

Theoretical progress on $|V_{us}|$ on lattice

Takashi Kaneko

*High Energy Accelerator Research Organization (KEK), Ibaraki 305-0801, Japan
School of High Energy Accelerator Science, The Graduate University for Advanced Studies
(Sokendai), Ibaraki 305-0801, Japan
E-mail: takashi.kaneko@kek.jp*

Recent lattice studies on (semi-)leptonic kaon decays towards a precise determination of $|V_{us}|$ are reviewed. Attention is given to recent unquenched calculations and consistency of their results with chiral perturbation theory.

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1. Introduction

Accurate knowledge of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{us}|$ is important because it gives the basic parameter λ in the Wolfenstein parametrization of the CKM matrix and is relevant to a stringent test of CKM unitarity $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$. The K_{l2} and K_{l3} decays provide two precise determinations of $|V_{us}|$, where their dominant uncertainty originates from theoretical evaluations of hadronic matrix elements, namely f_K/f_π and K_{l3} form factors.

Lattice QCD can provide a non-perturbative estimate of these matrix elements from first principles. Due to the limitation of the computational resources, however, some simulation parameters have to be different from those of the real world. The use of finite lattice spacing a and spatial extent L is unavoidable but its effects can be systematically reduced. It is assumed in the simulations reviewed in this article that up and down sea quarks are degenerate. The use of relatively heavy masses $m_{ud,sim}$ for degenerate up and down quarks is much more problematic, because it could cause a large uncontrolled error by extrapolating lattice results to the physical mass m_{ud} . It is therefore vital to simulate quark masses $m_{ud,sim}$, where chiral perturbation theory (ChPT) can be safely used as a guide for the chiral extrapolation.

The staggered fermions are known to be computationally inexpensive [1], and led to a precise determination of f_K/f_π [2] from the MILC collaboration's simulations at $m_{ud,sim} \gtrsim m_s/10$ [3]. Their complicated flavor structure is however a serious obstacle to extensive calculations of more involved matrix elements, such as the K_{l3} form factors. While simulations with other discretizations were limited to relatively heavy quark masses, typically $m_{ud,sim} \gtrsim m_s/2$, at the time of the last conference KAON 2005, recent algorithmic improvements now enable us to explore $m_{ud,sim}$ comparable to that in the MILC's study. In any lattice studies, consistency between their data and ChPT is a crucial issue for a reliable chiral extrapolation.

In this article, we first review recent progress on f_K/f_π in Sec. 2, focusing on the latest update on the MILC's estimate and the status of studies with other discretizations. Section 3 is devoted to the K_{l3} form factors. We outline the calculation method and discuss the associated systematic errors. Finally, our conclusions are given in Sec. 4.

2. f_K/f_π

As pointed out in Ref.[4], $|V_{us}|$ can be extracted from K_{l2} and π_{l2} decays through the ratio of their decay rates

$$\frac{\Gamma(K \rightarrow l\bar{\nu}_l)}{\Gamma(\pi \rightarrow l\bar{\nu}_l)} = \frac{|V_{us}|^2 f_K^2 M_K (1 - m_l^2/M_K^2)^2}{|V_{ud}|^2 f_\pi^2 M_\pi (1 - m_l^2/M_\pi^2)^2} \left\{ 1 + \frac{\alpha}{\pi} (C_K - C_\pi) \right\}. \quad (2.1)$$

Radiative corrections parametrized by $C_{K,\pi}$ and the muonic decay rates lead to an uncertainty of $\lesssim 0.2\%$ in $|V_{us}|$. The determination of $|V_{ud}|$ from super-allowed nuclear β decays is accurate at the impressive level of 0.05%. Therefore the main uncertainty in $|V_{us}|$ comes from the theoretical input f_K/f_π .

Lattice QCD can, in principle, give a precise estimate of f_K/f_π , since each decay constant is calculated from simple (and hence less noisy) two-point functions and uncertainties due to the lattice scale and renormalization are canceled in the ratio. The dominant error arises from the continuum and chiral extrapolations. The original estimate $|V_{us}| = 0.2236(30)$ in Ref.[4] was obtained

with the MILC's estimate $f_K/f_\pi = 1.201(8)(15)$ [2] from their simulations using an improved staggered action (the so-called AsqTad action) at two lattice spacings $a = 0.09$ and 0.12 fm and with quark masses down to $m_{ud,sim} \sim m_s/10$ [3].

2.1 Update on result from staggered fermions

The MILC collaboration has been steadily updating their simulations. One of the main improvements in their latest report is that their simulations are extended to finer ($a = 0.06$ fm) and coarser lattice spacings ($a = 0.15$ fm) [5, 6].

It should be noted that simulations with the staggered quark action have the following theoretical and technical complications. By construction, a single staggered field describes four species of quark. This degree of freedom is called ‘‘taste’’. In simulations with single flavor (two degenerate flavors) of quarks, gauge configurations are generated by taking the fourth (square) root of the quark determinant in the Boltzmann weight, e.g.

$$Z_{N_f=1} = \int [dU] \det[D]^{1/4} \exp[-S_g], \quad (2.2)$$

where D is the Dirac operator for the *four-taste* staggered quark, S_g is the lattice gauge action of choice, and dU represents the path integral over the gauge fields. It is still actively debated whether the non-local Dirac operator corresponding to the rooted determinant leads to the correct continuum limit [7]. In addition, the explicit taste symmetry breaking at finite lattice spacings makes calculations of matrix elements complicated.

In the MILC's latest analysis [5, 6], they fit the quark mass and lattice spacing dependence of the pseudo-scalar meson masses and decay constants simultaneously using formulas from the so-called staggered ChPT [8], where effects due to the taste symmetry breaking are taken into account. Analytic and chiral logarithmic terms at NLO and a part of analytic terms up to NNNLO are included into their fitting function to achieve a reasonable value of χ^2/dof . Their two observations increase the reliability of their chiral and continuum extrapolations:

- their fit curve in the continuum limit exhibits a curvature towards the chiral limit as expected from NLO ChPT;
- they obtain low-energy constants (LECs) $L_4 = 0.1(4)$ and $L_5 = 2.0(4)$, which are consistent with a phenomenological estimate $L_4 = 0.0(8)$ and $L_5 = 2.3(1)$ [9].

They obtain their latest estimate

$$\frac{f_K}{f_\pi} = 1.208(2)(+7/-14), \quad (2.3)$$

where the first error is statistical and the second is systematic, and obtain $f_\pi = 128.6(0.4)(3.0)$ MeV and $f_K = 155.3(0.4)(3.1)$ MeV, which are in good agreement with experiment. The statistical error is remarkably reduced from their previous estimate in Ref.[2]. The uncertainty in f_K/f_π is now dominated by the systematics of the combined chiral and continuum extrapolation, which might be difficult to improve drastically without extending their simulations to a much wider range of $m_{ud,sim}$ and a . Independent calculations with different fermion discretizations are highly required to reduce the systematic uncertainties and to confirm that the rooted staggered theory has the correct continuum limit.

2.2 Status of studies with other discretizations

Unquenched simulations with other fermion discretizations also have a long history, leading up to recent studies with the following fermion actions:

- (improved) Wilson fermions

This traditional formulation is computationally cheap compared to chiral fermions (see below) and useful to simulate large and fine lattices. Its main drawback is the explicit chiral symmetry breaking at finite lattice spacing, which may distort the chiral behavior of the decay constants [10]. The clover action is an improved formulation by removing leading $O(a)$ discretization errors. These discretizations are employed in recent simulations in Refs.[11, 12, 13].

- twisted mass Wilson fermions

This is a variant of the Wilson fermions with the so-called twisted mass term [14], which simplifies the mixing pattern in the renormalization of lattice operators remarkably with computational costs comparable to Wilson fermion simulations. This mass term, however, leads to the explicit breaking of parity and isospin symmetry. Its effects have to be studied carefully, as in large-scale simulations by the ETM collaboration [15].

- chiral fermions

With the five dimensional domain-wall formulation [16], chiral symmetry is restored in the limit of infinitely large size L_s in the fifth dimension. It is however L_s/a times costly with respect to the above mentioned Wilson-type fermions. The RBC/UKQCD collaborations simulate three-flavor QCD with $L_s/a = 16$, which leads to the additive quark mass renormalization of a few MeV [17]. The (four dimensional) overlap fermions [18] are even more computationally demanding. However, it has almost exact chiral symmetry and hence is useful for phenomenological applications where chiral symmetry plays an important role. The JLQCD collaboration has started large scale simulations in two-flavor QCD [19].

The simulation cost for the above formulations with the commonly used Hybrid Monte Carlo (HMC) algorithm [20] rapidly increases as $m_{ud,sim}$ decreases: it scales as $\propto m_{ud,sim}^{-3}$ [21]. This is why previous simulations on relatively large and fine lattices [22, 23, 24, 25, 26, 27, 28] are limited to heavy quark masses typically $m_{ud,sim} \gtrsim m_s/2$, as shown in Fig. 1. However, recent algorithmic improvements [29, 30] enable us to simulate much smaller values of $m_{ud,sim}$, which are now comparable to those in the MILC's simulation with the staggered fermions.

In Fig. 2, we plot the pion decay constant obtained with Wilson-type and chiral fermions [15, 19, 26, 27]. We observe a reasonable agreement among the data suggesting that discretization errors are not large in this plot. More importantly,

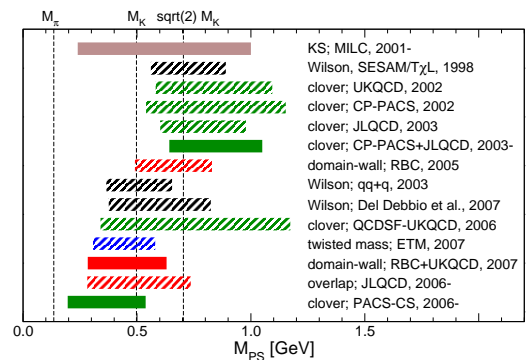


Figure 1: Region of pion mass simulated in large-scale unquenched calculations in two-flavor (shaded band) and three-flavor QCD (solid band).

data at $m_{ud,sim} \lesssim m_s/2$ from recent simulations show a curvature toward the chiral limit as suggested by the chiral logarithm at NLO, whereas the curvature is not clear at heavier $m_{ud,sim}$. While data at small $m_{ud,sim}$ are subject to effects due to finite lattice volume, the ETM collaboration [15] observe that their data with finite volume corrections [31] are described by the NLO ChPT formula reasonably well. They obtain

$$F = 121.3(7) \text{ MeV}, \quad \bar{l}_4 = 4.52(6), \quad (2.4)$$

which are consistent with lattice estimates of F from the MILC's simulation in p -regime [6] and JLQCD's one in ε -regime [32], and with a phenomenological estimate of \bar{l}_4 [33]. This suggests that recent simulations with Wilson-type and chiral fermions are now exploring $m_{ud,sim}$ sufficiently small to make contact with NLO ChPT.

Recent estimates of f_K/f_π in three-flavor QCD are collected in Fig. 3. The CP-PACS and JLQCD collaborations obtain a slightly smaller result than others [34], probably because their simulations are limited to $m_{ud,sim} \gtrsim m_s/2$ and f_π is overestimated due to the lack of the chiral logarithm. The PACS-CS collaboration employs the clover fermions with the Lüscher's domain-decomposed HMC [30]. A good agreement of their [35] and RBC/UKQCD's estimates [17] with Eq. (2.3) is very encouraging, though their simulations are still on-going and/or the quoted error is statistical only. These are not enough mature to be used to derive an world average, and we simply quote Eq. (2.3) as the current best estimate of f_K/f_π . It is, however, worth emphasizing that estimates of f_K/f_π are expected to be improved remarkably in the near future by on-going simulations with the Wilson-type and chiral fermions by various groups.

3. K_{l3} form factor

The K_{l3} decays provide a precise determination of $|V_{us}|$ through

$$\Gamma(K \rightarrow \pi l \bar{\nu}_l) = \frac{G_\mu^2}{192\pi^3} M_K^5 C^2 I |V_{us}|^2 |f_+(0)|^2 S_{EW} (1 + \Delta_{EM} + \Delta_{SU(2)}), \quad (3.1)$$

where C is the Clebsh-Gordon coefficient equal to 1 ($1/\sqrt{2}$) for neutral (charged) kaon decays. The short- and long-distance radiative corrections, denoted by S_{EW} and Δ_{EM} , and $SU(2)$ breaking corrections $\Delta_{SU(2)}$ are theoretical inputs, whereas the decay rate Γ and the phase space integral I are determined from experimental measurements. The uncertainties in $|V_{us}|$ due to these inputs are well below 1% [36].

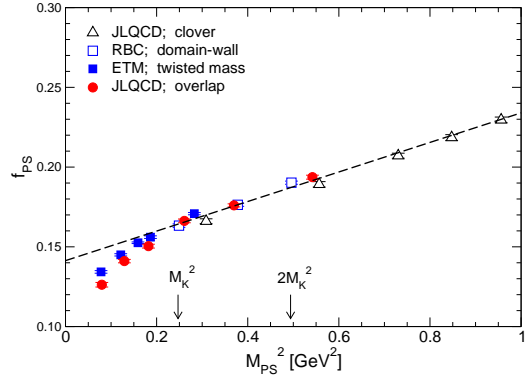


Figure 2: Pion decay constant as a function of pion mass squared. Dashed line shows a linear fit to open symbols.

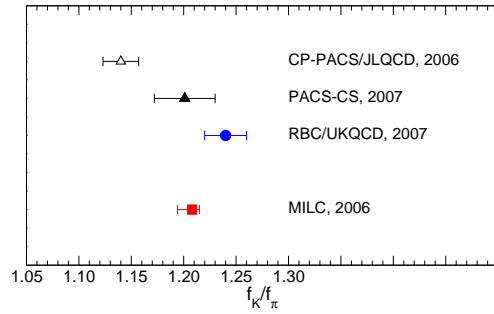


Figure 3: Recent unquenched estimate of f_K/f_π .

	N_f	quark action	a [fm]	L [fm]	M_{PS} [MeV]
JLQCD [41]	2	clover	0.09	1.8	$\gtrsim 600$
RBC [42]	2	domain-wall	0.12	1.9	$\gtrsim 490$
Fermilab/MILC/HPQCD [43]	3	AsqTad (d =clover)	0.12	2.5	$\gtrsim 500$
RBC/UKQCD [44]	3	domain-wall	0.12	1.9, 2.9	$\gtrsim 300$

Table 1: Simulation parameters in unquenched lattice calculations of $f_+(0)$. The clover fermions are used for valence down quarks in Ref.[43].

The dominant uncertainty of the present estimate of $|V_{us}|$ therefore arises from theoretical determination of the normalization of the vector form factor $f_+(0)$ defined from the $K \rightarrow \pi$ matrix element $\langle \pi(p') | \bar{s} \gamma_\mu u | K(p) \rangle = (p + p')_\mu f_+(q^2) + (p - p')_\mu f_-(q^2)$, where $q^2 = (p - p')^2$. The leading correction [37] in the chiral expansion

$$f_+(0) = 1 + f_2 + f_4 + O(p^6) \quad (3.2)$$

is practically free of uncertainties ($f_2 = -0.023$), because any poorly known LECs do not appear in f_2 thanks to the Ademollo-Gatto theorem [38].

However, the higher order correction f_4 contains LECs in the chiral Lagrangian both at $O(p^4)$ and $O(p^6)$. A phenomenological estimate $f_4 = -0.016(8)$ based on the quark model was obtained by Leutwyler and Roos (LR) [39], and has been used in previous determinations of $|V_{us}|$. There has been remarkable progress in studies based on ChPT, where the evaluation of the tree-level contribution with LECs in the $O(p^6)$ Lagrangian is the most crucial issue [40]. Recent estimates ranging from $f_4 = -0.007(9)$ to $+0.007(12)$ are slightly larger than the LR estimate due to a (partial) cancellation between loop and tree-level contributions.

Lattice QCD can provide a *non-perturbative* determination of $f_+(0)$, namely f_4 and higher order contributions. Unquenched calculations performed so far are listed in Table 1. These studies basically follow the strategy proposed in the first calculation in quenched QCD [45], which is outlined below.

The first step is to calculate the scalar form factor $f_0 = f_+ + (q^2/(M_K^2 - M_\pi^2)) f_-$ from three point functions, e.g.

$$C_\mu^{K\pi}(t, t', t'') = \langle O_\pi(t'') | V_\mu(t') | O_K(t)^\dagger \rangle, \quad (3.3)$$

where $O_{\pi(K)}(t)$ is the interpolation operator for pion (kaon) and $V_\mu(t)$ is the vector current at the timeslice t . With sufficiently large temporal separations $t'' - t'$ and $t' - t$, $C_\mu^{K\pi}(t, t', t'')$ is dominated by the ground state contribution, which is the matrix element $\langle \pi | V_\mu | K \rangle$ times unnecessary factors, such as the damping factor $e^{-M_K(t' - t)}$. These factors are canceled in the so-called double ratio [46]. For instance, a double ratio

$$\frac{C_4^{K\pi}(t, t', t'') C_4^{\pi K}(t, t', t'')}{C_4^{KK}(t, t', t'') C_4^{\pi\pi}(t, t', t'')} \rightarrow \frac{(M_K + M_\pi)^2}{4M_K M_\pi} |f_0(q_{\max})|^2 \quad (3.4)$$

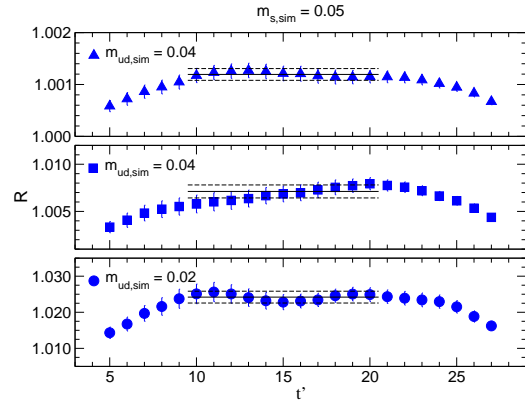


Figure 4: Double ratio Eq. (3.4) as a function of t' with fixed $t = 4$ and $t'' = 28$. Data are from Ref.[42].

can be determined precisely as shown in Fig. 4, and it gives f_0 at $q_{\max}^2 = (M_K - M_\pi)^2$ with an accuracy well below 1%. We can calculate f_0 at $q^2 \neq q_{\max}^2$ from different double ratios proposed in Refs. [45, 41, 42], which however involve three-point functions with nonzero meson momenta and hence are much noisier than the ratio Eq. (3.4).

Then, we interpolate f_0 to $q^2 = 0$. The q^2 dependence is parametrized using the monopole ansatz $f_0(q^2) = f_0(0)/(1 - \lambda_0 q^2)$ or polynomial forms up to quadratic order $f_0(q^2) = f_0(0) + \lambda_0 q^2 + \lambda_0' q^4$. These forms are also employed in analyses of experimental data. It turns out that the choice of the interpolation form does not cause a large uncertainty in the unquenched studies, since an accurate estimate of $f_0(q_{\max}^2)$ is available near the interpolation point $q^2 = 0$. It is also encouraging to observe in Fig. 5 that λ_0 from lattice studies shows a reasonable agreement with experimental measurements [36].

Finally, $f_+(0) = f_0(0)$ is extrapolated to the physical quark masses m_{ud} and m_s . In all unquenched calculations, a ratio motivated by the Ademollo-Gatto theorem

$$R = \frac{f_+(0) - 1 - f_2}{(M_K^2 - M_\pi^2)^2} \quad (3.5)$$

can be fitted to a rather simple polynomial form

$$R = c_0 + c_1 (M_K^2 + M_\pi^2). \quad (3.6)$$

It is possible that, since most simulations are limited to heavy quark masses $m_{ud,\text{sim}} \gtrsim m_s/2$, the NNLO (and higher order) chiral logs vary smoothly in this region and are well approximated by the analytic terms.

In order to get an idea about how small $m_{ud,\text{sim}}$ is needed to see the chiral logs in $f_+(0)$ clearly, data from the RBC/UKQCD's study is plotted as a function of $m_{ud,\text{sim}}$ in Fig. 6. The NLO chiral log f_2 rapidly increases at $m_{ud,\text{sim}} \lesssim m_s/2$. This suggests that precise lattice data in this region are essential for a reliable chiral extrapolation compatible with the existence of the chiral logs.

We note that the error of $f_+(0)$ may rapidly increase with decreasing $m_{ud,\text{sim}}$, because of longer auto-correlations of gauge configurations and larger q_{\max}^2 for the q^2 interpolation of $f_0(q^2)$. In future studies at small $m_{ud,\text{sim}}$, therefore, it is advisable to employ improved measurement methods, such as the all-to-all quark propagators to improve the accuracy of $f_0(q^2)$ [48]. The twisted boundary condition, which enables us to explore $q^2 \sim 0$ [49], and a model independent parametrization of the q^2 dependence of f_0 [50] are useful to reduce systematic uncertainties due to the q^2 interpolation.

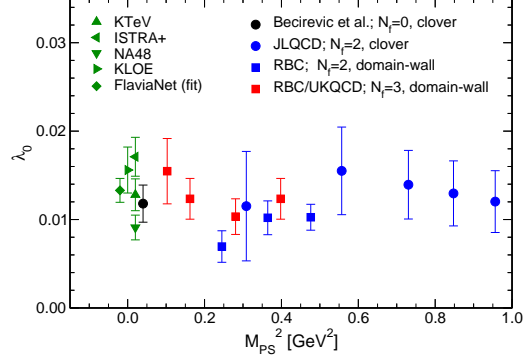


Figure 5: Lattice estimates of slope λ_0 [45, 41, 42, 44] together with experimental values [47].

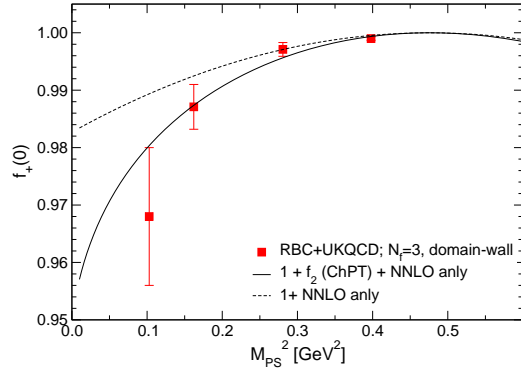


Figure 6: Vector form factor as a function of $m_{ud,\text{sim}}$. The solid line shows a fit curve Eq. (3.6). The NLO term f_2 is subtracted in the dashed line; namely the difference between two lines shows f_2 .

Figure 7 shows recent lattice estimates of $f_+(0)$. The RBC/UKQCD collaboration confirms that finite volume corrections at $L \sim 2$ fm are small down to $m_{ud,sim} \approx m_s/4$. This observation is encouraging since other unquenched studies are conducted with similar or larger lattice sizes. The nice consistency among lattice results may suggest that discretization and quenching errors are not large.

All lattice results are in good agreement with the LR value. We note that, however, estimates from ChPT are slightly higher due to the NNLO loop contributions. Therefore, the agreement between lattice and the LR value has to be examined carefully by precise lattice calculations at $m_{ud,sim} \lesssim m_s/2$, where chiral logs are expected to be seen clearly as discussed above.

4. Conclusions

From the MILC's estimate of f_K/f_π and the $K_{\mu 2}/\pi_{\mu 2}$ decay rates, we obtain $|V_{us}| = 0.2226 (+26/-15)$. The preliminary result of $f_+(0) = 0.9609(51)$ from the RBC/UKQCD's study and $|V_{us} f_+(0)| = 0.21673(46)$ from the FlaviaNet working group [36] lead to $|V_{us}| = 0.2255(13)$ which is consistent with the value quoted earlier. The latter estimate needs, however, further studies to increase the reliability of the chiral extrapolation of $f_+(0)$.

For both f_K/f_π and $f_+(0)$, we observe that precise data at sufficiently small ud quark masses, typically $m_{ud,sim} \lesssim m_s/2$, are needed for a reliable chiral extrapolation. Thanks to recent algorithmic improvements, several groups have already started large-scale simulations in this region of $m_{ud,sim}$ with different fermion discretizations. While their results are premature to be taken into account in the above estimates of $|V_{us}|$, lattice estimates of f_K/f_π and K_{I3} form factors are expected to be remarkably improved by these studies in the *near* future [51].

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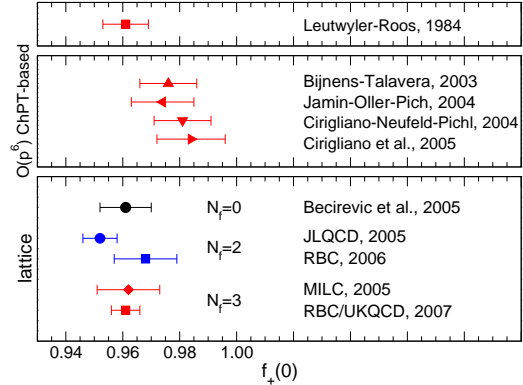


Figure 7: Vector form factor $f_+(0)$ from phenomenological model (top panel), ChPT (middle panel) and lattice QCD (bottom panel).

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