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Approximate forms of the density of states

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We compare MC calculations of the density of states in SU(2) pure gauge theory with the weak and strong coupling expansions. Surprisingly, the range of validity of the two approximations overlap significantly, however the large order behavior of both expansions appear to be similar to the corresponding expansions of the plaquette. We discuss the implications for the calculation of the Fisher's zeros of the partition function.

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1. Introduction

Understanding the large distance behavior of asymptotically free gauge theories in terms of the weakly coupled short distance degrees of freedom is a major challenge for theoretical physics. In pure gauge theory with the standard Wilson's action, the available numerical data on L^4 lattices indicates that there is no phase transition for $SU(2)$ or $SU(3)$ and the theory should be in the confining phase for all values of the coupling. Convincing arguments have been given [1, 2] in favor of the smoothness of the renormalization group flows between the two fixed points corresponding to the two limits. This suggests that it is possible to match the weak coupling and the strong coupling expansions of the lattice formulation. However, if we consider the two expansions, for instance for the average $SU(2)$ plaquette as a function of $\beta = 4/g^2$, there is a crossover region (approximately $1.5 < \beta < 2.5$) where none of the two expansions seem to work. This situation can probably be explained in terms of singularities in the complex β plane [3, 4] that at this point are not completely understood. In these proceedings, we discuss the weak and strong coupling expansions of the density of states for $SU(2)$ and compare them to Monte Carlo calculations. The density of states is the inverse Laplace (or Borel) transform of the partition function. Its logarithm can be interpreted as a "color entropy". This is discussed in section 2 where the basic concepts are defined.

For the one plaquette model, the density of states is a function that has better convergence properties than the partition function [5]. This is explained in section 3. We would like to know if this property persists on L^4 lattices. The comparison between weak and strong expansions and numerical calculations of the density of states for a 6^4 lattice are summarized in section 4. More details can be found in [6].

Knowing the density of states, we can calculate the partition function and its derivatives for any real or complex value of β. In particular, it can be used to determine the Fisher's zeros of the partition function [7, 8]. Locating these zeros in the complex β plane and their volume dependence is important to understand the large order behavior of the weak coupling expansion [4, 9, 10, 11] at zero temperature and the nature of the finite temperature transition [12]. Related questions have also been discussed in a poster presented at the same conference [13].

2. The density of states

In the following, we focus on a $SU(2)$ gauge theory with Wilson's action on a L^4 lattice and periodic boundary conditions. We denote the number of plaquettes $\mathcal{N}_p = 6 \times L^4$. The partition function $Z(\beta)$ is the Laplace transform of $n(S)$, the density of states:

$$Z(\beta) = \int_0^{2\mathcal{N}_p} dS\, n(S)\, \mathrm{e}^{-\beta S}\,, \tag{2.1}$$

with

$$n(S) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N) ReTr(U_p))) \tag{2.2}$$

We can interpret $\ln(n(S))$ as a "color entropy" (extensive). For cubic lattices with an even number of sites in each direction and a gauge group that contains -1, it is possible to change $\beta ReTrU_p$ into

$-\beta ReTrU_p$ by a change of variables $U_l \to -U_l$ on a set of links such that for any plaquette, exactly one link of the set belongs to that plaquette [14]. This implies

$$Z(-\beta) = e^{2\beta \mathcal{N}_p} Z(\beta) \tag{2.3}$$

and consequently

$$n(2\mathcal{N}_p - S) = n(S) \tag{2.4}$$

Thanks to this symmetry, we only need to know $n(S)$ for $0 \leq S \leq \mathcal{N}_p$. Note that $< S >= \mathcal{N}_p$ means $< TrU_p >$=0.

We define

$$f(x, \mathcal{N}_p) \equiv ln(n(x\mathcal{N}_p, \mathcal{N}_p))/\mathcal{N}_p \ . \tag{2.5}$$

The symmetry (2.4) implies that

$$f(x, \mathcal{N}_p) = f(2-x, \mathcal{N}_p) \tag{2.6}$$

The existence of the infinite volume limit requires that

$$lim_{\mathcal{N}_p \to \infty} f(x, \mathcal{N}_p) = f(x) \ , \tag{2.7}$$

with $f(x)$ volume independent. In the same limit, the integral (2.1) can be evaluated by the saddle point method. The maximization of the integrand requires

$$f'(x) = \beta \ . \tag{2.8}$$

3. The one plaquette case

In the case of the one plaquette model, the density of state simply follows from the explicit form of the Haar measure:

$$n_{1pl.}(S) = \frac{2}{\pi}\sqrt{S(2-S)} \tag{3.1}$$

At leading order, the large β behavior of the partition function is determined by the behavior of $n(S)$ near $S = 0$. The fact that $n(S) \propto \sqrt{S}$ for small S implies $Z \propto \beta^{-3/2}$ for large β. The $1/\beta$ corrections can be calculated by expanding the remaining factor $\sqrt{2-S}$ in powers of S. One then sees that a series with finite radius of convergence becomes an asymptotic series if we integrate over S from 0 to ∞ (instead of 0 to 2). In addition, the large order behavior of the asymptotic series is determined by the non-analyticity of $n_{1pl.}(S)$ at the maximal value of S (2 in this case).

These properties are in agreement with the general idea that the large order behavior of the weak coupling expansion is determined by the behavior at small negative coupling [15, 16]. In the present case, small negative g^2 means that β is very negative. In this limit, the largest possible values of S dominate the integral (in agreement with what we explained above). It would be interesting to understand if this property persists on L^4 lattices. Unfortunately, numerical values of the weak coupling expansion of the plaquette are not available for $SU(2)$ and we will have to rely on a model proposed in [9].

4. Approximate forms of $n(S)$

Numerical calculations of $n(S)$ can be obtained by patching plaquette distributions multiplied by the inverse Boltzmann weight at various values of β. In [6] we presented numerical data for L^4 lattices with $L = 4$, 6 and 8. For these values of L, finite volume effects are not too large and plaquette distributions are broad enough to allow a reasonably smooth patching. The volume dependence is resolvable only for small values of S where a behavior $ln(S)/V$ is observed for $ln(n(S))$. The coefficient of the singularity was calculated to be $(3/4)-(5/12)L^{-4}$ in reasonably good agreement with the numerical data.

The numerical results for $f(x)$ were compared with expansions that can be obtained from the strong and weak coupling expansions of the average plaquette. Intermediate orders in these expansions show a good overlap for values of S that correspond to the crossover (see Fig. 1). The convergence of the new series can be related empirically to those of the series for the average plaquette. The general picture that was obtained by trying with known series is that the converted series inherits the asymptotic behavior of the original series. The conversion of the series is performed using the saddle point equation. For the strong coupling, we expand about $x = 1$ (we remind that $x = S/\mathcal{N}_p$, see section 2). Graphs of the accuracy of the expansion at successive orders, show a crossing characteristic of a finite radius of convergence near $x = 0.5$. This is consistent with a crossing near $\beta \simeq 2$ for the plaquette (for $\beta = 2$, the average plaquette is about 0.47). For the weak coupling, we expand about $x = 0$. Accuracy graphs show consistent improvement as the order increases (with possible saturation) when $x < 0.4$ for $f(x)$ and $\beta > 3$ for the plaquette. However it should be kept in mind that the large order of the series for the plaquette has been modeled rather than calculated explictly. For details and graphs see [6].

The weak coupling expansion determines the logarithmic singularities of $ln(n(S))$ at both boundaries. When these singularities are subtracted we obtain a bell-shaped function that can

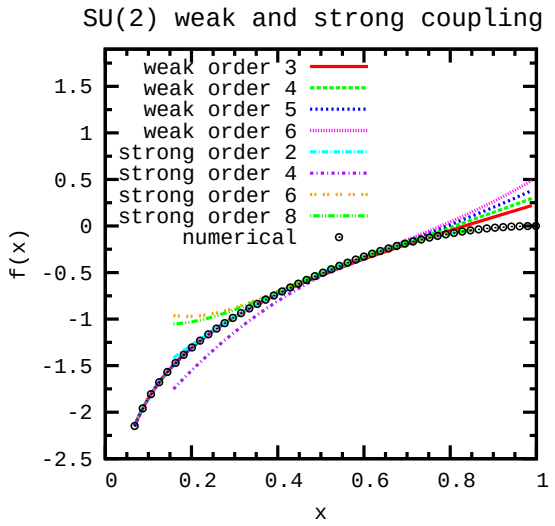

Figure 1: Weak and strong coupling expansion of f at a few intermediate orders.

be approximated by Legendre or Chebyshev polynomials. Empirically, the determination of the expansion coefficients based on the discrete orthogonality of the Chebyshev polynomials (rather then numerical interpolation followed by numerical integration) seems the most stable.

5. Calculation of Fisher's zeros

One motivation for this work is to improve our ability to determine the zeros of the partition function in the complex β plane. For reference, it is useful to understand the limitation of the reweighting MC method. In order to estimate the errors in the location of the complex zeros, we considered the changes in the location of the zeros of the real and imaginary part due to statistical fluctuations. We considered 200 sets of 40,000 values of S picked at random out of the large sample of values (bootstraps) generated for $\beta = 2.225$. For each of the 200 sets, we calculated the zeros of the real part on a small grid with typical distance between neighbooring points of the order of 10^{-3}. Using this procedure, 383895 zeros of the real part were found. We then studied the distribution of these zeros using a 200 by 200 grid in the β complex plane. The results are shown in Fig. 2. In this contour plot, the outer contours go through the bins that have 20 zeros, the first inner contours correspond to 60 zeros, the next to 100 zeros etc.. The circle of confidence [8] in the Gaussian approximation for 40,000 independent configurations as well as another estimate (red boxes in Fig. 2) of the region of confidence discussed in [10] are shown on this graph for reference. It is clear that as we get closer to the boundary of the region of confidence, the distributions get wider.

It is easier to look at horizontal sections of this distribution. We then have simple histograms with 200 bins. The results are shown in Fig. 3 for $Im\ \beta$ = 0.1, 0.115, 0.13 and 0.145. This allows us to observe the broadening of the four central peaks as $Im\ \beta$ increases. For instance, the two most central peaks are quite narrow up to $Im\ \beta$ = 0.1, but their width becomes comparable to their separation when $Im\ \beta\ > 0.13$. One should bare in mind that such distributions should be understood together with the interlaced distributions for the imaginary part which follow similar patterns. It is clear that complex zeros found in regions where there are broad distributions are unreliable. The improvement in this situation obtained by using the β independent density of states presented above is discussed in a poster [13].

6. Conclusions

We have calculated numerically the density of states for $SU(2)$ lattice gauge theory. The intermediate orders in weak and strong coupling agree well in an overlapping region of action values as shown in Fig. 1. However, the large order behaviors of these expansions appear to be similar to the corresponding ones for the plaquette. Volume effects can be resolved for small actions values. Corrections to the saddle point estimate need to be developed more systematically. Aprroximation of a subtracted quantity by Chebyshev polynomials looks very promising. We also plan to use this method to study abelian gauge theories and the large N behavior of $SU(N)$ gauge theories where interesting results based on the density of states have already been obtained [17].

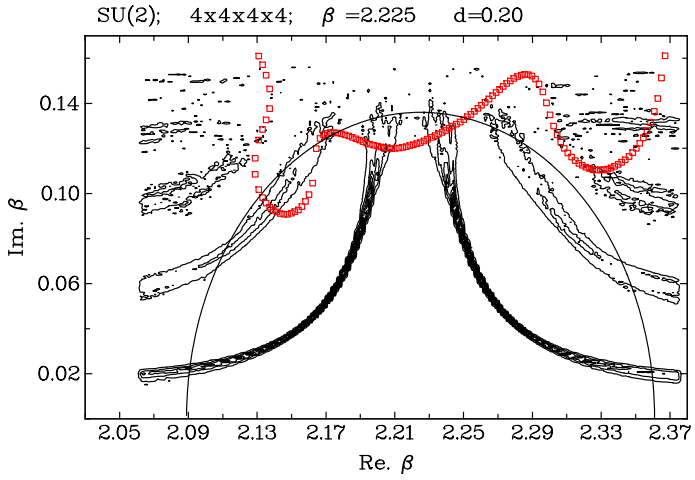

Figure 2: Distribution of zeros of the real part of the partition function in the complex β plane and regions of confidence described in the text.

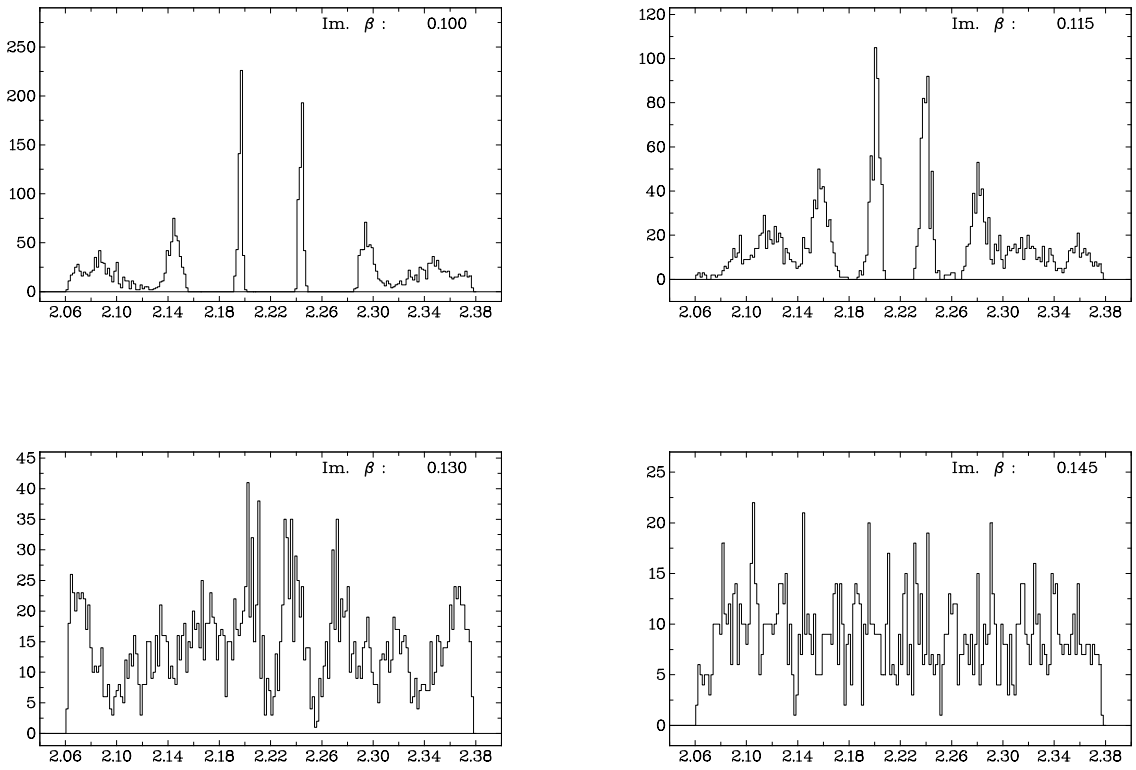

Figure 3: Horizontal sections of the previous graph described in the text.

Acknowledgments

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References

[1] E. T. Tomboulis, Confinement for all values of the coupling in four- dimensional SU(2) gauge theory, arXiv:0707.2179 2007.

[2] E. T. Tomboulis, Deriving confinement via RG decimations, *PoS*, LATTICE2007:336, 2007.

[3] J. B. Kogut, Progress in lattice gauge theory, *Phys. Rept.*, 67:67, 1980.

[4] L. Li and Y. Meurice, About a possible 3rd order phase transition at t = 0 in 4d gluodynamics, *Phys. Rev.*, D73:036006, 2006.

[5] L. Li and Y. Meurice, An example of optimal field cut in lattice gauge perturbation theory, *Phys. Rev.*, D71:054509, 2005.

[6] A. Denbleyker, Daping Du, Yuzhi Liu, Y. Meurice, and A. Velytsky, Series expansions of the density of states in su(2) lattice gauge theory, *Physical Review D (Particles and Fields)*, 78(5):054503, 2008.

[7] Nelson A. Alves, Bernd A. Berg, and Sergiu Sanielevici, Partition function zeros and the su(3) deconfining phase transition, *Phys. Rev. Lett.*, 64:3107–3110, 1990.

[8] Nelson A. Alves, Bernd A. Berg, and Sergiu Sanielevici, Spectral density study of the su(3) deconfining phase transition, *Nucl. Phys.*, B376:218–252, 1992.

[9] Y. Meurice, The non-perturbative part of the plaquette in quenched QCD, *Phys. Rev.*, D74:096005, 2006.

[10] A. Denbleyker, D. Du, Y. Meurice, and A. Velytsky, Fisher's zeros of quasi-gaussian densities of states, *Phys. Rev.*, D76:116002, 2007.

[11] A. Denbleyker, D. Du, Y. Meurice, and A. Velytsky, Fisher's Zeros and Perturbative Series in Gluodynamics, *PoS*, LAT2007:269, 2007.

[12] W. Janke, D. A. Johnston, and R. Kenna, Phase transition strength through densities of general distributions of zeroes, *Nucl. Phys.*, B682:618–634, 2004.

[13] A. Denbleyker, D. Du, Y. Liu, Y. Meurice, and A. Velytsky, poster presented at this conference.

[14] L. Li and Y. Meurice, Lattice gluodynamics at negative g**2, *Phys. Rev. D*, 71:016008, 2005.

[15] C. Bender and T. T. Wu, Anharmonic oscillator, *Phys. Rev.*, 184:1231, 1969.

[16] J. C. LeGuillou and J. Zinn-Justin, *Large-Order Behavior of Perturbation Theory*, North Holland, Amsterdam, 1990.

[17] Barak Bringoltz and Stephen R. Sharpe, Applying the Wang-Landau Algorithm to Lattice Gauge Theory, arXiv:0807.1275, 2008.