Strings in SU(N) gauge theories in 2+1 dimensions: beyond the fundamental representation

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We calculate energies and tensions of closed k-strings in (2+1)-dimensional SU(N) gauge theories with N=4,5,6,8. When we study the dependence of the ground state energy on the string length, we find that it is well described by a Nambu-Goto (NG) free bosonic string for large lengths. At shorter lengths we see deviations which we fit, and this allows us to control the systematic error involved in extracting the tension. We compare the resulting string tensions with Casimir scaling, which we find to be lower than our data by $1\% - 4\%$. Extrapolating our results to $N = \infty$ we see that our data fits more naturally to $1/N$ rather than $1/N^2$ corrections. Finally, we see that the full spectrum of the k-string states falls into sectors that belong to particular irreducible representations of $SU(N)$.

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1. Introduction

In this work we study $SU(N)$ gauge theories in $D = 2+1$ space-time dimensions and focus on the energies and tensions of closed strings that carry flux in $SU(N)$ representations whose N-ality k is larger than one. (For references on related lattice works see our companion contribution to these proceedings [1] as well as [2]-[5]). The reason we restrict ourselves to $D = 2+1$ dimensions is closely related to our motivation in the related study [6], where we tested the Karabali-Kim-Nair (KKN) analytic prediction [7] for the fundamental string tension σ. As a natural continuation to that work, we now aim to test how accurate is the following KKN prediction for the tension of a string in a general $SU(N)$ representation $\mathscr{R}$.

$$\sigma_{\mathscr{R}} = \sigma \cdot \frac{C_{\mathscr{R}}}{C_1}. \tag{1.1}$$

Here $C_{\mathscr{R}}$ is the quadratic Casimir of the representation, and $C_1 = (N^2 - 1)/2N$ is the Casimir of the fundamental representation. We note here that the above 'Casimir scaling' of string tensions is also predicted by other approaches to $2+1$ dimensions (for example see [8]). The KKN prediction lacks the physics of screening, and we thus take a practical point of view and regard Eq. (1.1) as an approximate prediction for the tensions of stable k-strings, and for the asymptotic energy per unit length of excited k-strings states with flux in an excited representation.

Besides comparing to Eq. (1.1), we are also interested in the way the k-string energy depends on the string length l (which will reveal its central charge), on the manner in which the planar limit is approached (i.e. whether the corrections to $N = \infty$ scale like $1/N$ or $1/N^2$), and on a curious pattern of degeneracies seen in previous studies of the k-string spectrum.

2. Methodology

We define the gauge theory on a discretized periodic Euclidean three dimensional space-time lattice, with spacing a and, typically, with $L_s^2 L_t$ sites. The action we use is the ordinary Wilson action, where the bare coupling β is related to the dimensionful coupling g^2 by $\lim_{a \to 0} \beta = \frac{2N}{ag^2}$. In the large–$N$ limit, the 't Hooft coupling $\lambda = g^2 N$ is kept fixed, and so we must scale $\beta = 2N^2/\lambda \propto N^2$ in order to keep the lattice spacing fixed (up to $O(1/N^2)$ corrections). We calculate observables by performing Monte-Carlo simulations of the Euclidean path integral, in which we use a mixture of Kennedy-Pendelton heat bath and over-relaxation steps for all the $SU(2)$ subgroups of $SU(N)$.

We measure the energy of flux tubes closed around a spatial torus, from the correlators of suitably smeared Polyakov loops that have vanishing transverse momentum [9, 4]. For each Hilbert space sector of given N-ality k we construct lattice operators that couple to states of that N-ality. These are given by $\text{Tr} U^k$, $\text{Tr} U^{k-1} \text{Tr} U \ldots, (\text{Tr} U)^k$, where U is the path-ordered product of smeared links around the spatial torus. We then construct the full correlation matrix and use it to obtain best estimates for the string states using a variational method applied to the transfer matrix $\hat{T} = e^{-aH}$ (see for example [9] and references therein).

Our study is logically divided into two. We first investigate the way the k-string energy E depends on its length l. In [6, 1] we have discussed the theoretical possibilities for $E(l)$, and we will not reiterate that discussion here, but rather just quote its conclusion : a natural way to fit our data for the energy is with

$$E_k^2(l) = E_{\text{NG}}^2 - \frac{C_k}{(l\sqrt{\sigma_k})^3} \quad ; \quad E_{\text{NG}}^2 = (\sigma_k l)^2 - \sigma_k \frac{\pi}{3}, \tag{2.1}$$

where E_{NG} is the ground state energy of a closed string in the Nambu-Goto string theory.

For the $k = 1$ case we found that our data is very well described by this ansatz with $C_1 \lesssim 0.3$. We now ask whether this situation persists for $k > 1$ strings as well. This will also tell us whether the k-strings belong to the same IR universality class as the $k = 1$ string. We perform these measurements for $SU(4)$ at $\beta = 28.00, 50.00$, $SU(5)$ at $\beta = 80.00$, $SU(6)$ at $\beta = 59.40, 90.00$, and $SU(8)$ at $\beta = 108.00, 192.00$. These bare couplings correspond to lattice spacings of $a \simeq 0.06, 0.08, 0.11$ fm, depending on N. The string lengths l ranged between ~ 0.45 fm and ~ 3 fm, again depending on the values of N and a.[1]

After we obtain an estimate for $E_k(l)$ we use it to extract string tensions from string energies which were measured on a set of lattices with increasingly small spacings in the range $a \simeq 0.05 - 0.2$ fm. This is done only for strings whose length obeys $l \gtrsim 3/\sqrt{\sigma} \simeq 1.4 - 1.5$ fm. This way of extracting tensions controls the systematic error involved in the usual neglect of the sub leading corrections to the Luscher term. Once we obtain the continuum string tensions, we extrapolate our results to the large-N limit. This is particularly interesting since there exists a controversy in the literature with respect to the possibility of having $1/N$ corrections in the k-string tensions [10, 3].

3. Results : length dependence of the k-string energies and their conformal anomaly

In the left panel of Fig. 1 we present the energy of the $k = 2$ string for $SU(5)$ at $\beta = 80.00$. The plot shows the energy divided by σl (here σ is the fundamental string tension which we obtain in [1]) vs. the length in physical units, $l\sqrt{\sigma}$. The string tension in lattice units is $a^2\sigma = 0.016874(12)$ which gives a lattice spacing of $a \simeq 0.058$ fm, and tells us that our string length stretches from $\simeq 0.6$ fm to $\simeq 1.85$ fm. The red line that goes through our data is a fit of the form Eq. (2.1) which results in the ratio $\sigma_2/\sigma = 1.5244(21)$ and $C_2 = 1.41(7)$ (the fit is good with $\chi^2/dof \simeq 3/4$). The coefficient C_2 is thus much larger than the corresponding one for $k = 1$, which was $0.0554(139)$ for this data [1]. This reflects the fact that the NG prediction is a much better approximation for $k = 1$ than it is for $k > 1$, which can be easily seen by comparing the left panel of Fig. 1 to the corresponding plot for $k = 1$ [1].

The results for all the gauge groups that we study are similar and can be encompassed in a single formula with $C_2 = 3(2)$. For higher values of k the results are less accurate, but we can still fit them and find that taking $C_3 = 4.5(2.0)$ and $C_4 = 5.5(1.5)$ for $k = 3$ and $k = 4$, respectively, describes all our data. In practice, provided that the lengths of our strings obey $l \gtrsim 3/\sqrt{\sigma}$, we see that the correction term in Eq. (2.1) is at most a 0.5% contribution to the energy.

We now examine the universality class of the string by fitting pairs of adjacent points in the left panel of Fig. 1, and in the corresponding data sets for all other values of N and k, with the form $E^2 = (\sigma_k l)^2 - \sigma_k \frac{\pi}{3} \times C_{\text{eff}}$. As the points we fit have larger and larger values of l, then C_{eff} should approach the central charge of the k-string. In the case of $k = 1$ we have very strong evidence that $C_{\text{eff}} \overset{l\to\infty}{\to} 1$ (see [6, 1] and references within). In the $k > 1$ the situation is harder to pin down and in addition there is a recent prediction [5] suggesting that $C_{\text{eff}} \overset{l\to\infty}{\to} \sigma_k/\sigma$. We present our results in the right panel of Fig. 1 for the cases $N = 4, 5, 6, 8$ and $k = 2$, where it is reasonably clear that C_{eff} decreases towards 1 as l increases. We have similar results for $k = 3, 4$, which are, however, less accurate due to the larger energies.

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[1]For more details on the lattice parameters of our field configurations see [1].

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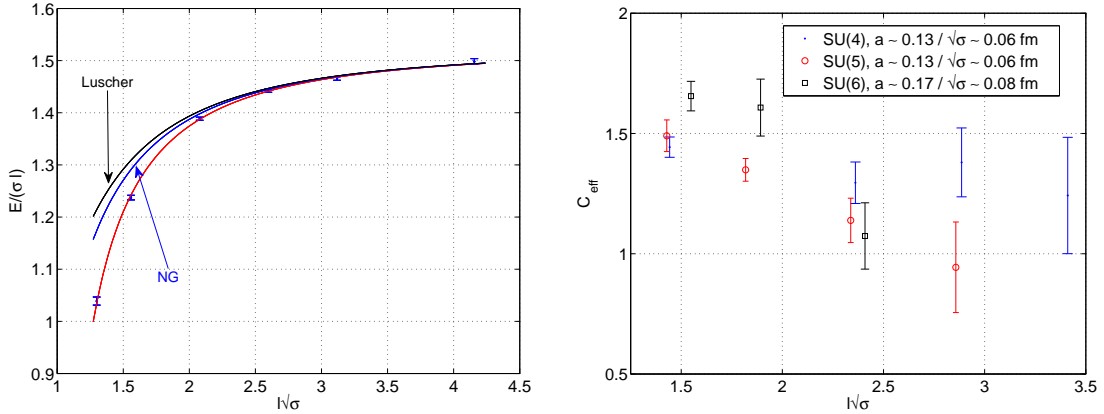
Figure 1: <u>Left:</u> Ground state energy of the $k = 2$ string in $SU(5)$ and $\beta = 80.00$. Our fit in red, and in blue(black) the NG(Luscher term) predictions. <u>Right:</u> The value of C_{eff} (see text) of the $k = 2$ strings in $SU(4,5,6)$ for $\beta = 50.00, 80.00, 90.00$, respectively (lattice spacings presented in the legend).

$r_k(N)$	$r_2(4)$	$r_2(5)$	$r_2(6)$	$r_2(8)$	$r_3(6)$	$r_3(8)$	$r_4(8)$
Lattice	1.3553(23)	1.5275(26)	1.6242(35)	1.7524(51)	1.8590(63)	2.1742(187)	2.3725(111)
Casimir	$1.3333\ldots$	1.5	1.6	$1.7142\ldots$	1.8	$2.1429\ldots$	$2.2857\ldots$

Table 1: The continuum extrapolation of $r_k(N) \equiv \frac{\sigma_k}{\sigma}(N)$ and the comparison with Casimir scalings.

Let us now pause to make the following comment. The decrease of C_{eff} becomes clear only above 1 fm, and its possible that this is the main cause for the difference between our conclusions and those of [5], where the maximum string length was ~ 0.9 fm.

4. Results : the string tensions in the continuum and the large-N extrapolation

We now use the empirically determined Eq. (2.1) to extract string tensions from string energies that we measure on a wide range of lattice spacings ranging between $a \simeq 0.2$ fm and $a \simeq 0.05$ fm. All the strings the we use have a length of at least 1.35 fm. The extrapolation of the ratios $r_k \equiv \sigma_k/\sigma$ to the continuum, and its comparison to the Casimir scaling is shown in Table 1.[2] All the continuum extrapolations had a acceptable χ^2/dof except for the $k = 2$ of $SU(6)$ where we find that our data is too scattered to be well fit by a smooth fitting ansatz. We proceed to perform two types of large-N extrapolation. The first is for $k = 2$ (left panel of Fig. 2) and the second is for $k = N/2$ (right panel of the figure). In both cases we present the Casimir scaling prediction in red, and two type of fits, that either allow or exclude $1/N$ corrections.

We begin by extrapolating $r_{k=2}$ to $SU(\infty)$. Since at $N = \infty$ one expects the $r_2 = 2$ we use the ansatz $r_2 = 2 - \frac{a}{N^p} - \frac{b}{N^{2p}}$ with $p = 1, 2$. For $p = 1$ our fit gives $a = 1.51(5)$ and $b = 4.3(2)$, but a $\chi^2/dof \simeq 2.2$. This high value of χ^2 comes from the data point of $SU(6)$ which, as mentioned above, suffers from a low confidence level. To check the sensitivity of the fit to this point, we drop it from the fit and find an acceptable χ^2 with similar values for the fit parameters a, b. When $p = 2$, however, we find no acceptable fit, and are led to drop the point with the lowest value of

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[2]The determination of the fundamental tension in these cases is described in [6]

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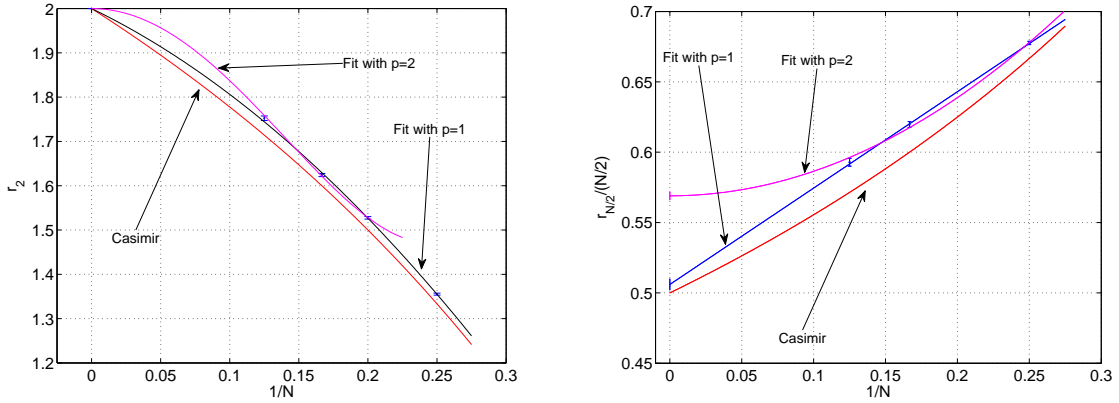
Figure 2: The extrapolation of $k = 2$ (left panel) and $k = N/2$ (right panel).

$N = 4$. This results in $a = 17.8(3)$ and $b = -150(9)$ and a still large $\chi^2/dof \simeq 2.1$. Also, the wavy behaviour of this fit (magenta line), suggests that the $p = 2$ ansatz is questionable.

Proceeding to extrapolate $\sigma_{k=N/2}$ to $SU(\infty)$, we begin with a cautionary remark. In this extrapolation, we use the tension of the $k = 4$ strings in $SU(8)$, which has a relatively large mass. This means that while its statistical error is also large, it may suffer from an even larger systematic error. Nonetheless, we proceed to fit our data with the ansatz $\frac{r_{N/2}}{N/2} = a + \frac{b}{N^p}$ with $p = 1, 2$. For $p = 1$ we find that $a = 0.506(4)$ and $b = 0.68(2)$ is a very good fit with $\chi^2/dof \simeq 0.27$. Also it is interesting to note that a is in fact consistent with the Casimir scaling prediction. In contrast, the $p = 2$ best fit has a high $\chi^2/dof \simeq 2.7$ (with $a = 0.569(3)$ and $b = 1.75(5)$).

5. Near degeneracies in the k-string excited state spectrum

We now use our data to revisit the issue of the near degeneracies in the k-string spectrum that were seen in [4, 12], with the clear advantage that our new data contains measurements of the string spectrum for a variety of string lengths l. We begin by focusing on the operators that couple best to the lowest states (as determined by our variational calculation), and calculate their overlap onto particular $SU(N)$ representation. We present the dependence of these overlaps on the string length for the five lowest states of the $k = 2$ string in $SU(6)$ (left panel of Fig. 3). This figure tells us that the ground state is always in the anti-symmetric representation, while the other states may change their 'representation content'. In particular, the first excited state is symmetric when the string is short, and becomes anti-symmetric for longer lengths. The opposite happens for the second excited state, and a similar pattern is seen for the third and fourth excited states.

To interpret these results we suggest the following simple model. Consider two non interacting NG free bosonic strings that carry fluxes in irreducible $SU(N)$ representations, and whose tensions scale according to Eq. (1.1). This model's spectrum for $SU(6)$ is presented in the right panel of Fig. 3,[3] where we see that it works well in predicting the switching of states as well as the approximate l at which this occur. We find that this model works also for other values of N and k.

Finally, note that whenever two levels cross there appears an approximate degeneracy in the spectrum, which we argue to be the one observed in [4, 12]. To check this, we looked at the

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[3]For a discussion on the excited state spectrum of the NG model see [1].
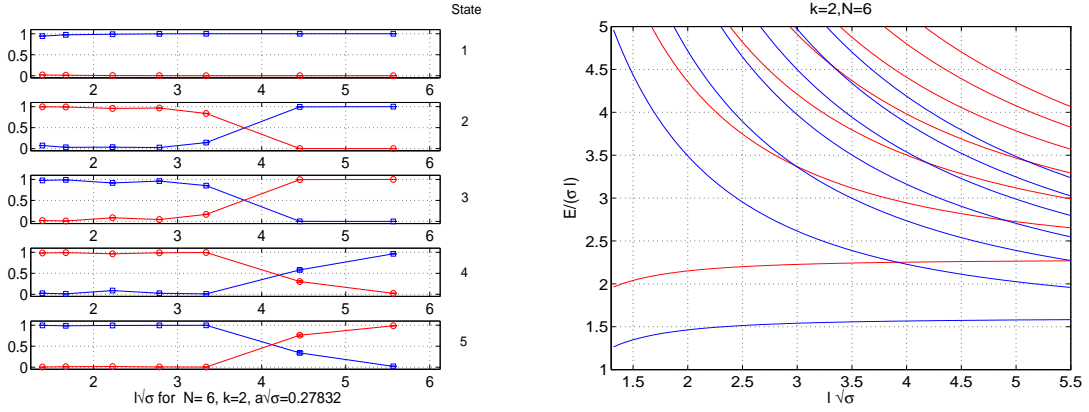
Figure 3: <u>Left:</u> Overlaps of the five lowest states in the $k = 2$ spectrum of $SU(6)$ at $\beta = 59.40$ ($a \simeq 0.12$ fm) vs. the string length. In blue(red) are the overlaps onto the antisymmetric(symmetric) representation. <u>Right:</u> The spectrum of energies vs. the string length for $SU(6)$ and $k = 2$ of the 2-NG model (see text). As in the left panel, blue(red) denotes the energies of the antisymmetric(symmetric) representation.

measured energies. Performing the variational calculation in the full basis gives the spectrum that we show in the left panel of Fig. 4. Performing the variational calculation with only a subspace
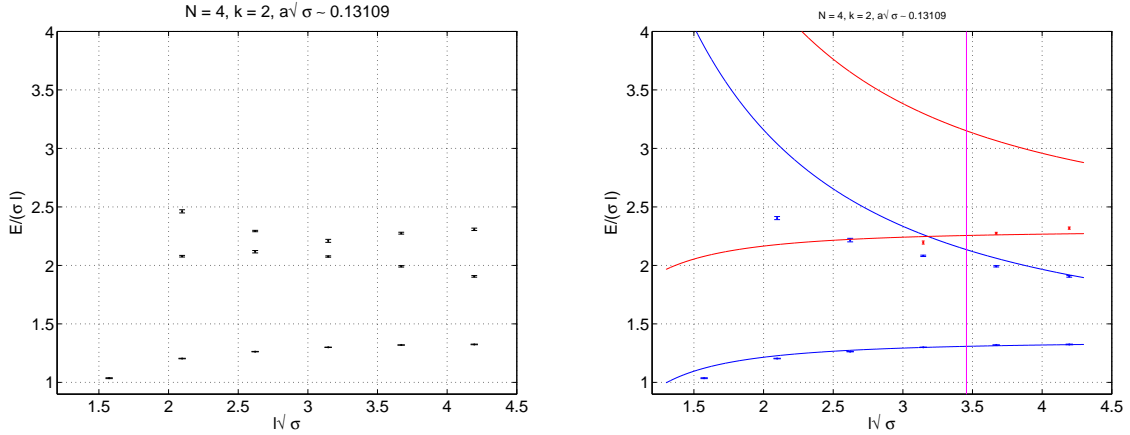
Figure 4: <u>Left:</u> The energies of the three lowest states in the $k = 2$ spectrum of $SU(4)$ at $\beta = 50.00$ ($a \simeq 0.06$ fm) vs. the string length. <u>Right:</u> Same as in the left panel, but obtained after the projection onto symmetric (in red) and antisymmetric (in blue) representations. The lines are the predictions of the 2-NG model (see text), and the vertical line in magenta denotes the string length analyzed in [12].

of operators that belongs to a single representation, we get the spectra in the right panel of Fig. 4, where we also plot the prediction of a simple 2-NG model for the lowest two states[4]. It is now clear that this model works quite well. Finally, the magenta vertical line denotes the length of the strings analyzed in [12] and we stress its proximity to the accidental degeneracy point at $l \simeq 3/\sqrt{\sigma}$.

6. Summary and future prospects

We have calculated energies of closed k-string in $SU(N)$ gauge theories in $2 + 1$ dimensions. We find that provided the strings are longer than ~ 1.4 fm, then the deviations of our data from the

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[4]The string tensions were chosen to fit our data.

Nambu-Goto (NG) free bosonic string are at most at the level of 0.5%. (This is in contrast to recent results [5] obtained in the Z_4 theory). For shorter strings we see significant deviations, which we fit and this allows us to control the systematic error involved in neglecting the $O(1/l^3)$ corrections that are sub leading to the Luscher term. Doing so, we extract tensions from the string energies from a range of lattice spacings and extrapolate the ratio $r_k \equiv \sigma_k/\sigma$ to the continuum. We find that r_k is $1\% - 4\%$ higher than the Casimir scaling law. We test different large-N extrapolations for r_2 and $r_{N/2}$ and in both cases find that our data naturally prefers a leading $1/N$ correction. Finally, a striking observation about the spectrum of the excited states is that they fall into separate sectors that correspond to irreducible representations of $SU(N)$. This demonstrates that the string spectrum contains information on the states' $SU(N)$ representation, that goes beyond their N-ality.

We stress here that the the results presented in this contribution do not enjoy the same level of confidence as our former $k = 1$ study [6], since there are several systematic errors that we did not control. The first is the effect of contamination from excited states on the energy estimates obtained from the correlation functions. For $k = 1$ we controlled these by performing double-cosh fits to our correlations and saw a shift downwards of $\sim 1 - 2$ standard deviations, away from the KKN prediction [6]. For $k > 1$ we expect larger contamination from excited states which may push r_k *toward* Casimir scaling. Work is now in progress to check for the size of this shift.

Other systematic errors that we currently investigate include the (k-string)/(k-anti-string) mixing, and the fact that the untraced *smeared* Polyakov loops are only approximately $SU(N)$ matrices. Treating these issues may improve the overlap of our operators onto the physical states, and will tighten our control on the classification of the string states according to $SU(N)$ representations.

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