Dispersive representation of the $K\pi$ vector form factor and fits to $\tau \to K\pi\nu_\tau$ and K_{e3} data

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Recently, we introduced several dispersive representations for the vector $K\pi$ form factor and fitted them to the Belle spectrum of $\tau \to K\pi\nu_\tau$. Here, we briefly present the model and discuss the results for the slope and curvature of $F_+(s)$ arising from the best fit. Furthermore, we compare the pole position of the charged $K^*(892)$ computed from our model with other results in the literature. Finally, we discuss the prospects of a simultaneous fit to $\tau \to K\pi\nu_\tau$ and K_{e3} spectra.

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1. Introduction

Decays of the τ-lepton into hadrons are an important source of information about a wealth of fundamental parameters in the standard model. An important example is the QCD coupling α_s that can be extracted from inclusive τ decays [1]. After the separation of Cabibbo-allowed and Cabibbo-suppressed decay modes into strange particles, the mass of the strange quark and the quark-mixing matrix element $|V_{us}|$ could also be determined [2]. More recently, the B-factories have gathered high-statistics data for exclusive channels. In this work we deal with $\tau \to K_S \pi \nu_\tau$ decays for which a spectrum became available from Belle [3]. In these decays, the $K\pi$ form factors can be studied. Furthermore, the isolated hadronic pair in the final state constitutes a clean environment to the study of $K\pi$ interactions. Therefore, information about $K\pi$ resonances can also be obtained.

The $K\pi$ form factors are key ingredients in the benchmark extraction of $|V_{us}|$ from K_{l3} decays [4]. They are defined as follows [5]

$$\langle \pi^-(p)|\bar{s}\gamma^\mu u|K^0(k)\rangle = \left[(k+p)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}(k-p)^\mu\right]F_+(q^2) + \frac{m_K^2 - m_\pi^2}{q^2}(k-p)^\mu F_0(q^2), \quad (1.1)$$

where $F_+(q^2)$ and $F_0(q^2)$ are the vector and scalar form factors respectively and $q^2 = (k-p)^2$. It follows from the definition that at $q^2 = 0$ we have $F_+(0) = F_0(0)$. It is then convenient to work with normalised form factors $\tilde{F}_{+,0}(q^2)$ such that

$$F_{+,0}(q^2) = F_{+,0}(0)\tilde{F}_{+,0}(q^2). \quad (1.2)$$

On the one hand, a reliable value for the normalisation at zero is crucial in order to disentangle the product $|V_{us}|F_{+,0}(0)$ that can be extracted from K_{l3} decays. In this respect, chiral perturbation theory and lattice QCD are the most trustworthy methods to obtain $F_{+,0}(0)$. On the other, the energy dependence of the form factors, encoded in $\tilde{F}_{+,0}(q^2)$, is needed when performing the phase space integrals for K_{l3} decays. Here, we tackle the latter aspect of the problem.

In the context of K_{l3} decays, where $m_l^2 < t \equiv q^2 < (m_K - m_\pi)^2$ it is customary to Taylor expand the form factors

$$\tilde{F}_{+,0}(t) = 1 + \lambda'_{+,0}\frac{t}{m_\pi^2} + \frac{1}{2}\lambda''_{+,0}\left(\frac{t}{m_\pi^2}\right)^2 + \cdots \quad (1.3)$$

From fits to the K_{l3} spectra one can obtain the constants $\lambda'_{+,0}$ and $\lambda''_{+,0}$. The study of $F_{+,0}(q^2)$ in $\tau \to K\pi\nu_\tau$, where $(m_K + m_\pi)^2 < s \equiv q^2 < m_\tau^2$, is welcome as it can further our knowledge of the energy dependence of the form factors. This can lead to a better determination of $\lambda'_{+,0}$ and $\lambda''_{+,0}$ as well as the phase space integrals that appear in the description of K_{l3} decays and, consequently, to an improvement in the determination of $|V_{us}|$.

In Section 2, we briefly review some of the results of Ref. [6] where dispersive representations of the vector form factor were used to fit the $\tau \to K\pi\nu_\tau$ spectrum from Belle [3]. We emphasise the comparison of our results with others found in the literature. In Section 3, we present an exploratory study based on a combined analysis of $\tau \to K\pi\nu_\tau$ and K_{e3} spectra aimed at better determining the phase space integrals required in K_{e3} decays.

2. The $K\pi$ vector form factor in $\tau \to K\pi\nu_\tau$ decays

The differential decay distribution for the process $\tau \to K(k)\pi(p)\nu_\tau$ can be written as [7]

$$\frac{d\Gamma_{K\pi}}{d\sqrt{s}} = \frac{G_F^2|V_{us}|^2 m_\tau^3}{32\pi^3 s} S_{EW} \left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left[\left(1 + 2\frac{s}{m_\tau^2}\right) q_{K\pi}^3 |F_+(s)|^2 + \frac{3\Delta_{K\pi}^2}{4s} q_{K\pi}|F_0(s)|^2\right], \quad (2.1)$$

where isospin invariance is assumed and we have summed over the two possible decay channels $\tau^- \to \nu_\tau \overline{K}^0 \pi^-$ and $\tau^- \to \nu_\tau K^- \pi^0$, that contribute in the ratio $2:1$ respectively. Furthermore, $S_{EW} = 1.0201$ [8] is an electro-weak correction factor, $\Delta_{K\pi} \equiv m_K^2 - m_\pi^2$, $s = (k+p)^2$, and $q_{K\pi}$ is the kaon momentum in the rest frame of the hadronic system,

$$q_{K\pi}(s) = \frac{1}{2\sqrt{s}}\sqrt{\left(s - (m_K + m_\pi)^2\right)\left(s - (m_K - m_\pi)^2\right)} \times \theta\left(s - (m_K + m_\pi)^2\right). \quad (2.2)$$

In Eq. (2.1) the prevailing contribution is given by $F_+(s)$. Note that since the $K\pi$ pair is in the final state, we now deal with the crossing-symmetric version of Eq. (1.1) which corresponds to an analytic continuation of $F_{+,0}(q^2)$ to the region $q^2 \geq s_{K\pi} = (m_K + m_\pi)^2$, where the form factors develop imaginary parts. This renders the approximate description given by Eq. (1.3) useless and, hence, one has to resort to more sophisticated treatments. The Belle collaboration [3] employed form-factors based on Breit-Wigner expressions to describe the effect of resonances, among which the $K^*(892)$ largely dominates. In Ref. [9], a description of $F_+(s)$ based on resonance chiral theory (RChT) was employed and, from fits to the Belle spectrum, the Taylor expansion as well as the masses and widths of the lowest vector resonances were determined. Finally, in Ref. [6] we have introduced several dispersive representations for $\tilde{F}_+(q^2)$.

The purpose of our study was twofold. First, from general principles of analyticity the form factors must fulfil a dispersion relation. Although in Ref. [9] the deviations from the analytic behaviour are only small corrections of order p^6 in the chiral expansion, it is interesting to corroborate this pattern by the use of a dispersive representation for $F_+(s)$. Second, a three-times-subtracted dispersive representation of the type used in Ref. [10] enables us to produce less model dependent results. To make the argument clearer let us quote the expression of $\tilde{F}_+(s)$ used in our best fit [6]

$$\tilde{F}_+(s) = \exp\left[\alpha_1 \frac{s}{m_{\pi^-}^2} + \frac{1}{2}\alpha_2 \frac{s^2}{m_{\pi^-}^4} + \frac{s^3}{\pi}\int_{s_{K\pi}}^{s_{cut}} ds' \frac{\delta_1^{K\pi}(s')}{(s')^3(s'-s-i0)}\right]. \quad (2.3)$$

In the last equation, the two subtraction constants α_1 and α_2 are obtained from a fit to the Belle spectrum. These constants are related to the Taylor expansion (1.3) as $\lambda_+' = \alpha_1$ and $\lambda_+'' = \alpha_2 + \alpha_1^2$. Concerning the phase $\delta_1^{K\pi}(s)$, up to the first inelastic threshold unitarity ensures that $\delta_1^{K\pi}(s)$ is the $K\pi$ P-wave scattering phase shift. For simplicity, in Eq. (2.3) we consider only the $K\pi$ channel. An advantage of the three-times-subtracted form of $F_+(s)$ is the fact that the integral over the phase is highly suppressed by the factor $(s')^3$ in the denominator of the integrand. Therefore, the high-energy portion of $\delta_1^{K\pi}$ weights little, laying emphasis to the elastic domain for which we can provide a reliable model. We vary the cut-off s_{cut} in the interval $(1.8\,\text{GeV})^2 < s_{cut} < \infty$ to quantify this suppression.

In practice, when using Eq. (2.3) one needs a functional form for the phase $\delta_1^{K\pi}(s)$. We take a form inspired by the RChT description of Ref. [9]. The phase reads

$$\delta_1^{K\pi}(s) = \tan^{-1}\left[\frac{\text{Im}\,\tilde{F}_+(s)}{\text{Re}\,\tilde{F}_+(s)}\right], \tag{2.4}$$

where

$$\tilde{F}_+(s) = \frac{m_{K^*}^2 - \kappa_{K^*}\tilde{H}_{K\pi}(0) + \gamma s}{D(m_{K^*}, \gamma_{K^*})} - \frac{\gamma s}{D(m_{K^{*\prime}}, \gamma_{K^{*\prime}})}. \tag{2.5}$$

In the last equation, the first piece on the right-hand side corresponds to the $K^*(892)$ whereas the second accounts for the $K^*(1410)$. The parameter γ is obtained from fits to data and $\tilde{H}_{K\pi}(s)$ is the one-loop integral (for its precise definition we refer to Ref. [9]). The denominators are given by

$$D(m_n, \gamma_n) \equiv m_n^2 - s - \kappa_n \text{Re}\,\tilde{H}_{K\pi}(s) - im_n\gamma_n(s), \tag{2.6}$$

where the constants κ_n are defined so that $-i\kappa_n\,\text{Im}\,\tilde{H}_{K\pi}(s) = -m_n\gamma_n(s)$ and the running width of a vector resonance is taken to be

$$\gamma_n(s) = \gamma_n\frac{s}{m_n^2}\frac{\sigma_{K\pi}^3(s)}{\sigma_{K\pi}^3(m_n^2)}, \tag{2.7}$$

with $\gamma_n \equiv \gamma_n(m_n^2)$ and $\sigma_{K\pi}(s) = 2q_{K\pi}/\sqrt{s}$. The parameters m_n and γ_n are determined by the fit. Concerning the phase (2.4), the main difference as compared with that of Ref. [9] is that the real part of $\tilde{H}_{K\pi}(s)$ is resummed into the functions $D(m_n, \gamma_n)$. This procedure shifts the values of our parameters m_n and γ_n with respect to the ones of Refs. [3, 9]. As we show below, the *physical* pole position of the resonances are not affected by this shift.

Although the main contribution to the decay $\tau \to K\pi\nu_\tau$ is given by the vector form factor, the scalar component in Eq. (2.1) cannot be neglected. A comprehensive coupled-channel description of $F_0(s)$ in the RChT framework plus dispersive constraints was given in Ref. [11]. Here, for $F_0(s)$ we take the last numerical update of Ref. [12]. Finally, in order to compare Eq. (2.1) with real data, one needs an ansatz for the number of events in the i-th bin with centre at b_c^i and width b_w. The theoretical number of events is then

$$N(i) = \mathcal{N}_T\frac{1}{2}\frac{2}{3}b_w\frac{1}{\Gamma_\tau\bar{B}_{K\pi}}\frac{d\Gamma_{K\pi}}{d\sqrt{s}}(b_c^i), \tag{2.8}$$

where the factors $1/2$ and $2/3$ are introduced to take into account that the $K_S\pi^-$ channel was analysed, $\mathcal{N}_T$ is the total number of events, Γ_τ is the total τ-lepton decay width and $\bar{B}_{K\pi}$ is a normalisation that for a perfect agreement between the data and the model would be the branching fraction $B_{K\pi} = \mathcal{B}(\tau \to K_S\pi^-\nu_\tau)$.

The best fit of Ref. [6] is obtained using Eq. (2.3) for $\tilde{F}_+(s)$, Eq. (2.4) for $\delta_1^{K\pi}(s)$ and the ansatz (2.8). First, let us compare our results [6]

$$\lambda_+' = (24.66 \pm 0.77) \times 10^{-3}, \qquad \lambda_+'' = (11.99 \pm 0.20) \times 10^{-4}, \tag{2.9}$$

with other recent determinations of these two constants found in Refs. [5, 9, 13, 14, 15]. These values are compared in Fig. 1. From Refs. [5, 13] we quote the results from the quadratic fit to K_{l3} data. The results from Ref. [9] are obtained from a fit to the Belle data set for $\tau \to K_S\pi^-\nu_\tau$, as

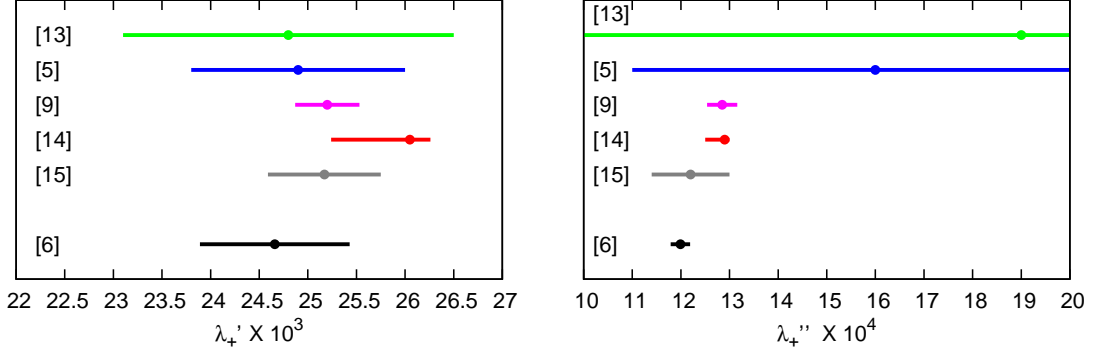**Figure 1:** Values for λ'_+ (left) and λ''_+ (right) from Refs. [5, 9, 13, 14, 15] compared to the ones from Ref. [6].

already commented. In Ref. [14] a coupled-channel dispersive representation constrained by scattering data was employed whereas in Ref. [15] a different single-channel dispersive representation was used to analyse K_{e3} data from the KTeV collaboration.

It emerges from Fig. 1 that the determinations of λ'_+ are in agreement. However, the results obtained from quadratic fits [5, 13], shown as the first and second entries, display larger uncertainties. The use of dispersive representations, as in Refs. [6, 14, 15], the data for $\tau \to K\pi\nu_\tau$ [9], or both [6], significantly reduces the uncertainty. The pattern repeats itself for λ''_+, but now the uncertainties in the case of Refs. [6, 9, 14, 15] are impressively smaller. This comparison reveals the potential of using dispersive representations for $F_+(s)$ and especially if combined with the $\tau \to K\pi\nu_\tau$ data.
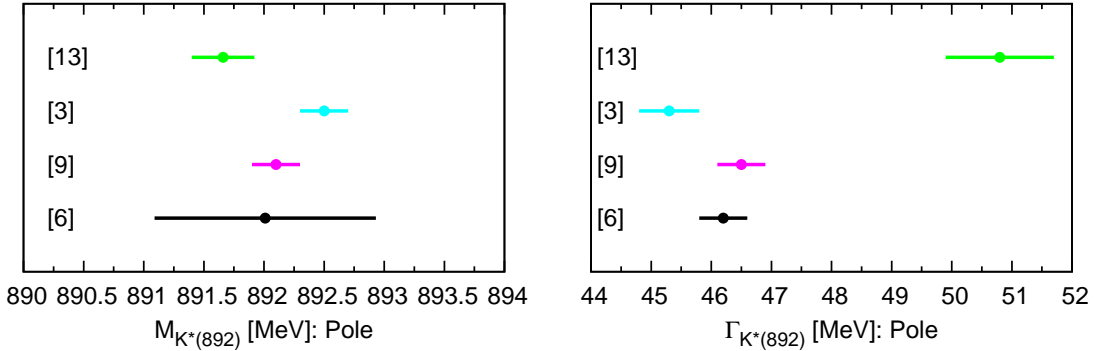

Figure 2: Mass (left) and width (right) of the charged $K^*(892)$ as defined from the pole position (see text) calculated for Refs. [3, 9, 6] compared with the value from the PDG [13]. For Ref. [3] we employed the values of the second fit of their Table 4.1 and quote solely statistical uncertainties.

Since in our description the phase of $F_+(s)$ is determined from the data, we are able to produce new values for the resonance pole positions. In this context, it is fundamental to distinguish between the physical pole position in the second Riemann sheet and the parameters m_n and γ_n in Eq. (2.6). In fact, the parameters depend strongly on the specific form of Eq. (2.6). On the contrary, the poles that arise from different models are compatible since they represent the most model independent

definition of a resonance [16]. For the $K^*(892)$, we have shown (see Table 5.1 of Ref. [6]) that although m_{K^*} and γ_{K^*} from Refs. [3, 9, 6] are rather different, the physical mass and width defined from the pole position s_p as $\sqrt{s_p} = M_R - (i/2)\Gamma_R$ are compatible. They are compared to the value quoted by the PDG [13] in Fig. 2. The values obtained from $\tau \to K\pi\nu_\tau$ point towards a smaller width.

3. Fitting $\tau \to K\pi\nu_\tau$ and K_{e3} spectra

As already mentioned, in the benchmark extraction of $|V_{us}|$ from K_{l3} decays the precise knowledge of the energy dependence of $F_{+,0}(s)$ is important to obtain reliable values for the phase space integrals. This problem is simplified in the case of K_{e3} decays, where the lepton in the final state is an electron. The scalar form factor contribution is suppressed by the square of the electron mass and, hence, the result is dominated by $F_+(s)$. We denote the phase space integral for the process $K^0 \to \pi^- e^+ \nu_e$ as $I_{K^0_{e3}}$ whose expression can be found in Ref. [4].

Using the best fit of Ref. [6], we obtain the following value for the integral: $I_{K^0_{e3}} = 0.15420(42)$. This result can be compared with the one quoted by the Flavianet Kaon WG in Ref. [5] from the average of quadratic fits: $I_{K^0_{e3}} = 0.15457(29)$. Although they are compatible, the latter is more precise in spite of the fact that the input λ'_+ and λ''_+ have larger uncertainties (see Fig. 1). This is due to the correlation $\rho(\lambda'_+, \lambda''_+)$ between the parameters in the two fits. In the quadratic fit to K_{e3} decays, λ'_+ and λ''_+ turn out strongly anti-correlated $\rho(\lambda'_+, \lambda''_+)_{K_{e3}} = -0.95$ whereas in our fit to $\tau \to K\pi\nu_\tau$ the correlation is large and positive $\rho(\lambda'_+, \lambda''_+)_\tau = 0.926$.

We have performed an exploratory study in order to determine whether a combined analysis of $\tau \to K\pi\nu_\tau$ and K_{e3} data could yield a more precise result for $I_{K^0_{e3}}$. For want of a true data set for the K_{e3} decays we made use of a simulation aimed at reproducing the situation of KLOE's data analysis [17]. To that end, using the expressions of Ref. [4], we constructed an ansatz for the number of events similar to that of Eq. (2.8). Then, assuming that the number of events follow a Poisson distribution, we generated 7.5×10^5 events that were split into 30 histograms. A quadratic fit to the generated data set yielded results very similar to the ones of Ref. [17].

Table 1: Main results of the simultaneous fit using Eq. (2.3) for $F_+(s)$. See text for details.

$\lambda'_+ \times 10^3$	$25.10 \pm (0.43)_{fit} \pm (0.07)_{s_{cut}}$
$\lambda''_+ \times 10^4$	$12.13 \pm (0.17)_{fit} \pm (0.13)_{s_{cut}}$
$\bar{B}_{K\pi}[\%]$	$0.430 \pm (0.014)_{fit} \pm (0.005)_{s_{cut}}$
$(B_{K\pi})[\%]$	0.427
$\rho(\lambda'_+, \lambda''_+)$	0.845
$\chi^2/d.o.f.$	$427/438$

With this data set, we carried out a simultaneous fit of the generated K_{e3} data and the Belle spectrum of $\tau \to K\pi\nu_\tau$ using our representation, Eq. (2.3), for $F_+(s)$. The main results are shown in Table 1 where we explicitly indicate the systematic error due to s_{cut}. Since the parameters are much better constrained at low-energies, it is possible to keep the normalisation $\bar{B}_{K\pi}$ in Eq. (2.8) as a free parameter. The result obtained from the fit is compatible with the world average $\mathscr{B} =$

$0.418 \pm 0.011\%$ [18] and the integrated value, denoted $(B_{K\pi})$ in Tab. 1, is very close to $\bar{B}_{K\pi}$. Furthermore, in this fit the uncertainty in λ'_+ is reduced and, more important, is mainly driven by statistics. From the results of this fit we obtain $I_{K^0_{e3}} = 0.15444(24)$, which has a smaller uncertainty than the result of Ref. [5]. Of course, we are by no means recommending this value, since it is based on a simulated data set. However, it is clear that the prospects are very positive since the statistics for $\tau \to K\pi\nu_\tau$ will soon be improved with the forthcoming spectrum from the BaBar collaboration [18] thus reducing the uncertainty even further.

Acknowledgments

References

[1] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. B **373** (1992) 581; M. Davier *et al.*, Eur. Phys. J. C **56** (2008) 305 [arXiv:0803.0979 [hep-ph]]; M. Beneke and M. Jamin, JHEP **09** (2008) 044 [arXiv:0806.3156 [hep-ph]]; K. Maltman and T. Yavin, Phys. Rev. D **78** (2008) 094020 [arXiv:0807.0650 [hep-ph]].

[2] E. Gámiz *et al.*, Phys. Rev. Lett. **94** (2005) 011803 [arXiv:hep-ph/0408044]; JHEP **01** (2003) 060 [arXiv:hep-ph/0212230].

[3] D. Epifanov *et al.* [Belle Collaboration], Phys. Lett. B **654** (2007) 65 [arXiv:0706.2231 [hep-ex]].

[4] H. Leutwyler and M. Roos, Z. Phys. C **25** (1984) 91.

[5] M. Antonelli *et al.* [FLAVIAnet Working Group on Kaon Decays], [arXiv:0801.1817 [hep-ph]].

[6] D. R. Boito, R. Escribano, and M. Jamin, Eur. Phys. J. C **59** (2009) 821 [arXiv:0807.4883 [hep-ph]].

[7] M. Finkemeier and E. Mirkes, Z. Phys. C **72** (1996) 619 [arXiv:hep-ph/9601275].

[8] J. Erler, Rev. Mex. Fis. **50** (2004) 200, [arXiv:hep-ph/0211345].

[9] M. Jamin, A. Pich, and J. Portolés, Phys. Lett. **B 640** (2006) 176 [arXiv:hep-ph/0605096]; Phys. Lett. B **664** (2008) 78 [arXiv:0803.1786 [hep-ph]].

[10] A. Pich and J. Portoles, Phys. Rev. D **63** (2001) 093005 [arXiv:hep-ph/0101194].

[11] M. Jamin, J. A. Oller, and A. Pich, Nucl. Phys. B **622** (2002) 279 [arXiv:hep-ph/0110193]; Eur. Phys. J. C **24** (2002) 237 [arXiv:hep-ph/0110194]; JHEP **0402** (2004) 047 [arXiv:hep-ph/0401080].

[12] M. Jamin, J. A. Oller, and A. Pich, Phys. Rev. D **74** (2006) 074009 [arXiv:hep-ph/0605095].

[13] C. Amsler *et al.* [Particle Data Group], Phys. Lett. B **667** (2008) 1.

[14] B. Moussallam, Eur. Phys. J. C **53** (2008) 401 [arXiv:0710.0548 [hep-ph]].

[15] V. Bernard *et al.*, [arXiv:0903.1654 [hep-ph]]; E. Abouzaid *et al.*, in preparation.

[16] R. Escribano *et al.*, Eur. Phys. J. C **28** (2003) 107 [arXiv:hep-ph/0204338].

[17] F. Ambrosino *et al.* [KLOE Collaboration], Phys. Lett. B **636** (2006) 166 [arXiv:hep-ex/0601038].

[18] A. Wren, talk presented at Tau08, Novosibirsk, Russia; B. Aubert et al. [BaBar Collaboration], talk presented at ICHEP08, Philadelphia, Pensylvania, [arXiv:0808.1121v2 [hep-ex]].