PROCEEDINGS OF SCIENCE

Light-Cone Hamiltonian for a $q\bar{q}$ meson

B. Galow[*a], **H.J. Pirner**[ab] **and O. Schlaudt**[a]
[a]*Institut für Theoretische Physik, Universität Heidelberg, Germany*
[b]*Max-Planck-Institut für Kernphysik Heidelberg, Germany*
E-mail: galow@tphys.uni-heidelberg.de, schlaudt@tphys.uni-heidelberg.de, pirner@tphys.uni-heidelberg.de

The vacuum expectation value of a Wegner-Wilson loop representing a fast moving quark-antiquark pair defines the light-cone Hamiltonian for a $q\bar{q}$ meson. We solve the corresponding Schrödinger equation for various trial wave functions. The result shows how confinement determines the light-cone wave function for valence quarks in a rather model-independent way. The correct chiral-symmetry behavior of the pion mass is obtained when the self-energy of the quark is chosen properly.

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1. Introduction

One of the challenges in quantum chromodynamics (QCD) is to solve the relativistic bound state problem. In the light-cone Hamiltonian approach [1] light-cone wave functions are boost invariant and have a well-defined probability interpretation - in contrast to the Bethe-Salpeter equation. It is necessary to have reliable light-cone wave functions, especially if one wants to calculate exclusive reactions. Various approaches have been proposed to compute such wave functions. In ref. [2], Simula uses the usual equal-time Hamiltonian and transforms the resulting wave functions into the light-cone form with the help of kinematical on-shell equations. In ref. [3], Simonov and collaborators derive a light-cone Hamiltonian in a model with certain string degrees of freedom. More ambitious is the construction of an effective Hamiltonian including the QCD gauge degrees of freedom explicitly and then solving the bound-state problem. For mesons, this approach [4, 5] still needs many parameters which have to be fixed. Attempts have also been made to find the valence-quark wave function for mesons with a simple Hamiltonian [6].

A necessary input for the calculation of a two-body Fock state is an adequate potential in the light-cone Hamiltonian. For the equal-time Hamiltonian and heavy quarks the calculation of Wegner-Wilson loops provides the form of the non-perturbative potential at large distances. Numerical lattice simulations of QCD can give an accurate non-relativistic Hamiltonian. The continuum stochastic vacuum model [7, 8] allows one to calculate vacuum expectation values of Wegner-Wilson loops using perturbative and non-perturbative field-strength correlation functions as input. One can compute the loop expectation value $\langle W[C]\rangle$ in terms of a gauge-invariant bilocal gluon field-strength correlator integrated over the minimal surface by using the non-Abelian Stokes' theorem and the matrix cumulant expansion in the Gaussian approximation. The gluon field-strength correlator has perturbative and non-perturbative components. The stochastic vacuum model is used for the non-perturbative low-frequency background field, and the perturbative gluon exchange is used for the additional high-frequency contributions. The calculation of the expectation value of a Wegner-Wilson loop along the imaginary-time direction gives the heavy quark-antiquark potential with color-Coulomb behavior for small and confining linear rise for large sources' separations [9].

Since the computation of the VEV for the Wegner-Wilson loop can be done completely analytically, also other orientations of the loop can be chosen, e.g. a loop where the quark-antiquark pair moves along the z-direction. By transforming to Minkowski space-time, the dependence of the interaction potential on longitudinal and transverse separations of the pair can be obtained this way. Approaching light-like trajectories of the quark-antiquark pair, we have deduced in ref. [10] a light-cone Hamiltonian, which contains confinement from first principles.

This article gives a brief survey of our paper ref. [11], in which we would like to complete the Hamiltonian of ref. [10] by including quark self-energy effects, quark wave-function renormalization and spin-spin interactions phenomenologically and evaluate the eigenvalues of the full Hamiltonian for light and heavy mesons variationally.

2. The light-cone Hamiltonian

The light-cone Hamiltonian derived in ref. [10] for light valence quarks of mass μ has a simple confining potential, the magnitude of which is set by the string tension $\sigma = 0.18\,\mathrm{GeV}^2$. In the notations of reference [12], we introduce as dynamical variables the light-cone momentum fraction $\xi = k^+/P^+$ with $|\xi| < 1/2$ and its conjugate variable, namely the scaled longitudinal space coordinate $\sqrt{2}\rho = P^+ x_-$. The effective "distance" between the quarks is given by the scale-free light-cone longitudinal distance $\hat{\rho}$ and the transverse distance $x_\perp$ multiplied by the bound-state mass. Note that the transverse confinement scale is related to the self-consistent mass of the bound state M^2. The so-obtained light-cone Hamiltonian $H_{LC} = 2P^+P^-$ is Lorentz invariant under boosts, because the variables $\xi, \rho, k_\perp$, and $x_\perp$ are boost invariant. The transverse momentum and the longitudinal space coordinate are represented by the operators

$$\hat{k}_\perp = \frac{1}{i}\vec{\nabla}_\perp \tag{2.1}$$

and

$$\hat{\rho} = \frac{1}{i}\frac{d}{d\xi}, \tag{2.2}$$

so that the Hamiltonian reads ($\hbar = 1$):

$$H_{LC}^{q\bar{q}}(\mu^2) = M^2 = \frac{(\mu^2 + \hat{k}_\perp^2)}{1/4 - \xi^2} + 2\sigma\sqrt{\hat{\rho}^2 + M^2 x_\perp^2}. \tag{2.3}$$

The other, non-confining, potential has been worked out similarly, and is treated in ref. [11] in more details.

The best way to find the two-body wave function is to use as variables the light-cone momentum fraction ξ and the transverse quark-antiquark separation $x_\perp$. It is expected, that with the Hamiltonian of eq. (2.3), the meson masses, and especially the pion mass, are not described correctly. Additional terms are needed for a realistic valence-quark Hamiltonian. Indeed, it is a matter of a simple variational calculation to find out that the eigenvalues of the light-cone Hamiltonian in the form (2.3) are of the order of $M = 1.6$ GeV for $\mu^2 = 0$. This is obviously too high compared with good valence $q\bar{q}$-mesons like the vector mesons, which have an energy of 800 MeV for light quarks. First, the spin structure of the meson is not properly taken care of in the spin-independent expression above. Secondly, one expects quark self-energy corrections, which are especially important for small current quark masses.

In the literature, the quark self-energy has been deduced from the stochastic vacuum model in two calculations [13, 14]. In the first version [13]

$$\Delta_1\left(\mu^2\right) = -\frac{4\sigma}{\pi} \tag{2.4}$$

has been derived from the confining gluon field configurations interacting with the q-field. In the second version [14], Simonov takes into account the confined $q\bar{q}$ state with mass M and finds:

$$\Delta_2\left(\mu^2\right) = -\frac{4\sigma}{\pi}\phi(t) \tag{2.5}$$

with

$$\phi(t) = t\int_0^\infty dz z^2 K_1(tz) e^{-z}, \tag{2.6}$$

where

$$t = (\mu + M/2)a. \tag{2.7}$$

Here μ is the current quark mass, M is the uncorrected meson mass, $a = 0.302$ fm is the correlation length of the field-strength correlator. The dependence of the self-energy correction $\Delta_2\left(\mu^2\right)$ on the meson mass is shown in Fig. 1.

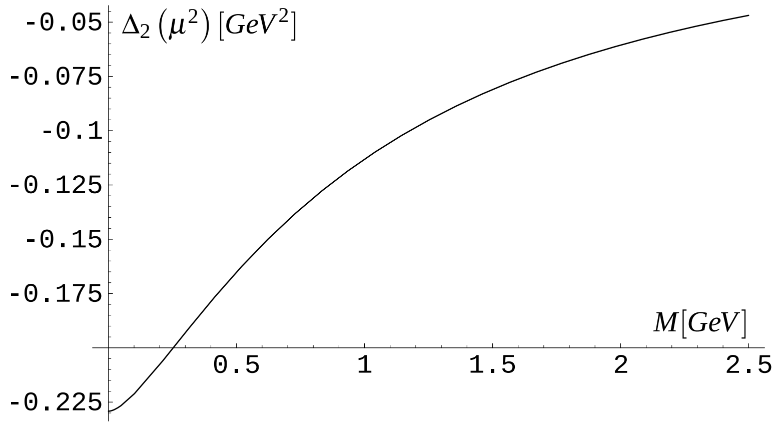

Figure 1: Self-energy correction $\Delta_2\left(\mu^2\right)$ for vanishing current quark mass $\mu = 0$ as a function of the uncorrected meson mass M.

This self-energy correction $\Delta_2\left(\mu^2\right)$ of eq. (2.5) agrees with the constant self-energy correction of eq. (2.4) $\Delta_1\left(\mu^2\right) = -0.23\,\mathrm{GeV}^2$ for $M = 0$. The self-energy correction is negative for light flavors, and vanishes for heavy quarks, i.e. for heavy-meson masses M. Such a functional behavior looks rather reasonable. Elimination of higher $q\bar{q}$-gluon states produces an attractive interaction.

3. Variational solution of the light-cone Hamiltonian

We evaluate the Hamiltonian for zero current quark masses $\mu^2 = 0$, but with quark self-energy correction $\Delta_1\left(\mu^2\right)$:

$$< \Psi|\hat{H}_{LC}(\Delta_1\mu^2)|\Psi >= M^2. \tag{3.1}$$

We compute the vacuum expectation value of the Hamiltonian eq. (2.3), using a variational method. Simple trial wave functions factorize in a longitudinal wave function $\phi(\xi)$ and a transverse wave function $\varphi(x_\perp)$. We take the following two trial wave functions $(i = 1, 2)$, where the first one has the conventional form of ξ-dependence:

$$\Psi_i(\xi, \vec{x}_\perp) = \phi_i(\xi) \cdot \varphi(x_\perp) \text{ for } i = 1, 2 \tag{3.2}$$

$$\varphi(x_\perp) = \frac{1}{\sqrt{\pi}x_0} \cdot \exp\left[-\frac{\vec{x}_\perp^2}{2x_0^2}\right] \tag{3.3}$$

$$\text{with } \phi_1(\xi) = \sqrt{6} \cdot \left(\frac{1}{4} - \xi^2\right)^{1/2} \tag{3.4}$$

$$\text{and } \phi_2(\xi) = \sqrt{\frac{8}{\pi}} \cdot \left(\frac{1}{4} - \xi^2\right)^{1/4}, \tag{3.5}$$

x_0 being the mean transverse extension of the meson.

The wave functions vanish at the kinematical boundaries ($\xi = \pm\frac{1}{2}$) which correspond to the limits of relative infinite longitudinal momenta in the non-relativistic description:

$$\Psi_i\left(\xi = -\frac{1}{2}, x_\perp\right) = \Psi_i\left(\xi = \frac{1}{2}, x_\perp\right) = 0. \tag{3.6}$$

In both cases, we get self-consistent transcendental equations for M, which can be solved numerically. In Fig. 2, we plot the resulting M as a function of the transverse-extension parameter x_0 of the trial wave functions Ψ_i. The trial wave function Ψ_1 leads to a smaller value of the meson mass, which lies in the expected range of light vector-meson masses. The higher mass corresponding to the trial wave function Ψ_2 comes about from the higher longitudinal momenta in this wave function. The rms-extensions $\sqrt{< x_\perp^2 >} = x_{0,i}$ of the mesons can be read off from the minima of both curves. We obtain

$$x_{0,1} = 0.8\,\text{fm and } x_{0,2} = 0.86\,\text{fm}. \tag{3.7}$$

The corresponding mass values are

$$M_1 = 0.85\,\text{GeV and } M_2 = 0.89\,\text{GeV}. \tag{3.8}$$

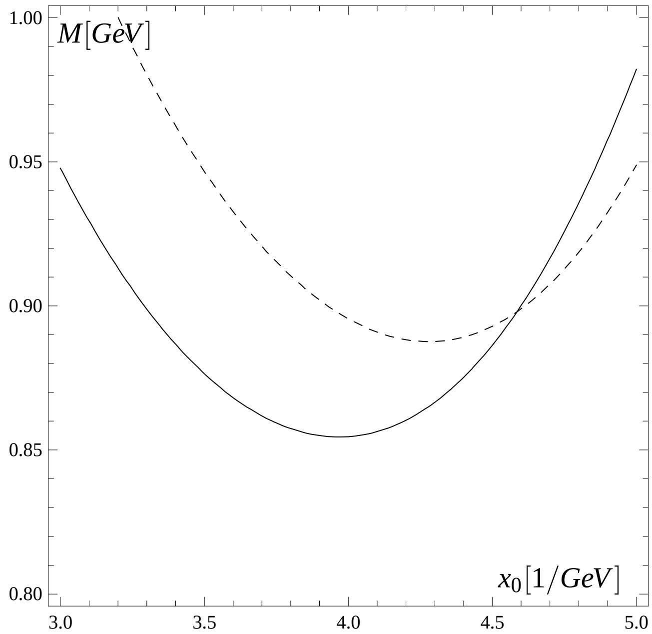

Figure 2: $M(x_0)$ for the trial wave functions Ψ_1 (full line) and Ψ_2 (dashed line). The Hamiltonian includes the self-energy correction $\Delta_1\left(\mu^2\right)$.

4. Chiral symmetry breaking

Chiral symmetry breaking has been a challenging aspect of the light-cone theory. It is known in equal-time theories that the vacuum is very complicated and higher Fock components of the quark-antiquark wave function are needed in order to reproduce the low-energy properties of the pion correctly. An interaction of the Nambu–Jona-Lasinio (NJL) type leads to a quark condensate, the excitations of which are massless Goldstone pions. In the light-cone approach, the most developed calculation uses the NJL-model with a vector interaction [15] and obtains very interesting differences of the light-cone wave function between the vector mesons and pions. In our framework, the complicated self-energy correction $\Delta_2\left(\mu^2\right)$ of the constituent quark can give the correct chiral-symmetry behavior of the pion mass. We apply the Feynman–Hellmann theorem [16, 17] to the light-cone Hamiltonian, which has dimension $[\mathrm{mass}]^2$,

$$\frac{\partial M_\pi^2}{\partial\mu} = \left\langle \frac{\partial H_{LC}}{\partial\mu} \right\rangle \tag{4.1}$$

and investigate what happens to the pion-mass squared $M_\pi^2 = 0$, when the current quark mass μ increases to finite values $\mu \neq 0$. Especially one may ask whether the Gell-Mann–Oakes–Renner relation still holds. How can the pion mass squared vanish linearly with the quark mass? A naive kinetic term cannot do that because then $\Delta M^2 \propto \mu^2$. In the Hamiltonian (2.3) with $\Delta_2\left(\mu^2\right)$ we have, however,

$$\left.\frac{\partial M^2}{\partial\mu}\right|_{\mu=0} = \frac{-\frac{4\sigma}{\pi}\left\langle\frac{1}{1/4-\xi^2}\right\rangle \left.\frac{\partial\phi(t)}{\partial\mu}\right|_{t=M_0/2a}}{1-\sigma\left\langle\frac{x_\perp^2}{\sqrt{\rho^2+M^2x_\perp^2}}\right\rangle}. \tag{4.2}$$

The function $\phi(t)$ is defined via $\Delta_2\left(\mu^2\right)$ by means of eq. (2.5). For M_0 we take the averaged meson mass of $\overline{M_0} = 0.67\,\mathrm{GeV}$, and for the transverse extension we take $x_0 = 0.8\,\mathrm{fm}$ of eq. (3.7). We get a linear dependence of the pion mass squared on the quark mass, which has a slope

$$\left.\frac{\partial M_\pi^2}{\partial\mu}\right|_{\mu=0} \approx 3.38\ \mathrm{GeV}. \tag{4.3}$$

We compare this value with the Gell-Mann–Oakes–Renner relation [18, 19]

$$M_\pi^2 = (-2\mu)\frac{<0|\bar{q}q|0>}{F_\pi^2}, \tag{4.4}$$

which amounts to a theoretical value for the same slope:

$$-\frac{2<0|\bar{q}q|0>}{F_\pi^2} \approx 3.20\ \mathrm{GeV}, \tag{4.5}$$

where the absolute value of the quark condensate is $(0.240\ \mathrm{GeV})^3$ and $F_\pi = 0.093$ GeV [20]. The relative difference between our light-cone calculation of eq. (4.3) and the empirical value $\frac{\partial M^2}{\partial\mu}$ of eq. (4.5) is only 6%. This is a very good result, but as one can see from eq. (4.2) it depends on the self-energy correction $\Delta\left(\mu_2^2\right)$. Besides the quantitative success, this result stimulates further studies of the self-energy correction in the light-cone theory. Here the new possibilities opening up by the AdS/QCD approach [21, 22] can play an important role.

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