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Solving some gauge systems with large number of colours

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Spectra of space reduced gauge theories are studied using Fock space methods. After short review of the SU(2) model we discuss in detail the non-abelian supersymmetric system with one fermion and one boson in the large N limit. The system turns out to be very rich exhibiting a phase transition and a strong-weak duality. Moreover it is equivalent, at strong 't Hooft coupling, to the XXZ chain of Heisenberg spins and, independently, to a lattice gas of q-bosons.

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Dimensional reduction of field theories is a useful trick which substantially simplifies the system, but nevertheless resulting models often inherit many nontrivial properties of their ancestors. In Dublin we have reported [1] on the numerical study of the spectrum of the supersymmetric Yang-Mills quantum mechanics with the SU(2) gauge group with the hamiltonian $i = 1,..,D-1, \quad a = 1,...,N^2-1$ [2]

$$H = \frac{1}{2}p_a^i p_a^i + \frac{g^2}{4} f_{abc} f_{ade} x_b^i x_c^j x_d^i x_e^j + \frac{ig}{2} f_{abc} \psi_a^\dagger \Gamma^k \psi_b x_c^k, \tag{1}$$

The spectrum of this system is very rich, c.f. Fig.1, and indeed exhibits many features of the parent theory [3]. Its salient characteristics include: 1) unbroken supersymmetry (the tails, or wings, on the plot denote dynamically formed supermultiplets), 2) coexistence of the discrete (red) and continuous (yellow) spectra. The discrete spectrum (blue and red) is a consequence of a, characteristic to the gauge theory, potentials with flat noncompact directions. On the other hand, the continuous spectrum (which fills the central, denoted by the yellow colour, channels on the plot) results from the supersymmetry driven cancelations of the transverse fluctuations which render the quantum induced barrier inactive. Finally 3) the fractional bulk value (1/4) of the Witten index was confirmed[4] and is the consequence of the continuous spectrum extending all the way to E=0. In particular the (two) SUSY ground states belong to this continuum and are non-normalizable.

All these results were obtained using the straightforward Fock space methods [5]. Namely, the gauge invariant basis of the finite number of bosonic and fermionic (here 9+6) oscillators was constructed. Then the space was cut by limiting the total number of bosonic quanta, and the cutoff was subsequently removed. The discrete energies shown in Fig.1 correspond to $n_{max} \sim 20$ and have already converged to more than three significant digits.

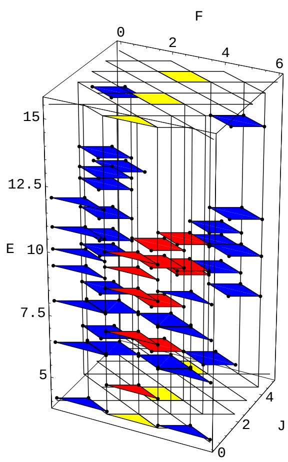

Figure 1: The discrete spectrum of the $D = 4$ SU(2) supersymmetric Yang-Mills quantum mechanics. States are grouped in supermultiplets, colored in red if the continuum spectrum is present in the channel, in blue otherwise.

At the same time there is much interest in studying the large N limit of gauge theories and their reduced counterparts. At large N mathematical structures of four dimensional models simplify, but also new analogies have been found in higher dimension. For example, while for $N = 2,3$ and $D = 4$ the non-SUSY part of (1) is the small volume limit of the Lattice YM theories, the infinite N limit is not only relevant to QCD, but for D=10, it becomes a celebrated matrix model for the M-theory [6]. The Fock space calculations, discussed above, become more tedious and CPU consuming with increasing N. Fortunately however, one can often calculate a hamiltonian matrix analytically strictly *at* infinite N. In this talk I would like to report results of such a study [7].

Consider the following supersymmetric hamiltonian of one boson and one fermion represented by the matrix valued creation and annihilation operators $a, a^\dagger$ and $f, f^\dagger$. $H = \{Q, Q^\dagger\}, Q = \sqrt{2}Tr[fa^\dagger(1+ga^\dagger)], Q^\dagger = \sqrt{2}Tr[f^\dagger(1+ga)a]$, or explicitly $H = Tr[a^\dagger a + g(a^{\dagger 2} a + a^\dagger a^2) + g^2 a^{\dagger 2} a^2] + Tr[f^\dagger f + g(f^\dagger f(a^\dagger + a) + f^\dagger(a^\dagger + a)f) + g^2(f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a)]$. It is a slightly more complicated version of the space reduced 1+1 dimensional supersymmetric Yang-Mills theory SYM_2. While the eigenstates of the latter are the gauge invariant plane waves [2], our model has more structure, as will be evident shortly.

Above hamiltonian conserves the fermion number $F = Tr[f^\dagger f]$. In the planar limit the Fock space is spanned by the single trace states and the H matrix can be easily calculated in the lowest fermionic sectors [7, 9]

$$
\begin{aligned}
\mathbf{F=0}, \quad |0,n\rangle = Tr[a^{\dagger n}]|0\rangle/\sqrt{N^n}, \qquad <0,n|H|0,n> &= (1+\lambda(1-\delta_{n1}))n, \\
<0,n+1|H|0,n> = <0,n|H|0,n+1> &= \sqrt{\lambda}\sqrt{n(n+1)}. \\
\mathbf{F=1}, \quad |1,n\rangle = Tr[a^{\dagger n} f^\dagger]|0\rangle/\sqrt{N^n}, \qquad <1,n|H|1,n> &= (1+\lambda)(n+1)+\lambda, \\
<1,n+1|H|1,n> = <1,n|H_2|1,n+1> &= \sqrt{\lambda}(2+n).
\end{aligned}
\tag{2}
$$

As in the previous case, we restrict the gauge invariant number of bosonic quanta $n < B$ and examine convergence of the spectrum with that cutoff (*cf.* first two columns of Figure 2). Indeed for $\lambda \neq 1$ the eigenvalues converge revealing a discrete, manifestly supersymmetric spectrum (see Fig.3, left). However in the vicinity of $\lambda = 1$ the convergence is slower and is replaced by the uniform fall off of all eigenvalues to zero at $\lambda = 1$. Such a behaviour is typical for a phase transition at $\lambda_c = 1$ which separates strong and weak coupling phases. In both regimes the spectrum is discrete while at the transition point the system looses it mass gap and the spectrum becomes continuous [1].

Moreover, we have found that exactly at the transition point another interesting phenomenon occurs[7]. Namely the supermultiplets rearrange while passing across λ_c and *a new* supersymmetric vacuum appears in the strong coupling phase, in the $F = 0$ sector. This is seen in the third column of Fig.1 where the first few levels from both (F=0 and F=1) sector are shown. For low cutoff, B,

the supersymmetry is broken (most noticeably in the vicinity of $\lambda = 1$), the levels forming supermultiplets in the weak coupling phase split, rearrange and rejoin at strong coupling. In that process one more level from $\lambda < 1$ becomes massless (and unpaired) in the $\lambda > 1$ region. All this is happening in the smaller and smaller neighbourhood of the critical point while we increase B, the whole structure collapsing to one point at infinite cutoff. The new vacuum can be explicitly constructed

$$
|0\rangle_2 = \sum_{n=1}^{\infty} \left(\frac{-1}{b}\right)^n \frac{1}{\sqrt{n}} |0,n\rangle, \quad b \equiv \frac{1}{\sqrt{\lambda}}. \tag{3}
$$

[1]This can be judged on the basis of the cutoff dependence, similarly as in SYMQM considered in Sect.1, *see also*[10].

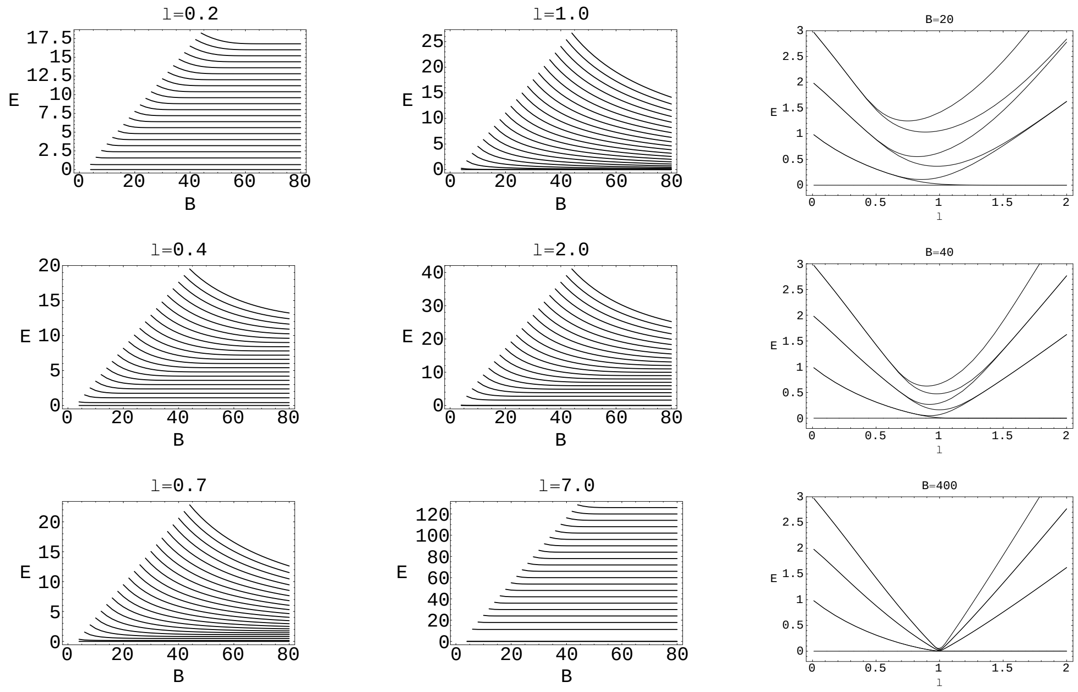

Figure 2: The cutoff dependence of the spectra of H, in the F=0 sectorfor a range of λ's

Indeed this state exists only for $\lambda > 1$ and supplements the the perturbative vacuum $|0\rangle_1 = |0\rangle$, which is there in both phases. Interestingly, the lowest two fermionic sectors reveal an exact duality between weak and strong coupling phases

$$\mathbf{F=0} \qquad b\left(E_n^{(F=0)}(1/b) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_{n+1}^{(F=0)}(b) - b^2\right). \tag{4}$$

$$\mathbf{F=1} \qquad b\left(E_n^{(F=1)}(1/b) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_n^{(F=1)}(b) - b^2\right)$$

Notice that the above mapping of energies takes into account additional vacuum state appearing at strong coupling.

Usually, dualities hint at a solubility of a system and indeed this is also the case with our Hamiltonian in the two lowest sectors. To show this we introduce a non-orthogonal basis [7] $|B_n\rangle = \sqrt{n}|0,n\rangle + b\sqrt{n+1}|0,n+1\rangle$, and a generating function $f(x)$ for the expansion of the eigenstates $|\psi >$ into the $|B_n\rangle$ basis: $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \leftrightarrow \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |B_n\rangle$. Action of H on $|B_n\rangle$ is so simple that the eigenequation $H\psi = E\psi$ is equivalent to the first order differential equation for $f(x)$

$$w(x)f'(x) + xf(x) - \varepsilon f(x) = bf(0) + f'(0),$$
$$w(x) = (x+b)(x+1/b), \qquad E = b(\varepsilon + b)$$

which can be readily solved in terms of the hypergeometric functions, $E = \alpha(b^2 - 1)$,

$$f(x) = \frac{1}{\alpha}\frac{1}{x+1/b}F(1,\alpha;1+\alpha;\frac{x+b}{x+1/b}), \quad b < 1,$$

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$$f(x) = \frac{1}{1-\alpha}\frac{1}{x+b}F(1,1-\alpha;2-\alpha;\frac{x+1/b}{x+b}), \quad b>1,$$

with the boundary condition $f(0)=0$ determining the eigenenergies E_n [7] (see also [8]).

As the additional check one can confirm construction (3) by setting $\alpha=0$ in the $b>1$ solution. and expanding $f_0(x)=\log[(b+x)/(b-1/b)]/(1+bx), \quad b>1$, into powers of $1/b$. Notice that this *cannot* be done for $b<1$ solution – there is no such state at weak coupling!

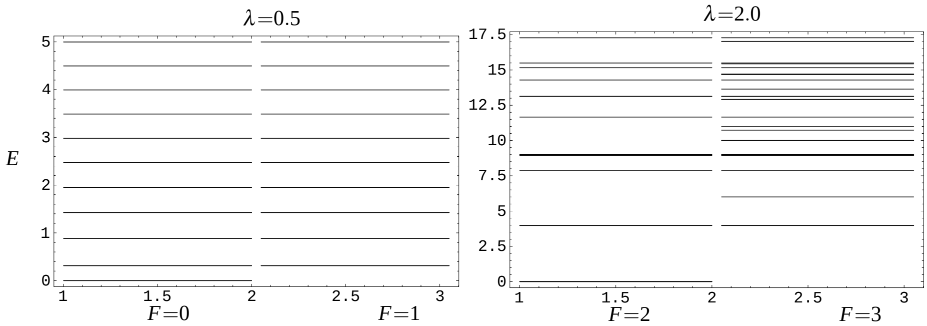

Figure 3: Low lying bosonic and fermionic levels in the first four fermionic sectors.

With more fermions the system becomes even richer, and new phenomena occur. In particular the supersymmetry results in a more involved pattern of energy levels (*cf.* Fig.3). While the SUSY pairing was complete among the $F=0$ and 1 states (Fig.3, left), it is no longer so with more fermions (Fig.3, right). For example, every state with two fermions has its (degenerate in energy) supersymmetric partner with three fermions. However there are states in the $F=3$ sector which do not have counterparts with $F=2$. Is SUSY broken? No, missing partners are in the four-fermions sector etc., *ad infinitum*. The lowest two sectors are special in a sense that there, and only there, complete representations of SUSY are accommodated. It is also evident from Fig.3 that the $F=0,1$ spectra are much more regular (almost, but not exactly, harmonic). Irregularities seen in the higher sectors are somewhat reminiscent of the many-body spectra with momentum modes taken into account. It seems that only because of the simplicity of the lowest two sectors the analytic approach was so successful there. Neither duality, nor the full analytic solution seems to exist in higher sectors [2].

On the other hand the phase transition occurs at the same value of 't Hooft coupling also with more fermions. Moreover, similarly to the $F=0,1$ cases, the supermultiplets rearrange and the *two* new non-trivial vacuum states appear as we move from weak to strong coupling phases [7].

Even though there are few analytic results for arbitrary value of 't Hooft coupling, in the strong coupling limit

$$H_{strong} \equiv \lim_{\lambda\to\infty}\frac{1}{\lambda}H = Tr(f^\dagger f) + \frac{1}{N}[Tr(a^{\dagger 2}a^2) + Tr(a^\dagger f^\dagger a f) + Tr(f^\dagger a^\dagger f a)], \quad (5)$$

the system considerably simplifies and additional analytical insight is possible [7]. Namely, the strong coupling hamiltonian (5) conserves a number of *both* fermionic and bosonic quanta. The Hilbert space splits now into sectors labeled by F and $B=Tr[a^\dagger a]$ with H becoming a finite matrix

[2]Some analytic results with two fermions are available tough[7].

in each (F,B) sector, *cf.* Table 1. Many properties of these splitting can be studied exploiting a fascinating interplay between supersymmetry and combinatorics leading to a number of exact results [11]. In particular we have found that there exists an infinite set of *magic* sectors with a single supersymmetric vacuum existing in (and only in) each of them. The magic sectors occur only at even F and $B = F \pm 1$, forming a magic staircase on the map (Table 1) of the whole Hilbert space. This generalizes results discussed above [3], for example the first row of Table 1 represents the $F = 0$ sector for any λ, i.e. without splitting into various B's, and indeed there is only one magic (F,B) sector corresponding to one new vacuum found in the strong coupling phase. Similarly, there are two magic sectors in the $F = 2$ column, confirming what was said above.

This intriguing result is explained by even more interesting equivalence to be discussed now. Consider a Heisenberg chain of spins located on a finite, one dimensional lattice

$$H^{(\Delta)}_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\,\sigma_i^z\sigma_{i+1}^z\right)$$

11	1	1	6	26	91	...	...	...	...	...	**16796**
10	1	1	5	22	73	201	497	1144	...	...	...
9	1	1	5	19	55	143	335	715	**1430**	...	**4862**
8	1	1	4	15	42	99	212	429	809	1430	2424
7	1	1	4	12	30	66	**132**	247	**429**	715	1144
6	1	1	3	10	22	42	76	132	217	335	497
5	1	1	3	7	**14**	26	**42**	66	99	143	201
4	1	1	2	5	9	14	20	30	43	55	70
3	1	1	**2**	4	**5**	7	10	12	15	19	22
2	1	1	1	2	3	3	3	4	5	5	5
1	**1**	1	**1**	1	1	1	1	1	1	1	1
0	1	1	0	1	0	1	0	1	0	1	0
B											
F	0	1	2	3	4	5	6	7	8	9	10

Table 1: Sizes of gauge invariant bases in the (F,B) sectors.

with the anisotropy parameter Δ. It turns out that our planar system at strong coupling (5) is equivalent to the above XXZ chain with (for details see[7])

$$L = F + B, \quad S^z = \sum_{i=1}^{L} s_i^z = F - B, \quad \text{and} \quad \Delta = \pm\frac{1}{2}$$

There are many consequences of this result, one of them being that there must exist a hidden supersymmetry of the above Heisenberg spin chain. In particular, since our SUSY generators change $F + B$ by one unit at strong coupling, the supersymmetry in question connects lattices with different sizes!

There exists a vast literature on lattice spin models. More than thirty years ago Baxter has found that, for $\Delta = -1/2$, the ground states with $S_z = \pm 1/2$ have particularly simple eigenenergy $E_0 = -\frac{3}{4}L$ for infinite L [12]. Recently his findings have been extended by Riazumov and Stroganov to any finite, odd L [13]. Our magic staircase appears at even F and $B = F \pm 1$ which suggests that

[3] Assuming there are no more phase transitions between $\lambda = 1$ and infinity.

the Riazumov-Stroganov states are nothing but our strong coupling vacua. Detailed inspection of the relations between H_{strong} and H_{XXZ} shows that indeed this is the case!

Amusingly, our system is also equivalent to another statistical model, namely that of the lattice gas of q-deformed bosons with the hamiltonian

$$H = B + \sum_{i=1}^{F} \delta_{N_i,0} + \sum_{i=1}^{F} b_i b^{\dagger}_{i+1} + b_i b^{\dagger}_{i-1}. \tag{6}$$

Skipping all the details (see again [7, 9]) we only mention that in view of this result the latter model, considered to be non-soluble until now [14], is in fact soluble as it becomes equivalent to the XXZ chain which is soluble *eg.* by the Bethe Ansatz. In fact a remarkable simplification occurs in the Bethe Ansatz when applied to H_{strong}. In particular, it can be used to construct analytically the first six SUSY vacua along our magic staircase [9].

Summarizing, we have shown that the Fock space methods successfully eliminate fermionic sign problem and can provide complete numerical solution (spectra and wave functions) for gauge systems with large (15 - 50) number of degrees of freedom (DOF). Applied in the planar limit they allow to diagonalize hamiltonians with an infinite number of colour DOF. Planar spectra of the space extended field theoretical systems can also be studied within this approach (see [15] for more references).

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References

[1] J. Wosiek and M. Campostrini, PoS LAT2005:273, 2006, hep-lat/0510030.

[2] J.D. Bjorken, *Quantum Chromodynamics* in Proc. Summer Inst. on Particle Physics, SLAC Report 224, Jan 1980; M. Claudson and M. B. Halpern, Nucl. Phys. **B 250** (1985) 689.

[3] M. Campostrini and J. Wosiek, Nucl. Phys. **B 703** (2004).

[4] S. Sethi and M. Stern, Comm. Math. Phys. **194** (1998) 675.

[5] J. Wosiek, Nucl. Phys. **B 644** (2002) 85; M. Campostrini and J. Wosiek, Phys. Lett. **B 550** (2002) 121.

[6] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys.Rev. **D 55** (1997) 5112.

[7] G. Veneziano and J. Wosiek, JHEP **01** (2006) 156; *ibid* **10** (2006) 033; *ibid* JHEP **11** (2006) 030.

[8] R. De Pietri, S. Mori and E. Onofri, JHEP **01** (2007) 018; M. Bonini, G.M. Cicuta and E. Onofri, J.Phys. **A40** (2007) F229-F234.

[9] G. Veneziano and J. Wosiek, hep-th/0603045v2; J. Wosiek, Acta Phys. Polon. **B 37** (2006) 3635.

[10] M. Trzetrzelewski and J. Wosiek, Acta Phys. Polon. **B35** (2004) 1615; hep-th/0308007.

[11] E. Onofri, G. Veneziano and J. Wosiek, Comm. Math. Phys. **274** (2007) 343.

[12] R. J. Baxter, Ann. Phys. (N.Y.) **70** (1972) 323.

[13] A.V. Razumov and Yu.G. Stroganov, J. Phys. **A34** (2001) 3185, cond-mat/0012141.

[14] N. M. Bogoliubov, R. K. Bullough and G. D. Pang, Phys. Rev. **B47** (1993) 11495.

[15] J.Wosiek, *in* Proc. HSQCD2008 Conference, Gatchina, Russia, June 30 - July 4, 2008.