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Numerical Investigation of the 2D $\mathcal{N}=2$ Wess-Zumino Model

Christian Wozar*, **Georg Bergner, Tobias Kästner, Sebastian Uhlmann, Andreas Wipf**
Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany
E-mail: `christian.wozar@uni-jena.de`

We study lattice formulations of the two-dimensional $\mathcal{N}=2$ Wess-Zumino model with a cubic superpotential. Discretizations with and without lattice supersymmetries are compared. We observe that the "Nicolai improvement" introduces new problems to simulations of the supersymmetric model. With high statistics we check the degeneracy of bosonic and fermionic masses on the lattice. Perturbative mass corrections to one-loop order are compared with continuum extrapolations of our lattice results in the weakly coupled regime. For intermediate couplings first results of fermionic masses in the continuum are presented where deviations from the perturbative result are observed.

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1. Introduction

The two-dimensional $\mathcal{N}=2$ Wess-Zumino model in the continuum shows no spontaneous supersymmetry breaking. However, a lattice formulation must break (part of) the supersymmetry explicitly due to the failure of the Leibniz rule on the lattice. In this model high statistics on large lattices are available and supersymmetry restoration effects can be analyzed numerically. The restoration of supersymmetry in the continuum limit must be treated carefully as has been demonstrated in supersymmetric quantum mechanics [1, 2]. One possible way to circumvent relevant supersymmetry breaking operators is the application of a blocking transformation to a free theory [3] leading to solutions similar to the Ginsparg-Wilson relation for the chiral symmetry [4].

A further suggestion keeps at least one supersymmetry on the lattice preserved and goes under the name of "Nicolai improvement" [5]. In former works [6, 7] such improved models using Wilson fermions were simulated, and discrepancies to the perturbative result as well as problems with the extraction of masses occurred at stronger couplings. In this work (see [8] for detailed analyses) the effects of the Nicolai improvement in the intermediate coupling regime are analyzed and compared to results of *unimproved* simulations. Additionally different fermion formulations (standard/twisted Wilson, SLAC) are explored.

2. The model

The continuum action with complex field $\varphi=\varphi_1+i\varphi_2$,

$$S_{\text{cont}}=\int d^2x\left(2\bar\partial\bar\varphi\partial\varphi+\tfrac{1}{2}|W'(\varphi)|^2+\bar\psi M\psi\right),\quad M=\gamma^z\partial+\gamma^{\bar z}\bar\partial+W''P_++\overline{W}''P_- \tag{2.1}$$

is invariant under *four real supercharges*. Taken together they satisfy the $\mathcal{N}=(2,2)$ superalgebra, and it has been argued that one supersymmetry can be preserved on the lattice [7].

The holomorphic superpotential here (see Fig. 1)

$$W(\varphi)=\tfrac{1}{2}m\varphi^2+\tfrac{1}{3}g\varphi^3 \tag{2.2}$$

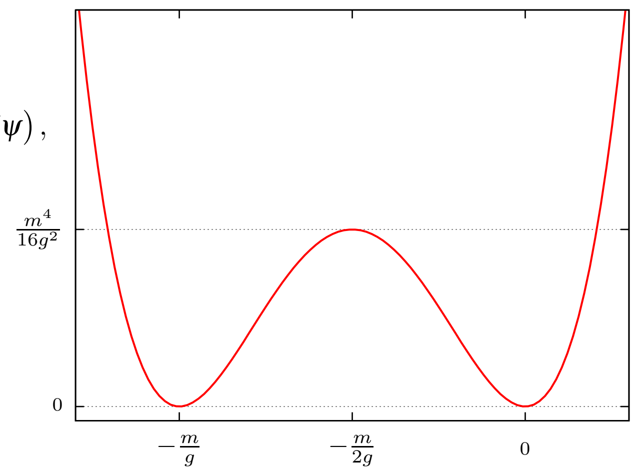


Figure 1: Classical potential $|W'(\varphi_1)|^2$ from (2.1) for vanishing imaginary part ($\varphi_2=0$). In the free theory limit ($g\to 0$) the left minimum is pushed towards minus infinity.

contains a mass parameter m and defines a dimensionless coupling $\lambda=\frac{g}{m}$. This theory possesses a discrete $\mathbb{Z}_2^4$ symmetry which in general is partially broken by a lattice discretization. A perturbative expansion in orders of λ around the free theory at $\lambda=0$ is possible and is used below.

2.1 Nicolai improvement

One real supersymmetry can be preserved on the lattice by using the action

$$S=\tfrac{1}{2}\sum_x\bar\xi_x\xi_x+\sum_{xy}\bar\psi_x M_{xy}\psi_y \tag{2.3}$$

in terms of the Nicolai variable $\xi_x = 2(\bar{\partial}\bar{\varphi})_x + W_x$ with $W_x = W'(\varphi_x)$, $W_{xy} := \partial W_x/\partial\varphi_y$ and

$$M_{xy} = \begin{pmatrix} W_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & \overline{W}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi_x}{\partial\varphi_y} & \frac{\partial\xi_x}{\partial\bar{\varphi}_y} \\ \frac{\partial\bar{\xi}_x}{\partial\varphi_y} & \frac{\partial\bar{\xi}_x}{\partial\bar{\varphi}_y} \end{pmatrix}. \tag{2.4}$$

In terms of the original field φ, the lattice action reads

$$S = \sum_x \Big(2\left(\bar{\partial}\bar{\varphi}\right)_x (\partial\varphi)_x + \tfrac{1}{2}\left|W_x\right|^2 + W_x(\partial\varphi)_x + \overline{W}_x(\bar{\partial}\bar{\varphi})_x\Big) + \sum_{xy} \bar{\psi}_x M_{xy}\psi_y. \tag{2.5}$$

This action only differs from a straightforward discretization by discretized *surface terms*

$$\Delta S = \sum_x \Big(W_x(\partial\varphi)_x + \overline{W}_x(\bar{\partial}\bar{\varphi})_x\Big) \tag{2.6}$$

which must vanish in the continuum limit.

2.2 The lattice discretization

To preserve the full supersymmetry of the free theory the same lattice derivatives for bosonic and fermionic degrees of freedom must be used. In this extensive study we compare three different lattice derivatives:

- Symmetric derivative $\left(\partial_\mu^{\mathrm{S}}\right)_{xy} = \frac{1}{2}(\delta_{x+\hat{\mu},y} - \delta_{x-\hat{\mu},y})$ with *standard Wilson* term $W_x = W'(\varphi_x) - \frac{r}{2}(\Delta\varphi)_x$ using $r=1$. The Wilson term must be added to W_x (and not to the derivative) in order to obtain an antisymmetric matrix $(\partial_\mu)_{xy}$. This results in a fermion matrix

$$M_{xy} = \begin{pmatrix} W''(\varphi_x)\delta_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & \overline{W}''(\bar{\varphi}_x)\delta_{xy} \end{pmatrix} - \frac{r}{2}\Delta_{xy}. \tag{2.7}$$

- Symmetric derivative ∂^{S} with *twisted (imaginary) Wilson* term $W_x = W'(\varphi_x) + \frac{ir}{2}(\Delta\varphi)_x$ resulting in

$$M_{xy} = \begin{pmatrix} W''(\varphi_x)\delta_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & \overline{W}''(\bar{\varphi}_x)\delta_{xy} \end{pmatrix} + \gamma_3\frac{r}{2}\Delta_{xy}. \tag{2.8}$$

  The choice $r = 2/\sqrt{3}$ reproduces the mass of the free theory up to $\mathscr{O}(a^4)$ at lattice spacing a as discussed in [1].

- *SLAC derivative* $\partial_{x\neq y} = (-1)^{x-y}\frac{\pi/N}{\sin(\pi(x-y)/N)}$, $\partial_{xx} = 0$ with fermion matrix

$$M_{xy} = \begin{pmatrix} W''(\varphi_x)\delta_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & \overline{W}''(\bar{\varphi}_x)\delta_{xy} \end{pmatrix}. \tag{2.9}$$

Using these discretizations, we have simulated the improved and unimproved (without surface terms) models applying a *combination of Fourier accelerated HMC with higher-order integrators*.

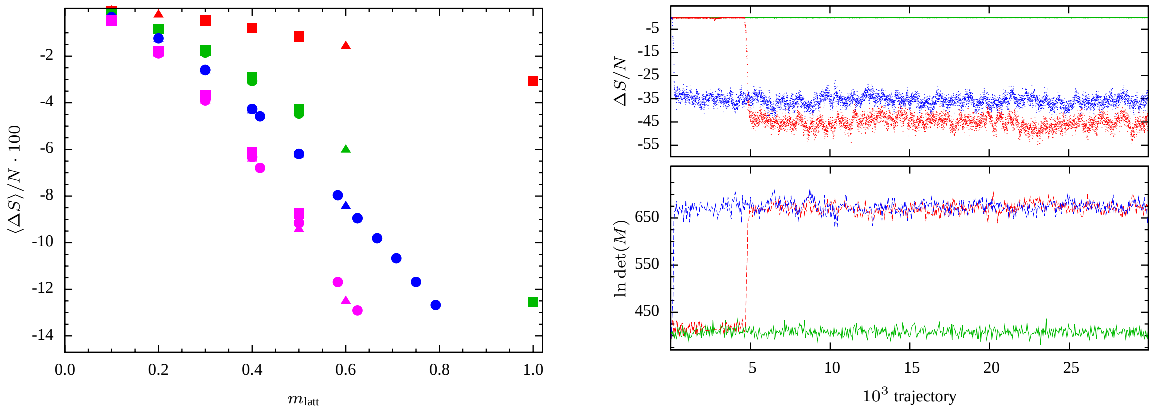


Figure 2: *Left:* Reduced improvement term $\Delta S/N$ for different lattice sizes: 9×9 (squares), 15×15 (triangles) and 25×25 (circles). Colors depict $\lambda = 0.8$ (red), 1.0 (green), 1.2 (blue), 1.5 (magenta). *Right:* MC history of improvement term and fermion determinant (SLAC improved, $N = 15\times 15$, $m_{\text{latt}} = 0.6$, $\lambda = 1.4$ (green), 1.7 (red), 1.9 (blue)).

3. Limitations of the Nicolai improvement

For simulations of the improved model including dynamical fermions the expectation value of the bosonic action is independently of λ fixed to

$$\langle S_{\text{B}} \rangle = N = \#\,\text{lattice points}. \tag{3.1}$$

Nevertheless the improvement term $\Delta S = \sum_x \left(W_x (\partial\varphi)_x + \overline{W}_x (\bar\partial \bar\varphi)_x \right)$ does not necessarily vanish. Therefore, we analyze ΔS with SLAC fermions at different couplings and for different lattice masses $m_{\text{latt}} = m/N_{\text{s}}$ (Fig. 2, left panel). Here N_{s} denotes the number of lattice points in spatial direction. In the continuum limit ($m_{\text{latt}} \to 0$) the improvement term consistently vanishes for every λ.

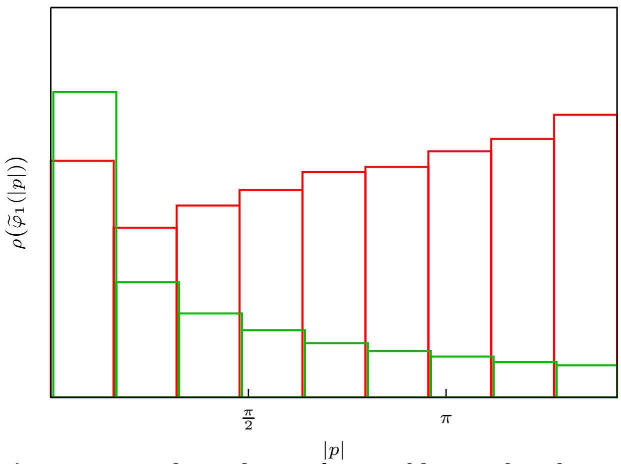


Figure 3: Mode analysis of ensembles in the physical (green, $\lambda = 1.4$) and unphysical (red, $\lambda = 1.7$) phase. Here ρ is the distribution function for the modulus of the lattice momentum averaged over 25,000 configurations (SLAC improved, $N = 15\times 15$, $m_{\text{latt}} = 0.6$).

For $\langle \Delta S \rangle / N > 14\%$ the behavior of improvement term and fermion determinant changes significantly (Fig. 2, right panel). The improvement terms dominates the bosonic action by more than one order of magnitude while $\langle S_{\text{B}} \rangle = N$ is still preserved. Additionally the fermion determinant grows drastically and so hinders the system from returning into the original region of configuration space. This instability can be explained by reconsidering the improved action

$$S_{\text{B}} = \frac{1}{2} \sum_x \left| 2(\partial\varphi)_x + \overline{W}_x \right|^2 = \sum_x \left(2\left(\bar\partial\bar\varphi\right)_x (\partial\varphi)_x + \tfrac{1}{2} \left|W_x\right|^2 \right) + \Delta S. \tag{3.2}$$

This action allows for two distinct behaviors of the fluctuating fields. The physically expected behavior consists of small fluctuations around the classical minima of the potential. Alternatively,

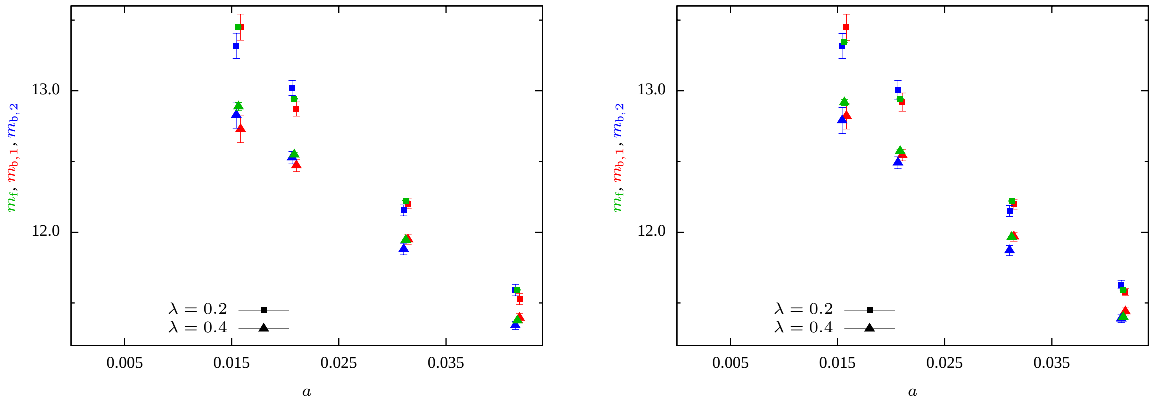


Figure 4: Masses for bosons (φ_1, φ_2, statistics 10^6–10^7 configs) and fermions (statistics 10^4 configs) for improved *(left)* and unimproved *(right)* model with standard Wilson fermions.

(3.2) allows for large fluctuations of kinetic and potential term to be compensated by the improvement term of opposite sign. In this situation, it is definitely no longer possible to extract meaningful physics.

Analyzing the distribution of the fields in momentum space at $\lambda = 1.4$ and $\lambda = 1.7$ (Fig. 3) shows that for too large couplings λ (or lattice masses m_{latt}) the simulation samples only *unphysical UV dominated* configurations. Therefore at strong coupling a *careful analysis of the improvement term* during the simulation must be ensured in order to achieve reasonable simulations.

4. Weak coupling

In the regime of weak couplings ($\lambda \leq 0.4$) we are able to match bosonic and fermionic masses so as to observe how well supersymmetry effects (e.g. the degeneracy of masses) are realized on the lattice. Furthermore continuum extrapolations of the different discretizations are compared to the result of continuum perturbation theory at one-loop order.

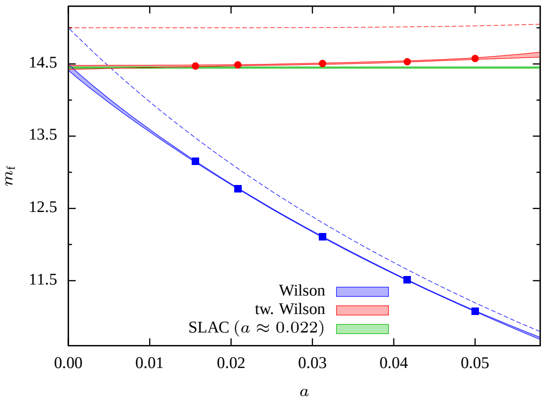
4.1 Signs of supersymmetry at finite lattice spacing

In an unbroken supersymmetric theory bosonic and fermionic masses coincide. In the lattice formulation the supersymmetry is broken explicitly (at least partially). This induces a possible breaking of the mass multiplets which is explored at different lattice spacings for $\lambda \in \{0.2, 0.4\}$, $m = 15$ (Fig. 4). Even with a statistics of up to 10^7 configurations the masses of bosons and fermions can *not be distinguished* in our simulations. Additionally improved and unimproved models give the same results (within error bars) for $\lambda \leq 0.4$.

This demonstrates that for Wilson type fermions in a region where the simulations do not show unphysical UV effects the Nicolai improvement is *not necessary*. A stable simulation with the unimproved model is likely to provide the same results, at least in the continuum limit.

4.2 Continuum extrapolation

For the free theory the lattice masses can be computed analytically. To make contact with perturbation theory which is carried out in the continuum it is crucial to get a stable continuum extrapolation even for the interacting case. Extrapolations from finite lattice spacing to the continuum using standard and twisted Wilson fermions for the improved model ($m = 15$, $\lambda = 0.3$) are shown in Fig. 5



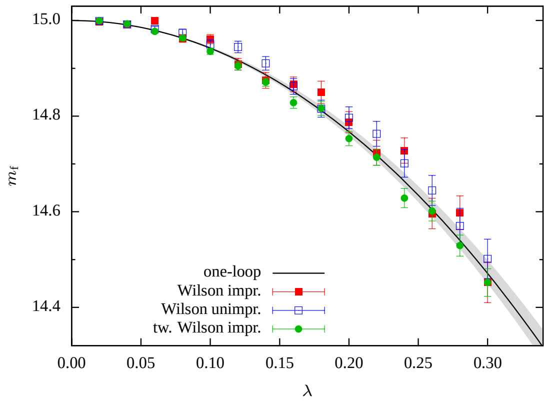


Figure 5: *Left:* The continuum extrapolation of fermionic masses for $\lambda = 0.3$ for the improved Wilson and twisted Wilson model. Here, the SLAC result is given for one single lattice size. For comparison the exact results for the free theory are also shown.
Right: Continuum extrapolation of fermionic masses for the weakly coupled regime in comparison to the perturbative result.

(left panel). These are based on lattice sizes $N_s \in \{20, 24, 32, 48, 64\}$ and demonstrate that both formulations yield the same continuum result. Additionally this result also coincides with a prediction by the SLAC model on a finite lattice ($N = 45 \times 45$). Furthermore the twisted Wilson fermions are much closer to the continuum limit than standard Wilson fermions for finite lattice spacing. Therefore our analysis of the intermediate coupling case uses only the twisted type of Wilson fermions and SLAC fermions.

4.3 Comparison to perturbation theory

For small λ we compare the perturbative one-loop result for the renormalized mass

$$m_{\text{ren}}^2 = m^2 \left(1 - \frac{4\lambda^2}{3\sqrt{3}}\right) + \mathcal{O}(\lambda^4) \quad (4.1)$$

to the continuum extrapolation of the lattice data (Fig. 5, right panel). All different formulations are seen to *coincide with perturbation theory*. Even for the unimproved model with Wilson fermions the correct (supersymmetric) continuum limit is reached within error bars.

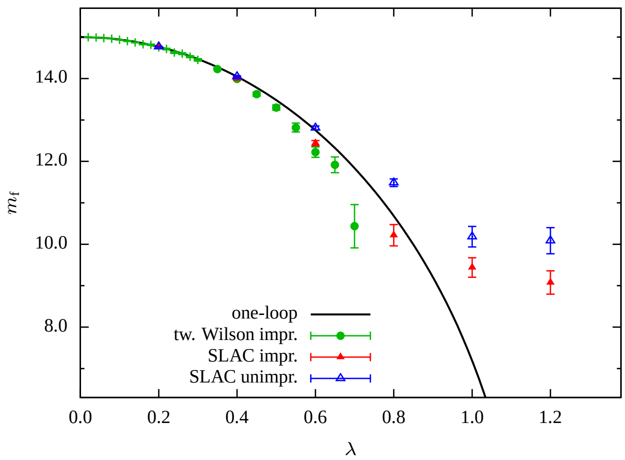


Figure 6: Masses of the improved and unimproved model with SLAC fermions on a 45×45 lattice and continuum extrapolated results for twisted Wilson fermions are compared with the perturbative one-loop result in the continuum.

5. Intermediate coupling

To explore the limitations of the one-loop calculation we have performed simulations with $\lambda \in [0, 1.2]$ (see Fig. 6). The continuum extrapolations of Wilson type fermions are only applicable up to $\lambda \leq 0.7$ using lattice sizes of $N_s \leq 64$ due to the improvement problems. To cope with this, we instead use SLAC fermions which allow for a much larger λ range on the accessible lattice sizes.

For $\lambda > 0.6$ the improved and unimproved model with SLAC fermions give slightly different results on a 45×45 lattice. To check which model is closer to the continuum limit additional simulations with $N = 63\times 63$ at $\lambda = 0.8$ have been performed (Tab. 1). The data unveil that the unimproved model suffers from stronger finite a effects. Therefore the correct continuum limit is reached for both SLAC models but the *improved SLAC model is closer to the continuum limit* on a finite lattice.

6. Conclusions and outlook

We have performed a detailed analysis of the Nicolai improvement in the Wess-Zumino model. This improvement introduces new problems due to the sampling of unphysical (high-momentum) states. Additionally with high statistics bosonic and fermionic masses can *not be distinguished* for Wilson type fermions at finite lattice spacing in the weak to intermediate coupling region for both the improved and unimproved formulations. Even without improvement the *correct continuum limit* is reached. Therefore the term "improvement" is somewhat misleading. Only for SLAC fermions in the intermediate coupling region the improved action is closer to the continuum limit on finite lattices.

N_s	improved	unimproved
45	10.22(26)	11.49(9)
63	10.54(15)	10.70(19)

Table 1: Fermionic masses for the SLAC derivative on two different lattice sizes for $\lambda = 0.8$.

More detailed results on this model including the discussion of discrete symmetries, the absence of finite size effects and the effects of negative fermion determinants can be found in [8].

In order to access the region of stronger couplings ($\lambda > 1.5$) further algorithmic improvements are necessary. With the help of the PHMC or RHMC algorithm and improved solvers and preconditioners we are confident to obtain strong coupling results in the near future. The elaborate algorithms will then be used to explore the $\mathcal{N}=1$ Wess-Zumino model in two dimensions where a spontaneous supersymmetry breaking is expected and to study supersymmetric $\mathbb{C}\mathrm{P}^N$ models in two dimensions.

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