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Worldline Approach to Chiral Fermions

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We propose to apply "worldline numerics" to a numerical calculation of quark determinants. The Gross-Neveu model with a U(1) chiral symmetry is considered as a first test. The worldline approach allows for an analytic renormalisation, and only finite parts of the determinant require a numerical calculation. It is shown that the discretisation of the worldlines, which is central to the numerical treatment, preserves chiral symmetry exactly. Numerical results for a kink configuration as a scalar background field are shown and compared with analytical results. The case of finite fermion chemical potential is also briefly discussed.

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1. Introduction:

Over the last decade, the QCD phase diagram as a function of temperature and baryon density has attracted intense investigations using computer simulations and collider experiments such as RHIC undertaken at the Brookhaven National Laboratory. From the simulation point of view, many efforts in lattice gauge theory in the recent past were devoted to control the severe sign problem. The proposals use a Taylor expansion with respect to the baryon chemical potential [1, 2, 3], imaginary values for the chemical potential μ [4, 5, 6] or overlap enhancing techniques [7]. Despite of these successes, our knowledge of the QCD phase diagram is still limited to rather small values of the chemical potential.

Further insights have come from QCD-like theories such as 2-colour QCD [8, 9, 10], from perturbative studies and QCD motivated models. In the latter case, one assumes that the phase at highest densities is homogeneous and argues that quark matter is organised in a colour superconducting state [11, 12, 13, 14]. Recent studies of the Gross Neveu model in the limit of many flavours have attracted a lot of interest since it was found that the high-density state of fermion matter forms an inhomogeneous "baryon crystal" [15].

Good chiral properties of the fermion action is of central importance for an investigation of quark matter at intermediate densities, since the high-density transition is driven by chiral dynamics. Unfortunately, lattice fermion actions necessarily suffer from the fermion doubling problem as firstly pointed out by Nielsen and Ninomiya [16]. Nowadays, *staggered fermions* [17], *domain wall fermions* [18] or *Neuberger fermions* [19], which are an explicit realisation of the Ginsparg-Wilson relation [20], are widely used in numerical simulations. Despite of these advanced formulations and great numerical efforts, it turns out cumbersome to achieve good chiral properties such as a sufficiently small pion mass.

Since the *worldline approach* to the quark determinant does not use a lattice discretisation of space-time, it circumvents many of these significant difficulties. Here, we will argue that the prospects of the worldline approach are (i) exact chiral symmetry but yet a fully numerical approach, (ii) analytic renormalisation and (iii) a clear description of Fermi surface effects.

The worldline method is a string-inspired approach to quantum field theory; see [21] for a review. It was further developed into a viable tool for an efficient calculation of functional determinants for arbitrary background fields [22]. Subsequently, *worldline numerics* has enjoyed a wide span of applications ranging from the Casimir effect [23, 24] and fermion induced quantum interactions [25] to the description of pair production in inhomogeneous fields [26]. A worldline lattice formulation has been presented in [27].

2. The chiral Gross-Neveu model

2.1 Setup of the model

The Gross-Neveu model in its original formulation is a two dimensional fermionic theory which shares with QCD the property of spontaneous chiral symmetry breaking and asymptotic freedom [28]. Due to Thies and Urlichs, the phase diagram is analytically known in the limit of many flavours N [15]. This model therefore provides for a benchmark test for any new numerical method which tries to extend its reach to very dense fermionic systems.

In the chiral version of this model, a pseudo-scalar field $\pi(x)$ acts as chiral partner of the scalar field $\sigma(x)$. The partition function is given by

$$\mathscr{Z} = \int \mathscr{D}\sigma\, \mathscr{D}\pi\, \exp\Big\{ -N S_{\rm fer} - N S_{\rm bos} \Big\}\,, \qquad S_{\rm bos} = \frac{1}{2g^2} \int d^2x \left[\sigma^2(x) + \pi^2(x) \right], \quad (2.1)$$

$$S_{\rm fer} = \frac{1}{2} {\rm tr}\, \ln \Big(-\partial^2 + \sigma^2 + \pi^2 - i \partial\!\!\!/ \sigma + \gamma_5 \partial\!\!\!/ \pi \Big)\,, \quad (2.2)$$

where g is the bare coupling constant and where we have used anti-hermitian Dirac matrices. The partition function is invariant under a $U(1)$ chiral rotation of the fields:

$$\begin{pmatrix} \sigma'(x) \\ \pi'(x) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma(x) \\ \pi(x) \end{pmatrix}\,. \quad (2.3)$$

In the large-N limit, one assumes that fluctuations of the mesons are negligible, and that the relevant field configurations can be obtained in leading-order saddle point approximation:

$$S_{\rm fer} + \int d^2x \frac{N}{2g^2} \left[\sigma^2(x) + \pi^2(x) \right] \stackrel{\sigma,\pi}{\longrightarrow} \text{min.} \quad (2.4)$$

2.2 The worldline approach to the GN model

The key ingredient of the worldline calculation of the fermionic part $S_{\rm fer}$ in (2.2) is the representation of $S_{\rm fer}$ in terms of an ensemble average of closed loops $x_\mu(\tau)$, $\tau = 0 \ldots T$, $x_\mu(0) = x_\mu(T)$, in Euclidean spacetime. In the *loop cloud* approach [22], the worldlines are generated according to the free probabilistic measure:

$$\delta\Big(x_{\rm cm}[x] - x_{\rm c} \Big)\, \exp\left\{ -\int_0^T d\tau\, \left[\frac{\dot{x}^2}{4} \right] \right\}\,, \quad (2.5)$$

where the loop centre of mass given by

$$x_{\rm cm}[x] = \frac{1}{T} \int_0^T d\tau\, x(\tau) \quad (2.6)$$

is constrained to $x_{\rm c}$. The fermion determinant is then represented by

$$S_{\rm fer} = \frac{1}{8\pi} \int_{1/\Lambda^2}^\infty \frac{dT}{T^2} \int d^2x_{\rm c} \left\langle \exp\left\{ -\int_0^T d\tau\, \left(\sigma^2 + \pi^2 \right) \right\} \Gamma(\sigma,\pi) \right\rangle\,, \quad (2.7)$$

$$\Gamma(\sigma,\pi) = {\rm tr}_\gamma\, \mathscr{P} \exp\left(i \int_0^T d\tau\, \left(\partial\!\!\!/ \sigma + i \gamma_5 \partial\!\!\!/ \pi \right) \right)\,, \quad (2.8)$$

where Λ is a UV cutoff.

2.3 Exact chiral symmetry

In numerical calculations, a closed loop $x(t)$ is represented by a finite number of points:

$$x_i \to x(t_i), \qquad i = 1 \ldots N_{\rm p}, \qquad d\tau = T/N_{\rm p}\,.$$

The spin factor $\Gamma(\sigma,\pi)$ is approximated by a path-ordered product

$$\Gamma_{\rm dis}(\sigma,\pi) = {\rm tr}_\gamma \prod_{x_i} \mathscr{P} \exp\left(i d\tau\, \left[\partial\!\!\!/ \sigma(x_i) + i \gamma_5 \partial\!\!\!/ \pi(x_i) \right] \right)\,, \quad (2.9)$$

The crucial observation is that, in spite of the discretisation, $\Gamma_{\rm dis}$ still is exactly chirally invariant. To show this, we define a unitary matrix U by $U \;=\; \cos(\theta/2) + i\sin(\theta/2)\;\gamma_5$, and show that

$$\partial\!\!\!/\sigma'(x) + i\,\gamma_5\,\partial\!\!\!/\pi'(x) \;=\; U\;\left(\partial\!\!\!/\sigma(x) + i\,\gamma_5\,\partial\!\!\!/\pi(x)\right)\;U^\dagger. \tag{2.10}$$

Because of the path ordering and the closeness of the (discretised) loops, we easily find that (see figure 1, left panel for an illustration)

$$\Gamma_{\rm dis}(\sigma',\pi') \;=\; \Gamma_{\rm dis}(\sigma,\pi)\;.$$

The other parts of the fermionic action (2.7) as well as the integration measure for the mesonic fields are trivially invariant (there is no anomaly in this model) leaving us with an exact chiral symmetry for the discretised theory.

2.4 Renormalisation

Another big advantage of the worldline approach to fermionic determinants is that the UV regularisation can be performed along the lines made explicit in the ab initio continuum formulation. Only finite parts of the determinant must be calculated by numerical means. This implies that one does not need to invoke any "order-a" improvement which is instrumental when conventional lattice fermions are considered. Let us illustrate the renormalisation procedure for the present case. Introducing the space-time average

$$M^2 \;=\; \frac{1}{L^2}\int d^2x\;[\sigma^2(x)+\pi^2(x)], \tag{2.11}$$

the fermionic action can be split into a UV divergent and a finite part:

$$S_{\rm fer} \;=\; S_0(M,\Lambda) \;+\; S_{\rm fer}^{\rm fin}[\sigma,\pi]\;, \tag{2.12}$$

$$S_{\rm fer}^{\rm fin}[\sigma,\pi] \;=\; \frac{1}{8\pi}\int_0^\infty \frac{dT}{T^2}\;\int d^2x\;\left\langle \exp\left\{-\int_0^T d\tau\,(\sigma^2+\pi^2)\right\}\;{\rm tr}_\gamma\;\mathscr{P}\exp\left(i\int_0^T d\tau\;(\partial\!\!\!/\sigma + i\gamma_5\partial\!\!\!/\pi)\right)\right.$$
$$\left. -2\exp\left\{-T\,M^2\right\}\;\right\rangle_x\;, \tag{2.13}$$

$$S_0(M,\Lambda) \;=\; \frac{L^2}{4\pi}\int_{1/\Lambda^2}^\infty \frac{dT}{T^2}\;\exp\left\{-T\,M^2\right\}\;. \tag{2.14}$$

With this construction, the part $S_{\rm fer}^{\rm fin}$ of the action which involves time consuming numerical simulations is UV and IR finite. Accordingly, we have removed the regulator in (2.13) by taking the limit $\Lambda\to\infty$. The part S_0 of the action contains the divergent pieces which can be calculated explicitly; dropping a field-independent constant, we obtain

$$S_0(M,\Lambda) \;=\; \frac{L^2}{4\pi}\left[\,M^2\;\ln\frac{M^2}{\Lambda^2}\;+\;(\gamma_E-1)\,M^2\right]\;+\;\mathscr{O}\left(\frac{M^2}{\Lambda^2}\right)\;, \tag{2.15}$$

where γ_E is Euler's constant. Adding the bare bosonic part of the action in (2.1), we can impose renormalization conditions, for instance, of Coleman-Weinberg type; this defines the renormalized coupling at an RG scale μ, $g^{-2}(\mu) := \partial^2 S/\partial\sigma^2|_{M^2=\sigma^2=\mu^2}$, finally yielding,

$$S_0(M,\Lambda) \;+\; S_{\rm bos} \;=\; \frac{L^2}{4\pi}\,M^2\;\left(\ln\frac{M^2}{M_0^2}-1\right)\;,\qquad M_0^2=\mu^2\,e^2\,e^{-2\pi/g^2(\mu)}, \tag{2.16}$$

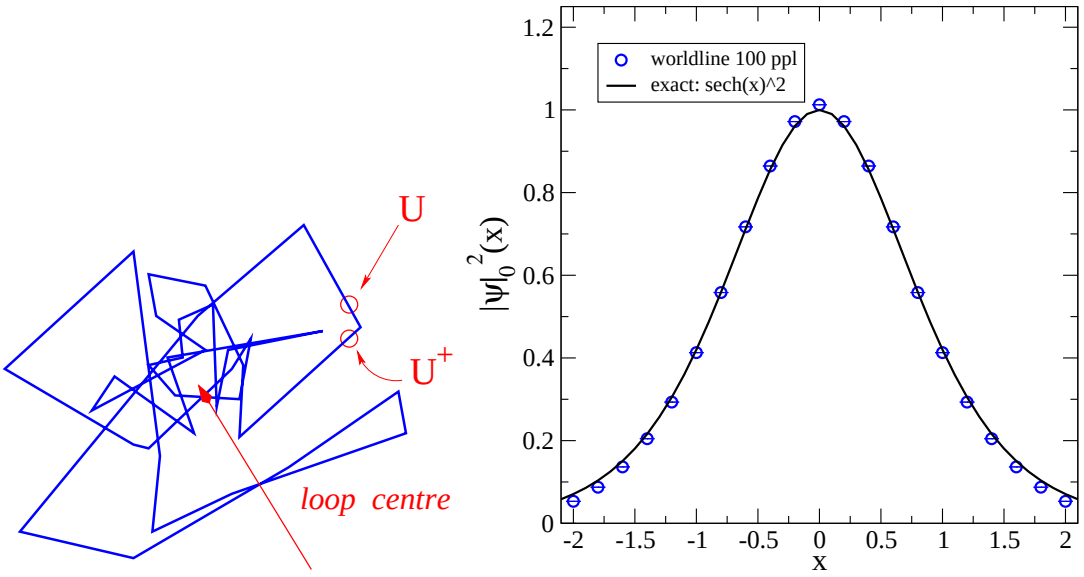

Figure 1: Illustration of chiral invariance of the spin factor (left); zero-mode wave function reconstructed from the heat kernel using free loop ensembles (right).

where we have traded the coupling $g(\mu)$ for an RG invariant mass scale M_0 in the large N limit, reflecting dimensional transmutation. This scale also denotes the large-N minimum of the action at zero temperature and density, $\sigma^2 = M_0^2 = \text{const}$.

2.5 A numerical benchmark test

For a benchmark test, we choose a kink configuration as a background field:

$$\sigma(t,x) \;=\; \sigma(x) = \tanh(x)\,, \qquad \pi(t,x) \;=\; 0\,.$$

This kink interpolates between the two homogeneous vacuum states $\sigma = \pm 1$ and is the basic building block of the 'baryonic crystal' of the Gross-Neveu model [15]. The Dirac structure decomposes into two Schrödinger problems for the heat-kernel traces

$$\mathrm{tr}_x \exp\{-TH_\pm\} = \frac{1}{\sqrt{4\pi T}} \Big\langle \exp\{-TH_\pm\} \Big\rangle_x\,, \qquad H_\pm \;=\; -\partial^2 + \sigma^2 \pm \frac{d}{dx}\sigma(x). \tag{2.17}$$

On the other hand, the heat kernel expectation value can be expressed in terms of the eigenmodes of the Hamiltonian $H_\pm$. In particular, $H_- = -\partial^2 + 1 - 2/\cosh^2 x$ gives rise to a zero mode,

$$\frac{1}{\sqrt{4\pi T}} \Big\langle \exp\{-TH_-\} \Big\rangle_x \;=\; |\psi_0(x)|^2 + \sum_i |\psi_i(x)|^2\, \mathrm{e}^{-TE_i}\,. \tag{2.18}$$

This choice therefore challenges the worldline approach, and the crucial question is whether a moderate number of free loops is able to grasp the zero-mode contribution. We have calculated the heat kernel expectation value for a range of propertime values T of $\mathscr{O}(10)$ using 50000 loops consisting of 100 points per loop, all of which include the point x. In this propertime range, the contributions of the excited states is small, and the modulus of the zero-mode wavefunction $|\psi_0|^2$

can be reconstructed from (2.18) by a fit. Our numerical findings for the modulus of the zero mode wavefunction are compared in figure 1, right panel, with the exact result. A rather rough discretisation of the loops already yields quite accurate results.

3. Finite densities

Let us consider the case of a time independent, but non-homogeneous scalar field (π is set to zero for illustration purposes), which is relevant for the high density crystal phase. The fermion determinant at finite temperatures $1/\beta$ and finite fermion chemical potential μ in the worldline approach is given by

$$S_{\rm fer} \;=\; \frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\sum_n\;\exp\left\{-T\left[\frac{2\pi}{\beta}(n+1/2)-i\mu\right]^2\right\}K(T)\,, \tag{3.1}$$

$$K(T) \;=\; \frac{1}{\sqrt{4\pi\,T}}\int dx_{\rm c}\left\langle\exp\left\{-\int_0^T d\tau\,\sigma^2\right\}\,{\rm tr}_\gamma\,\mathscr{P}\cosh\left(i\int_0^T d\tau\,\partial\!\!\!/\sigma\right)\right\rangle\,. \tag{3.2}$$

In order to make the physics of the Fermi surface transparent, we introduce the Laplace transform of the kernel $K(T)$ by

$$K(T) \;=\; 2\int_0^\infty dE\;E\;\exp[-TE^2]\,\rho(E)\,, \tag{3.3}$$

where $\rho(E)$ has the interpretation of the density of states. We obtain:

$$S_{\rm fer} \;=\; 2\int dE\;E\,\rho(E)\,\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\sum_n\;\exp\left\{-T\left[\frac{2\pi}{\beta}(n+1/2)-i\mu\right]^2\;-\;TE^2\right\}\,. \tag{3.4}$$

The technical advantage is that we have mapped the problem of dealing numerically with the Fermi surface onto the problem of a free particle theory with single particle energy E. Hence, it is well known how to evaluate the proper time integration T and the Matsubara sum n in (3.4). Decomposing the fermionic action into temperature dependent and independent parts, we find:

$$S_{\rm fer} \;=\; S_{\rm fer}^{\rm temp}\;+\;S_{\rm fer}^0 \tag{3.5}$$

$$S_{\rm fer}^{\rm temp} \;=\; \int dE\;E\;\rho(E)\left\{\ln\Big[1\;+\;\exp\{-\beta(E+\mu)\}\Big]\;+\;\ln\Big[1\;+\;\exp\{-\beta(E-\mu)\}\Big]\;\right\}. \tag{3.6}$$

$$S_{\rm fer}^0 \;=\; \int dE\;E\;\rho(E)\,\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\,\beta\,\frac{dk_0}{2\pi}\;\exp\Big\{-T\,\left[k_0^2+E^2\right]^2\Big\}\,. \tag{3.7}$$

Hence, the formulation offers a complete control over the physics associated with the Fermi surface, and even the low temperature and high density regime is accessible. For instance, the small temperature expansion (arbitrary chemical potential) of the baryon density is given by

$$\frac{1}{\beta}\frac{d}{d\mu}\,S_{\rm fer}^{\rm temp} \;=\; \int_0^\mu dE\;E\,\rho(E)\;+\;\frac{\pi^2}{6}T^2\;\frac{d}{dE}\left[E\rho(E)\right]|_{E=\mu}\;+\;\mathscr{O}(T^4)\,. \tag{3.8}$$

Fermi surface effects can thus be studied in a systematic fashion. Note, however, that the numerical calculation of the density of states $\rho(E)$ from the kernel $K(T)$ in (3.3) can be cumbersome.

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