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Measurement of shear viscosity in lattice gauge theory without Kubo formula

Masakiyo Kitazawa*
Department of Physics, Osaka University, Toyonaka 560-0043, Japan
E-mail: kitazawa@phys.sci.osaka-u.ac.jp

Masayuki Asakawa
Department of Physics, Osaka University, Toyonaka 560-0043, Japan
E-mail: yuki@phys.sci.osaka-u.ac.jp

Berndt Müller
Department of Physics, Duke University, Durham, NC 27708, USA
E-mail: muller@phy.duke.edu

Chiho Nonaka
Department of Physics, Nagoya University, Nagoya 464-8602, Japan
E-mail: nonaka@hken.phys.nagoya-u.ac.jp

A method to measure the transport coefficients on the lattice is proposed. We introduce a spatially-inhomogeneous momentum source to the Hamiltonian in order to generate a non-equilibrium but steady hydrodynamic flow. Once such hydrodynamic flow is created with appropriate spatial geometries, the transport coefficients are determined from expectation values of the energy-momentum tensor. A Monte Carlo simulation for SU(3) gauge theory is performed to measure the shear viscosity with this method. The effect of the momentum source is taken into account with the Taylor expansion up to the third order. Our numerical result shows, however, that the hydrodynamic flow is not generated up to this order in this formalism.

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*Speaker.

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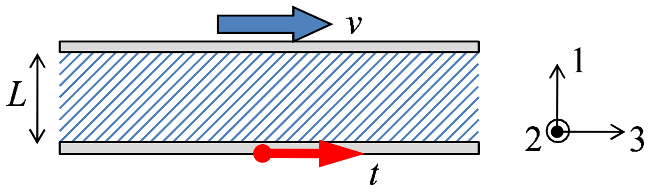

Figure 1: Experimental setting to measure the shear viscosity.

1. Introduction

The experimental results at Relativistic Heavy Ion Collider (RHIC) indicate that the time evolution of the quark-gluon plasma (QGP) near the critical temperature for deconfinement, T_c, is well described by the perfect hydrodynamic equation without viscosities [1]. In the framework of AdS/CFT correspondence, it is suggested that the matter described by the supersymmetric Yang-Mills theroy at strong coupling has a shear viscosity to entropy ratio $\eta/s = 1/4\pi$ [2]. This value is now conjectured as the universal lower bound for η/s. After these experimental and theoretical developments, the investigation of transport properties of the QGP near T_c is one of the most hot topics in these communities.

Lattice QCD Monte Carlo simulation provides a method to analyze properties of the QGP phase in the non-perturbative region directly from the first principle of QCD. There have been done several attempts to measure transport coefficients with numerical simulations on the lattice [3]. These analyses evaluate transport coefficients using the Kubo formulae, which relate transport coefficients to the low-energy behavior of the spectral functions for the energy-momentum tensor. In this method, one must determine the spectral function, which is a real time quantity, from the discretized lattice correlator in the Euclidean space. The reconstruction of the spectral function from the lattice correlator, however, is an ill-posed problem. Although there are considerable attempts to overcome this difficulty, lattice data used so far seem not fine enough for this reconstruction [4]. Moreover, it is non-trivial whether the spatial volume of lattices analyzed thus far is large enough to argue the hydrodynamics, which is a notion valid at long range.

In this paper, we propose a different strategy to evaluate the transport coefficients on the lattice. This idea is inspired by experimental methods to measure transport coefficients. While this idea can deal with various transport phenomena, in the following we limit our attention to the shear viscosity. An experimental setting to measure the shear viscosity is shown in Fig. 1: The target fluid is surrounded by two parallel boards with a distance L, and the upper board is moving with a velocity v. The lower board then feels a shear stress t from the fluid. By measuring this stress, the shear viscosity η of the fluid is determined to be $\eta = tL/v$.

In our strategy, we first attempt to generate a spatially-inhomogeneous system having a hydrodynamic flow on the lattice with a velocity profile similar to this experiment. Once such flow is created, the shear viscosity can be determined as the experiment with direct measurements of the velocity and shear stress on the lattice. Here, we remark that the system that we are considering now is not in equilibrium, but a steady state. The steady system can be treated in the imaginary time formalism, and thus this numerical simulation can be carried out on the lattice in principle.

The hydrodynamic flow on the lattice may be realized by adding a spatially-inhomogeneous momentum source to the Hamiltonian H,

$$H \to H_\lambda = H - \lambda T_{\rm source}, \tag{1.1}$$

where λ is a Lagrange multiplier. The source term $T_{\rm source}$ is given by a sum of the local momentum operators $P^i(\vec{x}) = T^{0i}(\vec{x})$, with $T^{\mu\nu}(\vec{x})$ being the energy-momentum tensor. In this study, we take the effect of $T_{\rm source}$ into account by the Taylor expansion with respect to λ. This expansion may be justified since an infinitely slow hydrodynamic mode is sufficient for our purpose.

In the following discussion, we consider the pure gauge theory for simplicity. We use the natural units $\hbar = c = 1$.

2. Hydrodynamic equation and shear viscosity

In order to make the outline of our strategy clear, let us first solve the hydrodynamic equation with a boundary condition corresponding to Fig. 1. The dissipative hydrodynamic equation is written by [5],

$$\partial_\nu T^{\mu\nu} = 0, \tag{2.1}$$

where each component of the energy-momentum tensor is given by the macroscopic quantities as

$$T^{\mu\nu} = -pg^{\mu\nu} + (\varepsilon + p)u^\mu u^\nu + \Delta T^{\mu\nu}, \tag{2.2}$$

$$\Delta T^{\mu\nu} = \eta(\Delta^\mu u^\nu + \Delta^\nu u^\mu) + \left(\frac{2}{3}\eta - \zeta\right) H^{\mu\nu}\partial_\rho u^\rho, \tag{2.3}$$

where ε and p are energy density and pressure, u^μ denotes the velocity field, η and ζ represent shear and bulk viscosities, respectively, $\Delta^\mu = \partial^\mu - u^\mu u_\nu \partial^\nu$, and $H^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}$. To reproduce the experimental geometry, now we consider a static system having a translational invariance along 2 and 3 directions. Furthermore, we assume that u^μ has only one non-vanishing spatial component, i.e.,

$$u^\mu(x^1) = (1, 0, 0, u^3(x^1)), \tag{2.4}$$

and that the amplitude is small, $|u^3(x^1)| \ll 1$. With this ansatz, all components in the l.h.s. of Eq. (2.1) trivially vanish except for

$$\partial_1 T^{31} = \eta\partial_1\partial^1 u^3 = 0. \tag{2.5}$$

Eq. (2.5) shows that $u^3(x^1)$ is linear as a function of x^1; by determining the velocities at two boundaries, $u^3(x^1)$ between them is just a straight line connecting these values. Substituting this solution to Eq. (2.2), we obtain

$$T^{31} = \eta\partial^1 u^3, \qquad T^{03} = (\varepsilon + p)u^3. \tag{2.6}$$

These solutions lead to the formula for η given by the momentum T^{03} and shear stress T^{31} as

$$\eta = (\varepsilon + p)\frac{T^{31}}{\partial^1 T^{03}} = (T^{00} + T^{11})\frac{T^{31}}{\partial^1 T^{03}}, \tag{2.7}$$

where we have neglected $u^3(x^1)$ dependences of ε and p, which are higher order effects. We note that Eq. (2.5) also shows that T^{31} and $\partial^1 T^{03}$ take constant values in this system. Using Eq. (2.7), we can determine the shear viscosity on the lattice by measuring the expectation values of $T^{\mu\nu}(\vec{x})$ under the presence of the hydrodynamic flow.

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3. Momentum source

The energy-momentum tensor in the pure gauge theory is written by the microscopic degrees of freedom as

$$T^{\mu\nu} = 2\mathrm{tr}\left[F^{\mu}_{\rho}F^{\rho\nu} + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right], \tag{3.1}$$

where $F^{\mu\nu}$ is the field strength. In the quantum statistical mechanics, the expectation value of Eq. (3.1) is given by $\langle T^{\mu\nu}(\vec{x})\rangle = \mathrm{Tr}[T^{\mu\nu}(\vec{x})\mathrm{e}^{-H/T}]/Z$, where H is the hamiltonian for the gauge field. The expectation values in the system having the hydrodynamic flow may be calculated by adding a source term $\lambda T_{\mathrm{source}}$ to the Hamiltonian as in Eq. (1.1), which leads to

$$\langle T^{\mu\nu}(\vec{x})\rangle_{\lambda} = \frac{1}{Z_{\lambda}}\mathrm{Tr}\left[T^{\mu\nu}(\vec{x})\mathrm{e}^{-(H-\lambda T_{\mathrm{source}})/T}\right], \tag{3.2}$$

where $Z_{\lambda} = \mathrm{Tr}[\mathrm{e}^{-(H-\lambda T_{\mathrm{source}})/T}]$. We put the subscript λ to the l.h.s. of Eq. (3.2), to emphasize that the expectation value is taken with the source. To realize the velocity profile satisfying Eq. (2.4) on the lattice with a finite volume with a periodic boundary condition, we put momentum sources on two planes at $x^1 = 0$ and $L_x/2$ pointing different directions, i.e.,

$$T_{\mathrm{source}} = T(0) - T(L_x/2), \tag{3.3}$$

$$T(r) = \int dx^2 dx^3 T^{03}(r, x^2, x^3), \tag{3.4}$$

with L_x representing the length of the lattice along x^1 direction.

We note that $T^{\mu\nu}$ written by the microscopic degrees of freedom, Eq. (3.1), and those written by hydrodynamic quantities, Eq. (2.2), represent the same physical quantity, while the latter is the coarse grainined version of the former. Statistical mechanics tells us that the expectation values of Eq. (3.1) after the coarse graining follow the hydrodynamic equation Eq. (2.1) at sufficiently long scale. In order to discuss the dissipative phenomena in our strategy, we must first check that such *hydrodynamic* flow is created on the lattice; in other words, we must confirm that $\langle T^{03}(x^1)\rangle_{\lambda}$ behaves linearly as a function of x^1 at distance sufficiently far from the sources. Equation (2.7) is then applied to such a region. The hydrodynamic behavior, on the other hand, can be violated at short range. Such behavior of $\langle T^{03}(x^1)\rangle_{\lambda}$ may be observed near the sources. Measurements of such deviations would in turn make it possible to estimate the length scale where the hydrodynamic picture is valid.

4. Taylor expansion

To calculate Eq. (3.2) for the momentum operator on the lattice, we write it in the path-integral formalism,

$$\langle T^{0i}(x)\rangle_{\lambda} = -\frac{1}{Z_{\lambda}}\int \mathscr{D}A_{\mu}\, iT^{E}_{0i}(x)\exp\left[-S^E - i\lambda\left(T^E(0) - T^E(L_x/2)\right) + \lambda^2(\cdots)\right], \tag{4.1}$$

where S^E is the Euclidean action, $T^E_{\mu\nu}(x)$ is the energy-momentum tensor in the Euclidean space, with x denoting the Euclidean four vector, and

$$T^E(r) = T\int_0^{1/T} d\tau \int dx_2 dx_3 T^E_{03}(x)|_{x_1=r}. \tag{4.2}$$

$N_x \times N_{yz}^2 \times N_t$	L_x[fm]	$N_{\mathrm{conf.}}$
$64 \times 32^2 \times 6$	3.13	23,000
$128 \times 32^2 \times 6$	6.27	24,000
$192 \times 32^2 \times 6$	9.41	13,000

Table 1: Simulation parameters.

The imaginary time $x^4 = \tau$ is introduced in this procedure, while $T^{\mu\nu}(\vec{x})$ in Eq. (3.2) is time-independent; they are related as $T^E_{4i} = iT^{0i}$. There appears terms proportional to λ^2 in the argument of the exponential in Eq. (4.1), which however we abbreviated to save space.

In this study, we incorporate the effect of the source by the Taylor expansion,

$$\langle T^{\mu\nu}(x)\rangle_\lambda = T^{\mu\nu}_{(0)}(x) + \lambda T^{\mu\nu}_{(1)}(x) + \frac{\lambda^2}{2!}T^{\mu\nu}_{(2)}(x) + \frac{\lambda^3}{3!}T^{\mu\nu}_{(3)}(x) + \cdots. \tag{4.3}$$

The coefficients of the zeroth and first order terms for $\langle T^{03}(x)\rangle_\lambda$ are calculated to be, respectively,

$$T^{0i}_{(0)}(x) = -\frac{1}{Z}\int \mathscr{D}A_\mu iT^E_{0i}(x)\exp\left[-S^E\right] = i\langle T^E_{0i}(x)\rangle_0, \tag{4.4}$$

$$T^{0i}_{(1)}(x) = \frac{1}{Z}\int \mathscr{D}A_\mu iT^E_{03}(x)T_{\mathrm{source}}\exp\left[-S^E\right] = -\langle T^E_{03}(x)T^E(0)\rangle_0 + \langle T^E_{03}(x)T^E(L_x/2)\rangle_0), \tag{4.5}$$

where $\langle\cdots\rangle_0$ denotes the expectation value with $\lambda = 0$. Higher order coefficients than the first order include multi-point correlation functions with $\lambda = 0$.

Now let us consider $\langle T^{03}(\vec{x})\rangle_\lambda$. First, we notice that the two-point spatial correlator, $C(x) \equiv \langle T^E_{03}(x)T^E(0)\rangle_0$, approaches a constant at long distance, corresponding to the massless mode. In the first order term Eq. (4.5), this constant cancels between the two correlators and does not contribute to the long range behavior. The other part in $C(x)$ vanishes at long distance. Thus, $T^{03}_{(1)}(x) = C(x) - C(L_x/2 - x)$ cannot reproduce the linear behavior required by the hydrodynamic equation at $L_x \to \infty$, and this order is never responsible for the hydrodynamic behavior at long range. Furthermore, the even order terms in Eq. (4.3) should vanish since $\langle T^{03}(\vec{x})\rangle_\lambda$ is an odd function of λ.

From these arguments, we understand that we need *at least* the third order term to argue the hydrodynamic mode in the Taylor expansion. The thrid order term $T^{03}_{(3)}(\vec{x})$ includes, for example, the four-point correlator

$$\langle T^E(0)^2 T^E_{03}(x)T^E(L_x/2)\rangle_0. \tag{4.6}$$

Since this term could contain the length scale $L_x/2$ as a function of x^1, it possibly reproduces the linear function required by the hydrodynamics, indeed.

5. Numerical results

To calculate the shear viscosity numerically, we have performed a Monte Carlo simulation for the SU(3) Yang-Mills theory. We take $\beta = 6.499$ corresponding to the lattice spacing $a = 0.049$fm [6]. The simulation is performed for three different volumes $N_x \times N_{yz}^2 \times N_\tau$ with a periodic boundary

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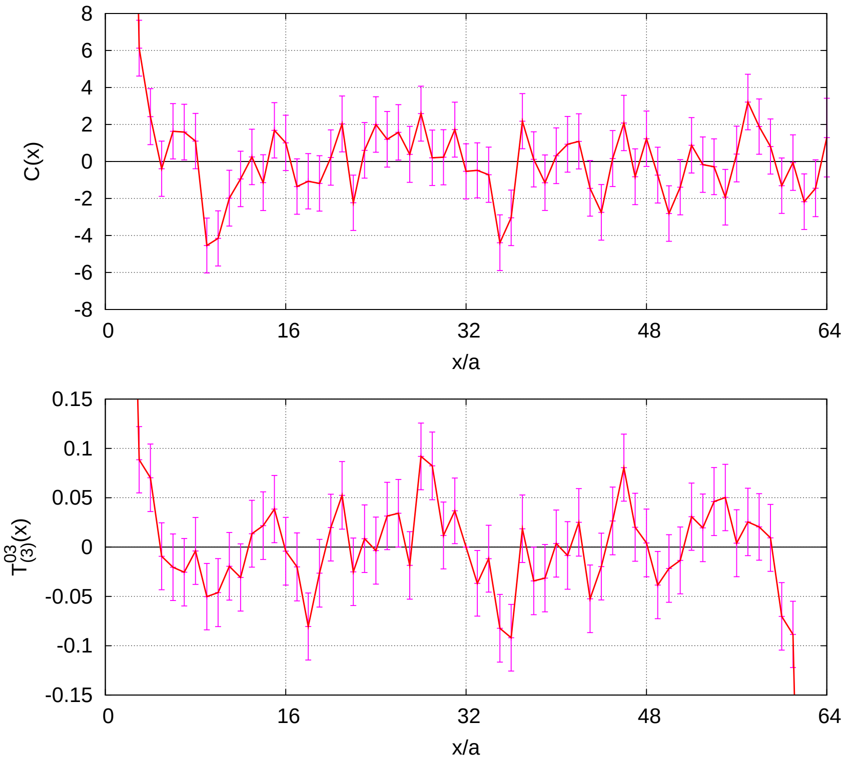

Figure 2: First and Third order coefficients of Taylor expansion for $\langle T^{03}\rangle_\lambda$ on the lattice of size $128\times 32^2\times 6$. The sources are at $N_x = 0$ and 64.

condition. The parameters and numbers of configurations are summarized in Table. 1. In order to see the long range behavior, we took one of the spatial direction, N_x, large. All lattices have the same temporal length $N_\tau = 6$, which corresponds to $T \simeq 2.5T_c$. We generated gauge configurations with the heatbath and overrelaxation algorithms; each configuration is separated by $20-60$ steps of one heatbath and four overrelaxation updates. These calculations are carried out on 128-node partition at bluegene@KEK. For the momentum operator $T^E_{0i}(x)$ on the lattice, we have chosen the definition Eq. (3.1) with the clover operator for the field strength.

In the upper panel of Fig. 2, we show the behavior of one of the correlator in Eq. (4.5), $C(x^1)$, on the lattice with $N_x = 128$. The horizontal axis represents x^1 in the lattice unit. The figure shows that $C(x^1)$ takes nonzero values only near the source at the origin, and damps quickly as x^1 becomes large. In particular, this function is zero within the statistical error for $x^1 \gtrsim 8a \simeq 0.4$fm. As discussed before, this term should be a constant in the range where the hydrodynamics is valid. The sudden damp of this term thus does not contradict the validity of hydrodynamic equation at the length scale of order 1fm, as suggested by the success of hydorodynamic models at RHIC.

The lower panel of Fig. 2 shows the third order coefficient $T^{03}_{(3)}(x^1)$. One sees that $T^{03}_{(3)}(x^1)$ is again zero within the statistical error except in the vicinity of the sources at $x^1/a = 0$ and 64. This result shows that the momentum flow does not manifest itself at all even at the third order of Taylor expansion. Without the hydrodynamic flow, we cannot extract the shear viscosity through Eq. (2.7). Finally, we note that the short range behavior of the momentum flow near the sources is governed by the microscopic dynamics, and we cannot conclude anything about the transport properties from this feature.

6. Summary

With a steady hydrodynamic flow satisfying Eq. (2.4), the shear viscosity is related to the energy-momentum tensor through Eq. (2.7). We have proposed to measure the shear viscosity using Eq. (2.7), by creating a steady hydrodynamic flow on the lattice with the momentum source Eq. (3.3). A numerical simulation has been performed. The effect of the source has been incorporated by the Taylor expansion up to the third order. The numerical result shows, however, that the momentum flow damps quickly near the source, and that the hydrodynamic flow at long range cannot be described by the present formalism. Since we did not observe the hydrodynamic mode on the lattice, the transport coefficients were not able to be measured up to the third order.

To realize the measurement of the shear viscosity using Eq. (2.7), we must first understand the reason why any hydrodynamic modes are not observed in the present analysis. A possibility is just a lack of the statistics. Increasing the number of configurations would clarify this point. It is also possible that the finite order terms in the Taylor expansion can never describe the hydrodynamic flow. If this is the case, Monte Carlo simulations without the Taylor expansion are required. These analyses will be performed elsewhere.

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