A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes

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The unitarity method for calculating one-loop amplitudes provides algorithms of polynomial complexity. This is primarily beneficial for the computation of multi-leg one loop amplitudes and it is therefore of great interest to develop a numerical implementation of the unitarity method. We describe a recently-developed, efficient, semi-numerical unitarity method for the computation of the cut-constructible part of one-loop amplitudes.

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1. Introduction

At the LHC the observation of hard processes with many jets, gauge bosons and perhaps heavy new particles and their subsequent theoretical interpretation is of primary importance. The quantitative analysis of these processes requires the knowledge of the corresponding cross-sections and correlations at next-to-leading order (NLO) accuracy in perturbative QCD. The standard method of calculation based on Feynman diagrams becomes very cumbersome for multi-leg processes. In gauge theories the conventional Feynman diagram method produces intermediate results which are much more complicated then the final answer. The number of Feynman graphs grows very fast with the number of external legs. For a tree-level N-gluon scattering the number of individual Feynman graphs is approximately $N^{(N-3)}$ (within 5% accuracy up to 16 gluons) [1]. This decomposition generates a large number of terms. With a growing number of external particles it becomes a forbidding task to simplify the resultant expression analytically. Recently more numerical techniques have been developed (see e.g. ref. [2]). However, because of the stronger than exponential growth in the number of Feynman diagrams these brute-force methods become computationally very intensive for amplitudes with six or more legs.

Unitarity methods have been suggested as an alternative, more efficient procedure for loop calculations a long time ago [3, 4, 5]. Their use in the context of gauge theories is especially beneficial [6, 7]. The computing time is governed by the efficiency in computing tree amplitudes and by the number of cuts. In these applications, the unitarity cut is four dimensional. This allows the use of helicity method and surprisingly simple analytic answers [8] have been derived. The four-dimensional unitarity method reconstructs only the so-called cut-constructible part of the amplitude. The remaining rational part is obtained using known properties of the collinear limit . In supersymmetric theories, which have improved ultra-violet behaviour, the rational part vanishes.

Recently, new ideas (for a review see [9]) on twistors [10], multi-pole cuts (generalized unitarity) [11], recursion relations [12, 13, 14, 15], unitarity in D-dimension [16, 17] and the use of algebraic parametric integration technique [18] have made the unitarity cut method even more promising. It appears that ultimately one can find an efficient algorithm which can be used to calculate the full one-loop amplitudes in terms of tree-level amplitudes.

Here we briefly describe a semi-numerical four dimensional unitarity method [19] which expands the algebraic method of ref. [18] by developing a numerical scheme. The numerical algorithm evaluates only the cut-constructible part of the one-loop amplitudes. [1]

2. Structure of the one loop amplitude

The generic D-dimensional N-particle one-loop amplitude (fig. 1) is given by [2]

$$\mathscr{A}_N(p_1, p_2, \ldots, p_N) = \int [dl] \, \frac{\mathscr{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N} \, , \tag{2.1}$$

where p_i represent the momenta flowing into the amplitude, and $[dl] = d^D l$. The numerator structure $\mathscr{N}(p_1, p_2, \ldots, p_N; l)$ is generated by the particle content and is a function of the inflow mo-

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[1]This scheme has been extended recently into a semi-numerical D-dimensional unitarity method [20]. For other promising methods for calculating the rational parts see Refs. [21, 22].

[2]We restrict our discussion to (color) ordered external legs. The extension to more general cases is straightforward.
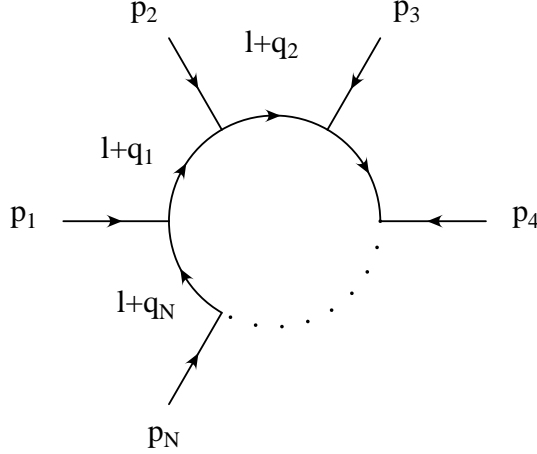
Figure 1: The generic *N*-point loop amplitude.

menta and the loop momentum. Since the whole amplitude has been put on a common denominator, the numerator can also include some propagator factors. The dependence of the amplitude on other quantum numbers has been suppressed. The denominator is a product of inverse propagators

$$d_i = d_i(l) = (l+q_i)^2 - m_i^2 = \left(l - q_0 + \sum_{j=1}^{i} p_i \right)^2 - m_i^2 \, , \tag{2.2}$$

where the 4-vector q_0 parameterizes the arbitrariness in the choice of loop momentum. The one-loop amplitude in $D = 4 - 2\varepsilon$ can be decomposed in a basis set of scalar master integrals giving

$$\mathscr{A}_N(p_1, p_2, \ldots, p_N) = \sum_{1 \le i_1 \le N} a_{i_1}(p_1, p_2, \ldots, p_N) I_{i_1} \, , + \sum_{1 \le i_1 < i_2 \le N} b_{i_1 i_2}(p_1, p_2, \ldots, p_N) I_{i_1 i_2}$$

$$+ \sum_{1 \le i_1 < i_2 < i_3 \le N} c_{i_1 i_2 i_3}(p_1, p_2, \ldots, p_N) I_{i_1 i_2 i_3} + \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} d_{i_1 i_2 i_3 i_4}(p_1, p_2, \ldots, p_N) I_{i_1 i_2 i_3 i_4} \, , \tag{2.3}$$

where the master integrals are given by $\quad I_{i_1 \cdots i_M} = \int [d\,l] \, \frac{1}{d_{i_1} \cdots d_{i_M}} \, .$ Analytic expressions for *D*-dimensional master integrals with massless internal lines are reported in ref. [7]. The corresponding results for divergent integrals with some massive internal lines are reported in ref. [23]. The maximum number of master integrals is determined by the dimensionality, *D*, of space-time; for the physical case this gives up to 4-point master integrals. The unitarity cut method is based on the study of the analytic structure of the one-loop amplitude. The coefficients are rational functions of the kinematic variables and will, in general, depend on the dimensional regulator variable $\varepsilon = (4 - D)/2$. When all the coefficients of the master integrals are calculated in 4 dimensions we obtain the "cut-constructible" part of the amplitude. The remaining "rational part" is generated by the omitted $\mathscr{O}(\varepsilon)$ part of the master integral coefficients. For a numerical procedure we need to recast the study of the analytic properties of the unitarity cut amplitudes into an algebraic algorithm which can be implemented numerically. In ref. [18] it was proposed that one focus on the integrand of the one-loop amplitude,

$$\mathscr{A}_N(p_1, p_2, \ldots, p_N|l) = \frac{\mathscr{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N} \, . \tag{2.4}$$

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This is a rational function of the loop momentum. We can re-express the rational function in an expansion over 4-, 3-, 2- and 1-propagator pole terms. The residues of these pole terms contain the master integral coefficients as well as structures (so-called spurious terms) which reside in the "trivial" space, the subspace orthogonal to the "physical" space. The physical space is the subspace spanned by the external momenta of the corresponding master integral. The spurious terms are important as subtraction terms in the determination of lower multiplicity poles. After integration over the loop momenta, Eq. (2.3) is recovered. This approach transforms the analytic unitarity method into the algebraic problem of partial fractioning a multi-pole rational function and allows for a numerical implementation.

3. Parameterization of the loop momentum on the unitarity cuts

We restrict ourselves to a 4-dimensional space. Given the master integral decomposition of Eq. (2.3) we can partial fraction the integrand of any 4-dimensional N-particle amplitude as

$$\mathscr{A}_N(l) = \sum_{1 \leq i_1 \leq N} \frac{\overline{a}_{i_1}(l)}{d_{i_1}} \cdot + \sum_{1 \leq i_1 < i_2 \leq N} \frac{\overline{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \leq i_1 < i_2 < i_3 \leq N} \frac{\overline{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq N} \frac{\overline{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} . \quad (3.1)$$

To calculate the numerator factors, one evaluates the residues by taking the inverse propagators equal to zero. The residue has to be taken by constructing the loop momentum $l_{ij\cdots k}$ such that $d_i(l_{ij\cdots k}) = d_j(l_{ij\cdots k}) = \cdots = d_k(l_{ij\cdots k}) = 0$. Then the residue of a function $F(l)$ is given by

$$\text{Res}_{ij\cdots k}[F(l)] \equiv \Big(d_i(l) d_j(l) \cdots d_k(l) F(l) \Big) \Big|_{l = l_{ij\cdots k}} . \quad (3.2)$$

The quadruple and triple pole residues are now given by as

$$\overline{d}_{ijkl}(l) = \text{Res}_{ijkl}\Big(\mathscr{A}_N(l) \Big), \quad \overline{c}_{ijk}(l) = \text{Res}_{ijk}\left(\mathscr{A}_N(l) - \sum_{l \neq i,j,k} \frac{\overline{d}_{ijkl}(l)}{d_i d_j d_k d_l} \right), \quad (3.3)$$

with similar expressions for the double $\overline{b}_{i_1 i_2}(l)$ and single $\overline{a}_{i_1}(l)$ pole residues. As an illustration we briefly outline how to construct the residue functions for quadruple cuts.

4. The quadrupole residue

To calculate the box coefficients we choose the loop momentum l_{ijkl} such that four inverse propagators are equal to zero,

$$\overline{d}_{ijkl}(l_{ijkl}) = \text{Res}_{ijkl}\Big(\mathscr{A}_N(l) \Big) \quad d_i(l_{ijkl}) = d_j(l_{ijkl}) = d_k(l_{ijkl}) = d_l(l_{ijkl}) = 0 . \quad (4.1)$$

We will drop the subscripts on the loop momentum in the following. Because we have to solve the unitarity constraints explicitly, we have to choose a specific parameterization. In ref. [19] the van Neerven-Vermaseren basis [24] is used which gives a very natural parameterization of the "trivial" and the "physical" space in terms of dual vectors constructed from the inflow momenta for a given cut type. We can decompose the loop momentum as

$$l^\mu = V_4^\mu + \alpha_1 n_1^\mu . \quad (4.2)$$

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The variable α_1 will be determined such that the unitarity conditions $d_i = d_j = d_k = d_l = 0$ are fulfilled. V_4 is a well defined vector in the "physical" space constructed from the three independent inflow momenta; n_1 is the unit vector of the one-dimensional trivial space. One finds two complex solutions

$$l_\pm^\mu = V_4^\mu \pm i \sqrt{V_4^2 - m_l^2} \times n_1^\mu \,, \tag{4.3}$$

which are easily numerically implemented. We note that the four propagators are on-shell and the amplitude factorizes for a given intermediate state into 4 tree-level amplitudes $\mathscr{M}^{(0)}$. The residue of the amplitude in Eq. (4.1) is given in terms of tree amplitudes as

$$\begin{aligned}
\mathrm{Res}_{ijkl}\left(\mathscr{A}_N(l^\pm)\right) &= \mathscr{M}^{(0)}(l_i^\pm; p_{i+1}, \ldots, p_j; -l_j^\pm) \times \mathscr{M}^{(0)}(l_j^\pm; p_{j+1}, \ldots, p_k; -l_k^\pm) \\
&\times \mathscr{M}^{(0)}(l_k^\pm; p_{k+1}, \ldots, p_l; -l_l^\pm) \times \mathscr{M}^{(0)}(l_l^\pm; p_{l+1}, \ldots, p_i; -l_i^\pm) \,, \tag{4.4}
\end{aligned}$$

where the loop momenta l_n^μ are complex on-shell momenta and there is an implicit sum over all states of the cut lines (such as e.g. particle type, color, helicity). The tree-level 3-gluon amplitudes, $\mathscr{M}_3^{(0)}$, are non-zero because the two cut gluons have complex momenta [11]. Any remaining dependence of the residue $\overline{d}_{ijkl}$ on the loop momentum enters through its component in the trivial space, $\overline{d}_{ijkl}(l) \equiv \overline{d}_{ijkl}(n_1 \cdot l)$. The number of powers of the loop momentum l in the numerator structure is called the rank of the integral. After integration we find that $(n_1 \cdot l)^2 \sim n_1^2 = 1$. Thus rank one is the maximum rank of a spurious term (which by definition vanishes upon integration over l). Hence the most general form of the residue is $\overline{d}_{ijkl}(l) = d_{ijkl} + \tilde{d}_{ijkl}\, l \cdot n_1$. Using the two solutions of the unitarity constraint, Eq. (4.3), we now can determine the two coefficients of the residue

$$d_{ijkl} = \frac{\mathrm{Res}_{ijkl}\left(\mathscr{A}_N(l^+)\right) + \mathrm{Res}_{ijkl}\left(\mathscr{A}_N(l^-)\right)}{2}, \quad \tilde{d}_{ijkl} = \frac{\mathrm{Res}_{ijkl}\left(\mathscr{A}_N(l^+)\right) - \mathrm{Res}_{ijkl}\left(\mathscr{A}_N(l^-)\right)}{2i\sqrt{V_4^2 - m_l^2}} \,. \tag{4.5}$$

After the subtracting the quadruple cut contributions from the amplitude we can repeat the procedure for the triple, double and single cuts.

5. Numerical results

As an application in ref. [19] the 4-, 5- and 6-gluon scattering amplitudes at one-loop have been recalculated with the new method. The cut-constructible parts of the ordered amplitudes are also known analytically making a direct comparison possible. Also, the 6-gluon amplitude was numerically evaluated using the integration-by-parts method [2]. To compare with the analytic results 100,000 flat phase space events has been generated for the $2 \to (n-2)$ gluon scattering. The events are required to have cuts in order to avoid soft and collinear regions in the momenta of the outgoing gluons. The evaluation time for 10,000 events is: for a $2 \to 2$ gluon ordered helicity amplitude 9 seconds, for a $2 \to 3$ gluon ordered helicity amplitude 35 seconds and for a for a $2 \to 4$ gluon ordered helicity amplitude 107 seconds. Note that using the integration-by-parts method of ref. [2] the evaluation time for 10,000 events would be approximately 90,000 second. The six-gluon evaluation is only three times slower than the five gluon evaluation and eleven times slower than the four gluon amplitude.

6. Summary and Outlook

The numerical unitarity method provides an efficient method to evaluate next-to-leading order corrections to multi-leg hard scattering amplitudes. It is applicable for processes including massive and massless particles as well as bosons and fermions. Very recently it has been generalized to *D*-dimensions so one can reconstruct the full amplitude including the rational part [20]. We expect that in the future it will be used in a number of important physics applications.

References

[1] R. Kleiss and H. Kuijf, *Nucl. Phys.* **B312** (1989) 616.

[2] R. K. Ellis, W. T. Giele and G. Zanderighi, JHEP **0605**, 027 (2006) [hep-ph/0602185].

[3] R. E. Cutkosky, J. Math. Phys. 1 **429** (1960)

[4] G. 't Hooft and M. Veltman, Diagrammar, CERN Report 73-9, Geneva (1973)

[5] W. L. van Neerven, Nucl. Phys. B **268**, 453 (1986).

[6] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425**, 217 (1994) [hep-ph/9403226].

[7] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435**, 59 (1995) [hep-ph/9409265].

[8] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **513**, 3 (1998) [hep-ph/9708239].

[9] Z. Bern, L. J. Dixon and D. A. Kosower, arXiv:0704.2798 [hep-ph].

[10] E. Witten, Commun. Math. Phys. **252**, 189 (2004) [hep-th/0312171].

[11] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725**, 275 (2005) [hep-th/0412103].

[12] F. A. Berends and W. T. Giele, Nucl. Phys. B **306**, 759 (1988).

[13] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **715**, 499 (2005) [hep-th/0412308].

[14] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005) [hep-th/0501052].

[15] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005) [hep-th/0501240].

[16] Z. Bern and A. G. Morgan, Nucl. Phys. B **467**, 479 (1996) [hep-ph/9511336];

[17] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B **645**, 213 (2007) [hep-ph/0609191]; R. Britto and B. Feng, hep-ph/0612089.

[18] G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) [hep-ph/0609007].

[19] R. K. Ellis, W. T. Giele and Z. Kunszt, arXiv:0708.2398 [hep-ph].

[20] W. T. Giele and Z. Kunszt and K. Melnikov, arXiv:0801.2237 [hep-ph].

[21] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, *Phys. Rev.* **D74** (2006) 036009, [hep-ph/0604195].

[22] R. Britto, B. Feng, arXiv:0711.4284 [hep-ph].

[23] A repository of one-loop master integrals can be found at http://qcdloop.fnal.gov/ . See also R.K. Ellis and G. Zanderighi, arXiv:0712.1851; J.R. Andersen, T. Binoth, G. Heinrich and J.M. Smillie, arXiv:0709.3513.

[24] W. L. van Neerven and J. A. M. Vermaseren, Phys. Lett. B **137**, 241 (1984).