compensator multiplets which break superconformal invariance through a gauge fixing condition. When there are only chiral multiplets coupled to supergravity [18], this gauge fixing condition can be generically solved, so that a lagrangian has been found for an arbitrary coupling of the chiral multiplets. In the presence of a linear multiplet, there is no such a generic solution of the gauge fixing condition, which must be solved case by case. Therefore, there is no generic lagrangian for the coupling of supergravity to linear multiplets. We shall not consider this problem here, like we did not in [8, 10, 13]. In both cases we were only interested in studying the $\mathcal{N} = 1$ supersymmetrization of the two different $d = 4$ \mathcal{R}^4 terms. The coupling of a linear multiplet to these terms can be determined following the procedure in [20].

5. \mathcal{R}^4 terms and extended $d = 4$ supersymmetry

We have just achieved the $\mathcal{N} = 1$ supersymmetrization of (2.4) by coupling this term to a chiral multiplet. A similar procedure may be taken in $\mathcal{N} = 2$ supergravity, since there exist $\mathcal{N} = 2$ chiral superfields which must be Lorentz and SU(2) scalars but can have an arbitrary U(1) weight, allowing for supersymmetric U(1) breaking couplings.

Such a result should be more difficult to achieve for $\mathcal{N} \geq 3$, because there are no generic chiral multiplets. But for $3 \leq \mathcal{N} \leq 6$ there are still matter multiplets which one can couple to the Weyl multiplet. Those couplings could eventually (but not necessarily) break U(1) R -symmetry and lead to the supersymmetrization of (2.4).

An even more complicated problem is the $\mathcal{N} = 8$ supersymmetrization of (2.4). The reason is the much more restrictive character of $\mathcal{N} = 8$ supergravity, compared to lower \mathcal{N} . Besides, its multiplet is unique, which means there are no extra matter couplings one can take in this theory. Plus, in this case the R -symmetry group is SU(8) and not U(8): the extra U(1) factor, which in $\mathcal{N} = 2$ could be identified with the remnant $\mathcal{N} = 1$ R -symmetry and, if broken, eventually turn the supersymmetrization of (2.4) possible, does not exist. Apparently there is no way to circumvent in $\mathcal{N} = 8$ the result from [17]. In order to supersymmetrize (2.4) in this case one should then explore the different possibilities which were not considered in [17]. Since that article only deals with the term (2.4) by itself, one can consider extra couplings to it and only then try to supersymmetrize. This procedure is very natural, taking into account the scalar couplings that multiply (2.4) in the actions (2.11), (2.12).

We now proceed with trying to supersymmetrize (2.4) but, first, we review the superspace formulation of $\mathcal{N} \geq 4$ supergravities and also some known higher order superinvariants in these theories.

5.1 Linearized $\mathcal{N} \geq 4, d = 4$ supergravity in superspace

In this section we review the superspace formulation of pure $\mathcal{N} \geq 4$ linearized supergravity theories and some of the known higher-order superinvariants, including a little discussion on the symmetries they should preserve. We will only be working at the linearized level, for simplicity.

One typically decomposes the $U(\mathcal{N})$ R -symmetry into $SU(\mathcal{N}) \otimes U(1)$ and considers only $SU(\mathcal{N})$ for the superspace geometry. $U(1)$ is still present, but not in the superspace coordinate indices. The only exception is for $\mathcal{N} = 8$; the more restrictive supersymmetry algebra requires in this case the R -symmetry group to be $SU(8)$, and there is no $U(1)$ to begin with. We always work therefore in this section in conventional extended superspace with structure group $SL(2; \mathbb{C}) \otimes SU(\mathcal{N})$.

The field content of $\mathcal{N} \geq 4$ supergravity is essentially described by a superfield W^{abcd} [21], totally antisymmetric in its $SU(\mathcal{N})$ indices, its complex conjugate \bar{W}_{abcd} and their derivatives.

Still at the linearized level, one has the differential relations

$$\begin{aligned} \nabla_{Aa} W^{bcde} &= -8\delta_a^{[b} W_A^{cde]}, \quad \nabla_{Aa} W_B^{bcd} = 6\delta_a^{[b} W_{AB}^{cd]}, \quad \nabla_{Aa} W_{BC}^{bc} = -4\delta_a^{[b} W_{ABC}^c], \\ \nabla_{Aa} W_{BCD}^b &= -\delta_a^b W_{ABCD}, \quad \nabla_{Aa} W_{BCDE} = 0, \end{aligned} \quad (5.1)$$

and

$$\nabla_A^a W_{BCDE} = 2i\nabla_{\underline{B}\dot{A}} W_{\underline{CDE}}^a, \quad \nabla_A^a W_{BCD}^b = i\nabla_{\underline{B}\dot{A}} W_{\underline{CD}}^{ab}, \quad \nabla_A^a W_{BC}^{bc} = -i\nabla_{\underline{B}\dot{A}} W_{\underline{C}}^{abc}, \quad \nabla_A^a W_B^{bcd} = iN_{\underline{B}\dot{A}}^{abcd}. \quad (5.2)$$

This last relation defines the superfield N_{AA}^{abcd} which, therefore, also satisfies

$$N_{AA}^{abcd} = \nabla_{A\dot{A}} W^{abcd}, \quad \nabla_{Aa} N_{\underline{B}\dot{B}}^{bcde} = -8\delta_a^{[b} \nabla_{\underline{A}\dot{B}} W_{\underline{B}}^{cde]}. \quad (5.3)$$

Here we should notice that these relations are valid for N_{AA}^{abcd} , but not for its complex conjugate $\bar{N}_{\dot{A}\dot{A}}^{abcd}$. In other words, $\nabla_{Aa} \bar{N}_{\underline{B}\dot{B}}^{bcde}$ is another independent relation, like its hermitian conjugate $\nabla_{\dot{A}a} N_{\underline{B}\dot{B}}^{bcde}$, as we will see below [21].

The spinorial indices in the differential relations (5.1) are completely symmetrized. Indeed, at the linearized level the corresponding terms with contracted indices vanish, through the Bianchi identities

$$\nabla_{\dot{A}}^A W_{ABC}^a = 0, \quad \nabla_{\dot{A}}^{\dot{B}} N_{\underline{B}\dot{B}}^{bcde} = 0, \quad \nabla_{\dot{A}}^A W_{ABCD} = 0. \quad (5.4)$$

For $\mathcal{N} \leq 6$, W^{abcd} is a complex superfield which together with \bar{W}_{abcd} describes at $\theta = 0$ the $2 \binom{\mathcal{N}}{4}$ real scalars of the theory. In $\mathcal{N} = 8$ supergravity, the superfield W^{abcd} represents at $\theta = 0$ the $\binom{8}{4} = 70$ scalars of the full nonlinear theory. On-shell, it satisfies the reality condition [21]

$$W^{abcd} = \frac{1}{4!} \varepsilon^{abcdefgh} \bar{W}_{efgh}. \quad (5.5)$$

Since $N_{AA}^{abcd} = \nabla_{A\dot{A}} W^{abcd}$, from the previous relation one also has *on-shell*, in linearized $\mathcal{N} = 8$ supergravity,

$$N_{AA}^{abcd} = \frac{1}{4!} \varepsilon^{abcdefgh} \bar{N}_{\dot{A}\dot{A}}^{efgh}. \quad (5.6)$$

Among the derivatives of W^{abcd} there is the superfield W_{ABCD} , which from the differential relations (5.1) is related to W^{abcd} at the linearized level by $W_{ABCD} \propto \nabla_{Aa} \nabla_{Bb} \nabla_{Cc} \nabla_{Dd} W^{abcd} + \dots$. The Weyl tensor appears as the $\theta = 0$ component of W_{ABCD} :

$$W_{ABCD}| = \mathcal{W}_{ABCD}. \quad (5.7)$$

Also $W_{BCD}^b|$ is the Weyl tensor of the \mathcal{N} gravitinos, $W_{BC}^{bc}|$ is the field strength of $\binom{\mathcal{N}}{2}$ vector fields and $W_B^{bcd}|$ are the $\binom{\mathcal{N}}{3}$ Weyl spinors.

In $\mathcal{N} = 6, 7$ supergravity there exist extra $\binom{\mathcal{N}}{6}$ vector fields, described by $\overline{W}_{BCbcdefg}|$. In $\mathcal{N} = 5, 6, 7$ supergravity there also exist additional $\binom{\mathcal{N}}{5}$ Weyl spinors, described by $\overline{W}_{Bbcdef}|$.⁴ In $\mathcal{N} = 8$ supergravity these superfields do not represent new physical degrees of freedom, because then we have the following relations:

$$\overline{W}_{Bbcdef} = \frac{1}{2}\epsilon_{bcdefgha}W_B^{gha}, \quad \overline{W}_{BCbcdefg} = \frac{1}{6}\epsilon_{bcdefgha}W_{BC}^{ha}. \quad (5.8)$$

The differential relations satisfied by these superfields can be derived, in $\mathcal{N} = 8$, from (5.8) and the previous relations (5.1) and (5.2). For $\mathcal{N} \leq 6$ supergravities, which are truncations of $\mathcal{N} = 8$, these relations are obtained from the $\mathcal{N} = 8$ corresponding ones, but considering that (5.5), (5.6) and (5.8) are not valid anymore (i.e. by considering W^{abcd} and \overline{W}_{abcd} as independent superfields). This is the way one can derive the differential relations which are missing in (5.1) and (5.2), like $\nabla_{Aa}\overline{W}_{bcde} = -\frac{2}{3}\overline{W}_{Aabcde}$, and so on.

Again for $4 \leq \mathcal{N} \leq 8$, on-shell (which in linearized supergravity is equivalent to setting the $SU(\mathcal{N})$ curvatures to zero), one has among others the field equations

$$\nabla_{\dot{A}}^A W_{AB}^{ab} = 0, \quad (5.9)$$

$$\nabla^{A\dot{A}} N_{A\dot{A}}^{abcd} = 0. \quad (5.10)$$

At the component level, at $\theta = 0$ (5.10) represents the field equation for the scalars in linearized supergravity. Equations (5.5), (5.6) and (5.10) are only valid on-shell, and are logically subjected to α' corrections. Plus, most of the equations in this section include nonlinear terms that we did not include here, but which can be seen in [21].

5.2 Higher order superinvariants in superspace and their symmetries

Next we will be analyzing linearized higher order superinvariants in superspace.

There are known cases in the literature of apparent linearized \mathcal{R}^4 superinvariants in ten-dimensional type IIB supergravity which did not become true superinvariants. One may therefore wonder if that could not happen in our case. But in $d = 4$ the structure of the transformation laws and the invariances of the supermultiplets are relatively easier and better understood than in $d = 10$, which guarantees us that the existence of the full superinvariants from the linearized ones is not in jeopardy, although they may not fully preserve their symmetries. We summarize here the explanation which can be found in [22].

⁴In $\mathcal{N} = 7$ supergravity there also exists an additional $\left(\binom{\mathcal{N}}{7} = 1\right)$ gravitino. Indeed, the $\mathcal{N} = 7$ and $\mathcal{N} = 8$ multiplets are identical.

For $\mathcal{N} \leq 3$, one can get a full nonlinear superspace invariant from a linearized one simply by inserting a factor of E , the determinant of the supervielbein. This is also true for $\mathcal{N} \geq 4$, but here some remarks are necessary, as fields which transform nonlinearly may be present. In these cases, the classical equations of motion of the theory are invariant under some global symmetry group G . The theory also has a local H invariance, H being the maximal compact subgroup of G . The supergravity multiplet includes a set of abelian vector fields with a local $U(1)$ invariance. Because of this invariance, the $U(1)$ potentials corresponding to the vector fields cannot then transform under H and must be representations of G .

In all these cases in the full nonlinear theory the scalar fields, represented in superspace by W^{abcd} , are elements of the coset space G/H . They do not transform linearly under G , but they still transform linearly under H . One can use the local H invariance to remove the non-physical degrees of freedom by a suitable gauge choice. In order for this gauge to be preserved, nonlinear G transformations must be compensated by a suitable local H transformation depending on the scalar fields. Because of this, linearized superinvariants can then indeed be generalized to the nonlinear case by inserting a factor of E , the determinant of the supervielbein, but they will not have the full G symmetry of the original equations of motion. If we want the nonlinear superinvariants to keep this symmetry, we must restrict ourselves to superfields which also transform linearly, like those which occur directly in the superspace torsions.

In full nonlinear $\mathcal{N} = 8$ supergravity [23] $G = E_{7(7)}$, a real non-compact form of E_7 whose maximal subgroup is $SL(2; \mathbb{R}) \otimes O(6, 6)$ but whose maximal compact subgroup is $H = SU(8)$. The 70 scalars are elements of the coset space $E_{7(7)}/SU(8)$. Nonperturbative quantum corrections break $E_{7(7)}$ to a discrete subgroup $E_7(\mathbb{Z})$, which implies breaking the maximal subgroup $SL(2; \mathbb{R}) \otimes O(6, 6)$ to $SL(2; \mathbb{Z}) \otimes O(6, 6; \mathbb{Z})$. $O(6, 6; \mathbb{Z})$ is the T -duality group of a superstring compactified on a six-dimensional torus; $SL(2; \mathbb{Z})$ extends to the full superstring theory as an S -duality group. In [24], evidence is given that $E_7(\mathbb{Z})$ extends to the full superstring theory as an U -duality group. It is this U -duality which requires (from a string theory point of view) that all the 70 scalars of the \mathbb{T}^6 compactification of superstring theory are on the same footing, even if originally, in the $d = 10$ theory, the dilaton is special.

Analogously, for $\mathcal{N} = 4$ supergravity coupled to m vector multiplets, we have $G = SL(2; \mathbb{R}) \otimes O(6, m)$, $H = U(1) \otimes O(6) \otimes O(m)$. The conjectured full duality group for the corresponding toroidally compactified heterotic string, with $m = 16$, is $SL(2; \mathbb{Z}) \otimes O(6, 22; \mathbb{Z})$.

The four-dimensional supergravity theories we have been considering can be seen as low energy effective field theories of toroidal compactifications of type II or heterotic superstring theories. The true moduli space of these string theories is the moduli space of the torus factored out by the discrete T -duality group Γ_T . For the case where the left-moving modes of the string are compactified on a p torus \mathbb{T}^p and the right-moving modes on a q torus \mathbb{T}^q [25], the moduli space is $\frac{SO(p, q)}{SO(p) \otimes SO(q)} / \Gamma_T$, with $\Gamma_T = SO(p, q; \mathbb{Z})$. In particular, for type II theories compactified on \mathbb{T}^6 , the moduli space is $\frac{SO(6, 6)}{SO(6) \otimes SO(6)} / \Gamma_T$, with $\Gamma_T = SO(6, 6; \mathbb{Z})$.

For heterotic theories, left-moving modes are compactified on \mathbb{T}^6 and right-moving modes on \mathbb{T}^{22} , resulting for the moduli space $\frac{SU(1, 1)}{U(1)} \times \frac{SO(6, 22)}{SO(6) \otimes SO(22)} / \Gamma_T$, with $\Gamma_T = SO(6, 22; \mathbb{Z})$. The factor $\frac{SU(1, 1)}{U(1)}$ is a separated component of moduli space spanned by a complex scalar including the dilaton, which lies in the gravitational multiplet and does not mix with the other toroidal moduli,

lying in the 22 abelian vector multiplets.

6. $\mathcal{W}_+^4 + \mathcal{W}_-^4$ and extended supersymmetry

In this section we turn our attention to the new higher order term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ and try to supersymmetrize it at the linearized level using different methods, according to [26].

We will only be working at the linearized level, for simplicity. Therefore we will not be particularly concerned with the string loop effects considered in the discussion on the string effective actions, because of their dilaton couplings which are necessarily highly nonlinear. We will be mainly concerned with the new \mathcal{R}^4 term in linearized supergravity, not worrying with the dilatonic factor in front of it to begin with (later this factor will be considered).

6.1 Superfield expression of $\mathcal{W}_+^4 + \mathcal{W}_-^4$

In [27], a general (for all \mathcal{N}) formalism for constructing four dimensional superinvariants by integrating over even-dimensional submanifolds of superspace ("superactions") was developed. Using this formalism we will try to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$ in $\mathcal{N} = 8$ supergravity, although the results can be easily extended to $4 \leq \mathcal{N} \leq 8$. For a more detailed treatment see [27, 28].

The superfield expression of $\mathcal{W}_+^4 + \mathcal{W}_-^4$ is given by

$$\left[\nabla_{A_1}^{c_1} \dots \nabla_{A_4}^{c_4} \nabla_{c_1}^{A_1} \dots \nabla_{c_4}^{A_4} \bar{W}_{d_1 d_2 d_3 d_4} W^{d_1 d_2 d_3 d_4} \right]^2 + \text{h.c.} \propto (W^{A_1 A_2 A_3 A_4} W_{A_1 A_2 A_3 A_4})^2 + \text{h.c.} \quad (6.1)$$

In order to understand this result, some preliminary basic calculations are necessary. From the differential relations (5.1), one can see that, at the linearized level, each W^{abcd} present cannot be acted by more than four (either dotted or undotted) spinorial derivatives:

$$\nabla_{Aa} \nabla_{Bb} \nabla_{Cc} \nabla_{Dd} \nabla_{Ee} W^{fghi} = 0, \quad \nabla_{\dot{A}}^a \nabla_{\dot{B}}^b \nabla_{\dot{C}}^c \nabla_{\dot{D}}^d \nabla_{\dot{E}}^e \bar{W}_{fghi} = 0. \quad (6.2)$$

From (6.2) one sees that, in order for (6.1) not to vanish, each W^{abcd} must be acted by four and only four spinorial derivatives. This way we see that from (6.1) one gets only a sum of products of four W_{ABCD} terms, eventually with different index contractions. Because of the uniqueness of \mathcal{W}^4 terms we mentioned - only (2.3), whose supersymmetrization is known [27, 28], and (2.4) -, the final result must be $\mathcal{W}_+^4 + \mathcal{W}_-^4$. Therefore, (6.1) represents the expression of $\mathcal{W}_+^4 + \mathcal{W}_-^4$ in terms of superfields, up to some numerical factor. The fact that one can write this or any other term as a superfield component does not necessarily mean that it can be made supersymmetric; for that one has to show how to get it from a superspace invariant. For (6.1), the most obvious candidate is

$$\int (W^{abcd} \bar{W}_{abcd})^2 d^{16} \theta + \text{h.c.} \quad (6.3)$$

By its structure (it requires integration over sixteen θ s), (6.3) is only valid for $\mathcal{N} = 8$ supergravity. One can write a similar expression but which is also valid for $4 \leq \mathcal{N} \leq 8$ by replacing W^{abcd} by some of its spinorial derivatives, while correspondingly lowering the number of θ s in the measure:

$$\int W^{abcd} \bar{W}_{abcd} W^{ABCD} W_{ABCD} d^8 \theta + \text{h.c.} \quad (6.4)$$

Although both (6.3) and (6.4) are equivalent in $\mathcal{N} = 8$ as linearized component expansions (up to some different numerical factor), they represent two distinct expressions at the nonlinear level.

To verify if these expressions are supersymmetric, we recall that at $\theta = 0$ the spinorial superderivatives equal the supersymmetry transformations: $\nabla_{Bb} | = Q_{Bb} |$, $\nabla_{\dot{B}}^b | = Q_{\dot{B}}^b |$. Using (5.1), (5.2) and (5.4), one can compute the supersymmetry variation of the result of the θ integrations, which from (6.2) in both cases is uniquely given by (6.1). This variation, at the linearized level, is

$$\begin{aligned} & \nabla_{\dot{A}}^a \left[(W^{BCDE} W_{BCDE})^2 + (W^{\dot{B}\dot{C}\dot{D}\dot{E}} W_{\dot{B}\dot{C}\dot{D}\dot{E}})^2 \right] \\ &= -8i \nabla_{B\dot{A}} (W^{FGHI} W_{FGHI} W^{BCDE} W_{CDE}^a) + 16i W^{FGHI} \nabla_{B\dot{A}} (W_{FGHI} W^{BCDE} W_{CDE}^a). \end{aligned} \quad (6.5)$$

This supersymmetry transformation is not a total derivative and cannot be transformed into one. Therefore neither (6.3) nor (6.4) represent a valid linearized superinvariant. This result is expected: it is just the confirmation of the prediction from [17] in $\mathcal{N} = 8$ which, as we said, is not easy to circumvent. Also as we saw at the linearized level (6.3) only gives $\mathcal{W}_+^4 + \mathcal{W}_-^4$ and no other terms. If (6.3) were supersymmetric, this would mean $\mathcal{W}_+^4 + \mathcal{W}_-^4$ was supersymmetric by itself, which would be bizarre since it does not represent a topological invariant in $d = 4$, like \mathcal{W}^2 does. Therefore the supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$, if it exists, must come in a different way.

6.2 Attempts of supersymmetrization without changing the linearized Bianchi identities

We now try to find out possible ways of supersymmetrizing $\mathcal{W}_+^4 + \mathcal{W}_-^4$ at the linearized level in $\mathcal{N} \geq 4, d = 4$ supergravity in superspace. The known solution to the superspace Bianchi identities [21] (equivalent to the x -space supersymmetry transformations) is only valid on-shell for pure supergravity (without any kind of string corrections). In principle, in order to supersymmetrize a higher-order term in the lagrangian one needs higher-order corrections to the superspace Bianchi identities (so one does to the x -space supersymmetry transformation laws), which should be of the same order in α' . In this section we attempt to supersymmetrize (2.4) assuming that the solution to the Bianchi identities for pure supergravity remains valid. This a matter of simplicity: the complete solution to the Bianchi identities involves, even without any α' corrections, many nonlinear terms which we haven't considered [21]. The α' corrections to the supersymmetry transformations are necessarily nonlinear and should affect and generate only nonlinear terms; it does not make sense to consider them if we are looking only for linearized superinvariants.

First we check if it is possible to make some change in (6.4) in order to make it supersymmetric. We notice that the result in (6.5) only tells us that (6.4) is not supersymmetric by itself; it does not mean that it is not part of some superinvariant. In fact, maybe there exists some counterterm Γ which can be added to (6.4) in order to cancel the supersymmetry variation (6.5), so that the sum of (6.4) and Γ is indeed supersymmetric. In order for Γ to exist, it must then satisfy, for some $\Gamma_{AA\dot{E}}^e$,

$$\nabla_{\dot{E}}^e \left[(W^{ABCD} W_{ABCD})^2 + (W^{\dot{A}\dot{B}\dot{C}\dot{D}} W_{\dot{A}\dot{B}\dot{C}\dot{D}})^2 + \Gamma \right] = \nabla^{AA} \Gamma_{AA\dot{E}}^e. \quad (6.6)$$

Together with (6.5) this is a very difficult differential equation, to which we did not find any solution in terms of known fields, both for Γ and $\Gamma_{AA\dot{E}}^e$.

The second possibility in order to try to cancel (6.5) is to multiply (6.4) by some factors $\Gamma, \bar{\Gamma}$, such that the product is supersymmetric. In this case $\Gamma, \bar{\Gamma}$ must satisfy, for some $\Gamma_{AA\dot{E}}^e$,

$$\nabla_{\dot{E}}^e \left[\bar{\Gamma} (W^{ABCD} W_{ABCD})^2 + \Gamma (W^{\dot{A}\dot{B}\dot{C}\dot{D}} W_{\dot{A}\dot{B}\dot{C}\dot{D}})^2 \right] = \nabla^{AA} \Gamma_{AA\dot{E}}^e. \quad (6.7)$$

The factors $\Gamma, \bar{\Gamma}$ must satisfy some restrictions, both by dimensional analysis (we want an α'^3 term) and by component analysis (we want to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$ in the Einstein frame (2.11) and (2.12), with a factor of $\exp(-4\phi)$ and at most some other scalar couplings resulting from the compactification from $d = 10$). Therefore the only acceptable (and actually very natural) factors $\Gamma, \bar{\Gamma}$ are simply functions of W^{abcd}, \bar{W}_{abcd} . In any case, again (6.7) is a very difficult differential equation, which we tried to solve in terms of each of the different known fields. We were not able to find any solution, both for $\Gamma, \bar{\Gamma}$ and $\Gamma_{AA\dot{E}}^e$, as one can see by considering (6.5), which cannot be canceled simply by taking factors of W^{abcd}, \bar{W}_{abcd} . Therefore one cannot supersymmetrize (2.4) using only the linearized (on-shell) solution to the Bianchi identities in pure supergravity. This result is unexpected and is not a confirmation of the prediction from [17] in $\mathcal{N} = 8$, which applies to (2.4) by itself and not when it is multiplied by a scalar factor. In the following subsection we will use the full nonlinear solution to the Bianchi identities, but still at $\alpha' = 0$.

6.3 Attempts of supersymmetrization with nonlinear $\alpha' = 0$ Bianchi identities

The generic effective action (1.9) has a series of terms which we designated by $I_i(\tilde{\mathcal{R}}, \mathcal{M})$. Some of these terms can be directly supersymmetrized: they constitute the "leading terms", each one of them corresponding to an independent superinvariant. The remaining terms are part of the supersymmetric completion of the leading ones.

In general it is very hard to determine the number of independent superinvariants. This problem becomes even more difficult in the presence of α' correction terms, because one single superinvariant includes terms at different orders in α' . For the complete supersymmetrization of a given higher-derivative term of a certain order in α' , typically an infinite series of terms of arbitrarily high order in α' shows up. This series may be truncated to the order in α' in which one is working, but when supersymmetrizing the terms of higher order in α' the contributions from the lower order terms must be considered. The reason is, of course, the α' dependence of the supersymmetry transformations. This has been explicitly shown for (2.3) and for $\mathcal{N} = 1, 2$ in [13, 15]. At any given order in α' , therefore, there are new leading terms (i.e. new superinvariants), and other terms which are part of superinvariants at the same order and at lower order.

Each time the supersymmetry transformation laws of single fields include linear terms, it should be possible to determine how to supersymmetrize an expression written only in terms of these fields already at the linearized level. A "leading term" of an independent superinvariant should then be invariant already at the linearized level. If this linearized supersymmetrization cannot be found for the term in question, but it still has to be made supersymmetric, it cannot be a "leading term", and must emerge only at the nonlinear level, as part of the supersymmetric completion of some other term. That must be the case of (2.4), which we have tried to supersymmetrize directly at the linearized level and we did not succeed. For the remainder of this section we will examine that possibility.

Since the α' corrections necessarily introduce nonlinear terms in the supersymmetry transformations, and since one should not consider any higher order term before considering all the corresponding lower order terms, before looking for higher-order corrections to the supersymmetry transformations one should first look at their nonlinear $\alpha' = 0$ terms. Here we will only be concerned with the nonlinear terms of the on-shell relations, i.e. of those relations which will probably acquire α' corrections: (5.5), (5.6) and (5.10).

The first two linearized equations, (5.5) and (5.6), refer to the 70 scalar fields of $\mathcal{N} = 8$ supergravity. As we mentioned, in the nonlinear theory these fields are given by the coset space $E_{7(7)}/SU(8)$; they transform nonlinearly under $E_{7(7)}$, but they still transform linearly under $SU(8)$ [23]. On shell, in superspace, at order $\alpha' = 0$, going from the linearized to the full nonlinear theory corresponds to replacing the constraint "SU(8) curvature=0" by " $E_{7(7)}$ curvature=0". A complete treatment can be found in [21].

The superspace equation (5.10) reflects the linearized field equation of the scalar fields in $4 \leq \mathcal{N} \leq 8$ supergravity, including the dilaton. For (1.9) the complete dilaton equation is given by

$$\nabla^2 \phi - \frac{1}{2} \sum_i e^{\frac{4}{2-d}(1+w_i)\phi} I_i(\tilde{\mathcal{R}}, \mathcal{M}) = 0. \quad (6.8)$$

At order $\alpha' = 0$, among the terms $I_i(\tilde{\mathcal{R}}, \mathcal{M})$ there should be those which contain field strengths corresponding to each of the vector fields present in the theory. Plus, still at order $\alpha' = 0$ there are couplings of the scalars to fermions, which we never considered explicitly but must be reflected in their field equations. In that order in α' , the $\mathcal{N} = 8$ nonlinear version of (5.10), the field equation for the scalars, is given by [21]

$$\begin{aligned} \nabla^{AA} N_{AA}^{abcd} &= W_{ef}^{\dot{A}\dot{B}} W_{\dot{A}\dot{B}}^{abcdef} + 12W^{AB[ab} W_{AB}^{cd]} - \frac{2}{3} iW_{efg}^{\dot{A}} W^{A[abc} N_{AA}^{d]efg} \\ &+ \frac{i}{12} W_{efg}^{\dot{A}} W^{Aefg} N_{AA}^{abcd} - \frac{3}{2} iW_{efg}^{\dot{A}} W^{Ae[ab} N_{AA}^{cd]fg} + 4 - \text{fermion terms}. \end{aligned} \quad (6.9)$$

As one can see, this expression does not contain any nonlinear term which is exclusively dependent on the Weyl tensor. As one can confirm in [21], the same is true for each of the differential relations considered in (5.1) and (5.2). Therefore we cannot expect (2.4) to emerge from the nonlinear completion of some (necessarily α'^3) linearized superinvariant. One must really understand the α' -corrections to the Bianchi identities. Since these corrections are necessarily nonlinear, this means one cannot supersymmetrize (2.4) at the linearized level at all. Here one must notice that never happened for the previously known higher-order terms, which all had its linearized superinvariant.

6.4 Corrections to the solution of the linearized Bianchi identities in $\mathcal{N} \geq 4, d = 4$ superspace: some considerations

In each of the three effective actions (2.10), (2.11), (2.12), only the $\mathcal{W}_+^2 \mathcal{W}_-^2$ term contains the transcendental coefficient $\zeta(3)$. This term must then have its own superinvariant, as no other term has such a coefficient. Therefore the changes in the supersymmetry transformation laws the other terms generate do not have such a coefficient and could not, by themselves, cancel the supersymmetry variation of (2.3).

Since the numerical coefficient in front of \mathcal{W}^2 in the $d = 4$ effective actions (2.11) and (2.12) is not transcendental, this term may eventually not need its own superinvariant and be part of some

other superinvariant, with a different leading bosonic term, maybe even of a lower order in α' , being related to \mathcal{W}^2 by an α' -dependent supersymmetry transformation. But even if such relation is valid in $d = 4$, that does not mean at all it should keep being valid in $d = 10$.

One can try to generate a higher-order (in α') term from a lower-order higher derivative superinvariant; maybe the higher-order term would lie on the orbit of its supersymmetry transformations. But in order to generate the higher-order term this way, one obviously needs to know the α' -corrected supersymmetry transformation laws.

One possibility would be to see if (2.4) could be obtained from the supersymmetrization of the \mathcal{R}^2 term, of order α' . But this term does not come from type II theories, which only admit α'^3 corrections and higher; it only comes from the heterotic theories. Therefore a \mathcal{R}^2 term must only be present as a correction to $\mathcal{N} = 4$ supergravity: it can also be written as an $\mathcal{N} = 8$ invariant, but in this case its stringy origin is not so obvious. Indeed, \mathcal{R}^2 terms show up from the \mathcal{R}^4 terms we are considering when we compactify string theory on a Calabi-Yau manifold [6], but for the moment we are only considering toroidal compactifications with maximal $d = 4$ supersymmetry.

There are other different terms one can consider. For instance, when going from the string frame (1.8) to the Einstein frame (1.9) with (1.7), one gets from a polynomial of the Riemann tensor a dilaton coupling and powers of derivatives of ϕ . The α'^3 effective action should contain, besides (2.3) and (2.4), the terms $((\nabla^\mu \nabla^\nu \phi) (\nabla_\mu \nabla_\nu \phi))^2$, $(\nabla^\mu \nabla^\nu \phi) (\nabla_\mu \nabla_\nu \phi) (\nabla^2 \phi)^2$ and $(\nabla^2 \phi)^4$. One therefore may expect the supersymmetrization of the higher derivative term $I(\mathcal{R})$ (which in the case we are interested includes $\mathcal{W}_+^4 + \mathcal{W}_-^4$) to lie in the orbit of some power of $\nabla^2 \phi$ or some other superinvariant of lower order in α' , so that one term may result from the other via an α' dependent supersymmetry transformation. If that is the case, one needs to find the α' corrections to the (on-shell) solution of the Bianchi identities, namely to the nonlinear versions of (5.5), (5.6) and (5.10).

Let's take for example the nonlinear dilaton field equation. Considering the pure gravitational α' corrections expressed in the effective actions (2.10), (2.11), (2.12), we are able to "guess" the expected corrections to (6.9), knowing the field content of W^{abcd} and its derivatives. Neglecting for the moment the numerical coefficients, one can see that some of the expected corrections to (6.9) (only the purely gravitational ones, i.e. those depending only on the Weyl tensor) are of the form

$$\begin{aligned} \nabla^{AA} N_{AA}^{abcd} \Big|_{\alpha'+\alpha^3} &\propto \alpha' W^{abcd} \left[W^{ABCD} W_{ABCD} + W^{\dot{A}\dot{B}\dot{C}\dot{D}} W_{\dot{A}\dot{B}\dot{C}\dot{D}} \right] \\ &+ \alpha'^3 W^{abcd} \left[(W^{ABCD} W_{ABCD})^2 + (W^{\dot{A}\dot{B}\dot{C}\dot{D}} W_{\dot{A}\dot{B}\dot{C}\dot{D}})^2 \right. \\ &\left. + (W^{ABCD} W_{ABCD}) (W^{\dot{A}\dot{B}\dot{C}\dot{D}} W_{\dot{A}\dot{B}\dot{C}\dot{D}}) \right] + \dots \end{aligned} \quad (6.10)$$

Of course this equation must be completed with other contributions, which may be derived, including the numerical coefficients, (2.11) and (2.12), once they are completed with the other leading α' corrections which do not depend only on the Riemann tensor.

It remains to be seen how are these corrections compatible with the superspace Bianchi identities. This would allow us to determine the α' corrections one needs to introduce in the other superspace field equations in order to the superspace Bianchi identities remain valid to this order in α' . This is a technically very complicated problem which we are not addressing here.

7. Conclusions

We showed in [10] that type IIA and heterotic string theories predict the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ to show up at one loop when compactified to four dimensions. Nonetheless, an older article [17] stated that this term, by itself, simply could not be made supersymmetric in $d = 4$. We worked out its $\mathcal{N} = 1$ supersymmetrization, by coupling it to a chiral multiplet.

In [26] we considered the more complicated problem of its $\mathcal{N} = 8$ supersymmetrization. We obtained the superfield expression of that term, given by (6.1), and we have shown that expression indeed was not part of a superinvariant. Since that term in $d = 10$ should come coupled to a dilaton, and it may acquire other scalar couplings after compactification to $d = 4$, in order to try to circumvent the argument of [17] we tried to construct a superinvariant which included this term, together with a proper scalar coupling, in general $4 \leq \mathcal{N} \leq 8$ superspace. We concluded that the supersymmetrization of this term at the linearized level, by itself, cannot be achieved, something which was always possible for the previously known higher-derivative string corrections.

We proposed some changes to the on-shell solution to the superspace Bianchi identities in order to include the lowest order α' -corrections. We did not present the whole set of possible α' -corrections to the Bianchi identities nor we tried to solve them in order to check the consistency of these corrections and to determine their coefficients.

As we mentioned before, the "no-go theorem" for the supersymmetrization of (2.4) given in [17] is based on $\mathcal{N} = 1$ chirality arguments. In order to circumvent these arguments, a reasonable possibility is to try to construct a superinvariant which violates the U(1) symmetry or (for $\mathcal{N} > 1$) some of the R -symmetry. But the superfield expression corresponding to (2.4) given by (6.1) is even U(1)-symmetric, as W_{ABCD} is U(1)-invariant. (This is more clearly derived in $\mathcal{N} = 1$ superspace [10], but it is easily understood if one thinks that from (5.7) $W_{ABCD}|$ is a component of the Riemann tensor.) The best one can aim at is to break U(1) or part of the $SU(\mathcal{N})$ by taking a different integration measure, as suggested in [27] and as we tried with (6.4). In $\mathcal{N} = 8$ superspace one can keep trying extra couplings of the scalar superfields W^{abcd} combined with different nonstandard integration measures. But it is easier if we are allowed to consider other multiplets than the gravitational, whose couplings automatically violate U(1). That is not possible in $\mathcal{N} = 8$ supergravity, both because there are no other multiplets than the gravitational to consider, and because the extra U(1) symmetry does not exist. We recall that $\mathcal{N} \leq 6$ theories have a $U(\mathcal{N})$ symmetry, which is split into $SU(\mathcal{N}) \otimes U(1)$, but the more restrictive $\mathcal{N} = 8$ theory has originally only an $SU(8)$ symmetry. This may be part of the origin of all the difficulties we faced when trying to supersymmetrize (2.4) in $\mathcal{N} = 8$.

But the main obstruction to this supersymmetrization is that, opposite to $\mathcal{W}_+^2 \mathcal{W}_-^2$, the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ does not seem to be compatible with the full R -symmetry group $SU(8)$. In ref. [29], a complete study has been made of all possible higher-order terms in $\mathcal{N} = 8$ supergravity, necessarily compatible with $SU(8)$, and (2.4) does not appear in the list of possible terms.

Indeed, as we saw in the discussion of section 5.2, only the local symmetry group of the moduli space of compactified string theories should be preserved by the four dimensional perturbative string corrections. As we saw, for \mathbb{T}^6 compactifications of type II superstrings this group is given by $SO(6) \otimes SO(6) \sim SU(4) \otimes SU(4)$, which is a subgroup of $SU(8)$. Most probably the perturbative string correction term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ only has this $SU(4) \otimes SU(4)$ symmetry. If that is the case,

in order to supersymmetrize this term besides the supergravity multiplet one must also consider U -duality multiplets [30], with massive string states and nonperturbative states. These would be the contributions we were missing.

But in conventional extended superspace one cannot simply write down a superinvariant that does not preserve the $SU(\mathcal{N})$ R -symmetry, which is part of the structure group. One can only consider higher order corrections to the Bianchi identities which preserve $SU(\mathcal{N})$, like the ones from (6.10), but these corrections would not be able to supersymmetrize (2.4). $\mathcal{N} = 8$ supersymmetrization of this term would then be impossible; the only possible supersymmetrizations would be at lower \mathcal{N} , eventually consider U -duality multiplets. If that is the case, in order to supersymmetrize this term besides the supergravity multiplet one must introduce U -duality multiplets, with massive string states and nonperturbative states. The fact that one cannot supersymmetrize in $\mathcal{N} = 8$ a term which string theory requires to be supersymmetric, together with the fact that one needs to consider nonperturbative states (from U -duality multiplets) in order to understand a perturbative contribution may be seen as indirect evidence that $\mathcal{N} = 8$ supergravity is indeed in the swampland, as proposed in [31]. This is a very fundamental topic of study, together with the recent claims of possible finiteness of $\mathcal{N} = 8$ supergravity. Plus, as we concluded from our analysis of the dimensional reduction of order α^3 gravitational effective actions, the new \mathcal{R}^4 term (2.4) has its origin in the dimensional reduction of the corresponding term in M-theory, a theory of which there is still a lot to be understood. We believe therefore that the complete study of this term and its supersymmetrization deserves further study.

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