Mass splitting of staggered fermion and $SO(2D)$ Clifford algebra

Morio Hatakeyama[*] **and Hideyuki Sawanaka**

Graduate School of Science and Technology, Niigata University
Ikarashi 2-8050, Niigata 950-2181, Japan
E-mail: hatake@muse.sc.niigata-u.ac.jp, hide@muse.sc.niigata-u.ac.jp

Hiroto So

Department of Physics, Niigata University
Ikarashi 2-8050, Niigata 950-2181, Japan
E-mail: so@muse.sc.niigata-u.ac.jp

We discuss mass splitting for degenerate tastes of staggered fermions based on an $SO(2D)$ Clifford algebra formulation. Rotationally invariant terms are added to the original staggered action with the formulation. We have four candidates for improved mass terms that can split the degenerate mass of staggered fermions. Among them, three type combinations are considered. We analyze the mass matrix and the propagator in two-dimensional free theories and find only one case that can identify with a light single Dirac mode.

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[*]Speaker.

1. Introduction

Staggered fermions are formulated in which species doublers of a Dirac field are interpreted as physical degrees of freedom, *tastes*, on lattice [1, 2]. However, it remains for a 4-fold degeneracy problem of tastes in four dimensions to be unsolved. To unfold the degeneracy, there is an approach called a *fourth-root trick* for the determinant in a staggered Dirac operator. Although studies on its theoretical basis are developed [3, 4, 5, 6], we have no local expression of one taste Dirac fermion after the fourth-root trick.

Avoiding the trick, there are pioneering works for solving the degeneracy tried by improved staggered fermion approaches [7, 8]. The improved actions generally include more operators than the original staggered one and are difficult to treat them [9]. For the control of their operators, we make use of staggered fermions on a D-dimensional lattice space based on an $SO(2D)$ Clifford algebra, and a discrete rotational symmetry can be represented by the algebra [10].

In this article, to split degenerate tastes, we add new four operators to the original staggered action in two dimensions. Only these four operators keep the discrete rotational symmetry in any dimension [10]. The total mass matrix analysis is insufficient because the matrix does not commute with the kinetic term. Therefore, we also analyze the propagator and the pole of the improved free staggered Dirac operator. It is found that only one combination in these operators is a good candidate after these analyses. More details can be found in Ref. [11].

2. Formulation of staggered fermions and rotational symmetry

Staggered fermions on the D-dimensional lattice space has been formulated by the $SO(2D)$ Clifford algebra [10]. The basic idea is that the dimension of the total representation space including spinor and taste spaces, 2^D is the same as that of an $SO(2D)$ spinor representation. 2^D is also the same as the number of sites in a D-dimensional hypercube. To avoid the double counting of sites, the lattice coordinate n_μ is noted by

$$n_\mu = 2N_\mu + c_\mu + r_\mu, \tag{2.1}$$

where N_μ is the global coordinate of the hypercube. In this case, a fundamental unit is $2a$, where a is a lattice constant, and is set to unity. $c_\mu = 1/2$ for any μ means the coordinate of a center in the D-dimensional hypercube and r_μ does the relative coordinate of a site to the center. The relative coordinate is the same as a weight of the spinor representation in $SO(2D)$.

Although our formulation can be generalized, we consider a free theory in a two-dimensional lattice, for simplicity. Relative coordinates of four sites around a plaquette are written by

$$(r_1, r_2) = (-1/2, -1/2), \ (-1/2, 1/2), \ (1/2, -1/2), \ (1/2, 1/2). \tag{2.2}$$

Actually, our staggered fermion is defined on sites (2.2) as

$$\Psi(n) \equiv \Psi_r(N) = \begin{pmatrix} \Psi_{(-1/2,-1/2)} \\ \Psi_{(-1/2,1/2)} \\ \Psi_{(1/2,-1/2)} \\ \Psi_{(1/2,1/2)} \end{pmatrix} (N) \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} (N). \tag{2.3}$$

It is noted that Ψ_1 and Ψ_4 are put on even sites and Ψ_2 and Ψ_3 are put on odd sites.

An $SO(4)$ Clifford algebra plays a crucial role in two-dimensional cubic lattice formulations [10]. The original staggered fermion action [1, 2] can be written as

$$S_{st} = \sum_{N,N',r,r',\mu,\vec{\tau}} \bar{\Psi}_r(N)(D_\mu^{\vec{\tau}})_{(N,N')}(\Gamma_{\mu,\vec{\tau}})_{(r,r')}\Psi_{r'}(N'), \tag{2.4}$$

where $\vec{\tau}$ is a two-dimensional vector with its components of $\pm 1/2$ and $D_\mu^{\vec{\tau}}$ for $\mu = 1,2$, is a generalized difference operator defined by

$$(D_\mu^{\vec{\tau}})_{(N,N')} \equiv \frac{1}{2^2} \sum_{\vec{\sigma}=0,1} (-1)^{(\vec{c}+\vec{\tau})\cdot\vec{\sigma}}(\nabla_\mu^{\vec{\sigma}})_{(N,N')}, \tag{2.5}$$

with

$$(\nabla_\mu^{\vec{\sigma}})_{(N,N')} = \begin{cases} \delta_{N,N'} - \delta_{N-\hat{\mu},N'} \equiv \nabla_\mu^-, & \sigma_\mu = 0, \\[2mm] \delta_{N+\hat{\mu},N'} - \delta_{N,N'} \equiv \nabla_\mu^+, & \sigma_\mu = 1. \end{cases} \tag{2.6}$$

$\vec{\sigma}$ is a two-dimensional vector dual to $\vec{\tau}$ and ∇_μ^+ (∇_μ^-) implies a forward (backward) difference operator along the μ-direction, respectively. The matrix $\Gamma_{\mu,\vec{\tau}}$ in our action (2.4) is composed of the $SO(4)$ Clifford algebra $\Gamma_{\mu,-\vec{c}} \equiv \gamma_\mu$ and $\Gamma_{\mu,-\vec{c}+\vec{e}_\mu} \equiv i\tilde{\gamma}_\mu$,

$$(\Gamma_{\mu,\vec{\tau}})_{(r,r')} \equiv \begin{cases} ((\sigma_3^{1/2+\tau_1} \otimes \sigma_3^{1/2+\tau_2}) \times (\sigma_1 \otimes \mathbf{1}))_{(r,r')}, & \mu = 1, \\[2mm] ((\sigma_3^{1/2+\tau_1} \otimes \sigma_3^{1/2+\tau_2}) \times (\sigma_3 \otimes \sigma_1))_{(r,r')}, & \mu = 2, \end{cases} \tag{2.7}$$

where $\vec{e}_\mu$ is the unit vector along the μ-direction. Here we denote the fundamental algebra, or the $SO(4)$ Clifford algebra as

$$\{\gamma_\mu,\gamma_\nu\} = \{\tilde{\gamma}_\mu,\tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu,\tilde{\gamma}_\nu\} = 0. \tag{2.8}$$

For a discrete rotation with angle $\pi/2$ around the center, the transformations of global and relative coordinates are denoted by $N \to R(N)$, $r \to R(r)$, and that of fermion is

$$\Psi(N) \to V_{12}\Psi(R(N)). \tag{2.9}$$

V_{12} is a rotation matrix about a spinor index in the $SO(4)$ base, up to a phase factor given by a form

$$V_{12} = \frac{e^{i\vartheta}}{2}\Gamma_5(\tilde{\gamma}_1 - \tilde{\gamma}_2)(1 + \gamma_1\gamma_2), \tag{2.10}$$

where $\Gamma_5 \equiv \gamma_1\gamma_2\tilde{\gamma}_1\tilde{\gamma}_2 = diag(1,-1,-1,1)$. Only the following four operators $\bar{\Psi}O_i\Psi$ for $i = 1$ - 4,

$$O_1 = \mathbf{1}, \quad O_2 = i\gamma_1\gamma_2 \equiv \Gamma_3, \quad O_3 = \tilde{\gamma}_1 + \tilde{\gamma}_2, \quad O_4 = \Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2), \tag{2.11}$$

are invariant under the rotation $V_{12}O_iV_{12}^\dagger$. Our analyses in the following sections concentrate on the improved staggered fermion action by these four matrices.

3. Analysis of mass matrices

To split masses in desired degenerate tastes we introduce four rotationally invariant operators which we denote as $\bar{\Psi}O_i\Psi$ [10], for the original staggered fermion action (2.4). The total mass matrix form which is invariant under the rotation by $\pi/2$ in two dimensions is given as

$$M_R = m_1\mathbf{1} + m_2\Gamma_3 + m_3(\tilde{\gamma}_1 + \tilde{\gamma}_2) + m_4\Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2), \tag{3.1}$$

where m_1, m_2, m_3 and m_4 are parameters of each operator in Eq. (2.11). M_R has four eigenvalues

$$m_1 - m_2 - \sqrt{2}m_3 + \sqrt{2}m_4, \quad m_1 - m_2 + \sqrt{2}m_3 - \sqrt{2}m_4,$$
$$m_1 + m_2 - \sqrt{2}m_3 - \sqrt{2}m_4, \quad m_1 + m_2 + \sqrt{2}m_3 + \sqrt{2}m_4. \tag{3.2}$$

A 4-component spinor should be separated into two 2-component spinors since a two-dimensional Dirac spinor is composed of a 2-component mode and we keep the rotational invariance even under a finite lattice constant[1]. Actually all possibilities of this separation are three cases and are listed in Table 1.

	parameter conditions	rotationally invariant mass term	mass eigenvalues
case 1	$m_2 = m_3 = 0$	$M_{R1} = m_1\mathbf{1} + m_4\Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)$	$m_1 \pm \sqrt{2}m_4$
case 2	$m_2 = m_4 = 0$	$M_{R2} = m_1\mathbf{1} + m_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)$	$m_1 \pm \sqrt{2}m_3$
case 3	$m_3 = m_4 = 0$	$M_{R3} = m_1\mathbf{1} + m_2\Gamma_3$	$m_1 \pm m_2$

Table 1: Three cases for the degenerate mass splitting into two spinors.

After the mass splitting, we can find the character of a Dirac spinor under the rotation,

$$\psi(x) \rightarrow Q\psi(R(x)), \tag{3.3}$$

where $Q = e^{(i\pi/4)\sigma_3} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$. Actually in cases 1 and 2 we can keep the property of a Dirac spinor on lattice. By contrast, $\Psi(N)$ acts as a vector not as a spinor in case 3. The properties of 2-component spinors under the rotation are summarized in Table 2 [2].

4. Pole analysis and 2-point functions

Our adding terms do not commute with the staggered Dirac operator. As a result, our analysis in the previous section is insufficient to split masses. We must proceed in the pole analysis of the theory because a pole mass is physical. The staggered Dirac operator in the momentum space is written as

$$D_{st}(p) = \sum_\mu \left\{ i\gamma_\mu \sin p_\mu + i\tilde{\gamma}_\mu(1 - \cos p_\mu) \right\}. \tag{4.1}$$

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[1]If one permits the rotational invariance only after taking the continuum limit, it is not necessary for degeneracy of a heavy mode and there are six more cases derived from Eq. (3.2).

[2]M_R and V_{12} can be diagonalized simultaneously because $[M_R, V_{12}] = 0$.

	V_{12}^{diag}	phase factor of V_{12}
case 1	$\begin{pmatrix} Q & 0 \\ 0 & e^{i\pi}Q^{\dagger} \end{pmatrix}$	$e^{i\vartheta} = e^{i\pi/2} = i$
case 2	$\begin{pmatrix} Q & 0 \\ 0 & e^{i\pi}Q^{\dagger} \end{pmatrix}$	$e^{i\vartheta} = e^{i\pi} = -1$
case 3	$\begin{pmatrix} Q^2 & 0 \\ 0 & e^{i\pi/2}(Q^{\dagger})^2 \end{pmatrix}$	$e^{i\vartheta} = e^{-i\pi/4} = (1-i)/\sqrt{2}$

Table 2: The properties of Dirac spinors under the rotation.

Our steps to find a pole mass are as follows: (i) set $p_1 = 0$ and $p_2 = i\kappa$ (pure imaginary) of the inverse propagator D^{-1} in the momentum representation where our rotationally invariant operators are included; (ii) calculate four eigenvalues λ of D^{-1}; (iii) find values of κ in setting $\lambda = 0$. Four values of κ equal to pole masses. As mentioned in sections 2 and 3, we keep the rotational invariance in our action and generate two Dirac spinors with different masses. We define $m_1, m_2' \equiv -im_2, m_3' \equiv -im_3$ and m_4 as real parameters to obtain real pole masses and then denote by m_l and m_h the light and heavy Dirac masses, respectively. For each three cases results in brief of the pole analysis are as follows.

● **case 1**
The pole mass is still splitting under $|m_4| < 1$. It is also found that we can take a limit $|m_h| \to \infty$ for arbitrary m_l by performing $\varepsilon \to 0$ in an expression $m_4^2 = 1 - \varepsilon$ $(0 < \varepsilon \ll 1)$.

● **case 2**
The pole mass remains degenerate because the improved term $m_3'(\tilde{\gamma}_1 + \tilde{\gamma}_2)$ is absorbed into the kinetic term.

● **case 3**
This case allows pole masses to split although the rotational property of the eigenmode is not a spinor from the discussion of the previous section.

Note that it is possible to take the light mass m_l to zero by tuning m_1 and m_4 only in case 1. In the case we obtain the equation for the pole mass as

$$16(1 - m_4^2)\sinh^4\frac{\kappa}{2} - 8(m_1^2 + 2m_4^2)\sinh^2\frac{\kappa}{2} + (m_1^2 - 2m_4^2)^2 = 0. \tag{4.2}$$

Then, solutions of Eq. (4.2) under the massless condition $m_1^2 = 2m_4^2$ are determined as

$$\sinh^2\frac{m_l}{2} = 0\,, \quad \sinh^2\frac{m_h}{2} = \frac{2m_4^2}{1 - m_4^2}. \tag{4.3}$$

In addition, to decouple the heavy mode, we can throw the mass up to infinity. Actually from Eq. (4.3), we can realize massless and infinity modes as Table 3 simultaneously. Although the

formal Γ_5 chiral projection which means even-site and odd-site separation of fermion modes is not consistent with the rotational invariance of a staggered Dirac action, it is found that infinity modes can be separately put on even or odd sites [3].

	massless modes		infinity modes	
$m_4 > 0$	$\begin{pmatrix} 1+\sqrt{2} \\ -1-\sqrt{2} \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1-\sqrt{2} \\ 1-\sqrt{2} \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
$m_4 < 0$	$\begin{pmatrix} 1-\sqrt{2} \\ -1+\sqrt{2} \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1+\sqrt{2} \\ 1+\sqrt{2} \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Table 3: Eigenvectors of the improved Dirac operator in case 1 with $m_1^2 = 2m_4^2$.

5. Summary and discussion

We have studied the mass splitting of two-dimensional staggered fermions based on the $SO(4)$ Clifford algebra. Introducing four rotationally invariant operators, we have analyzed three types of improved staggered Dirac operators and found one possibility (case 1) for taking a single mode in a two-dimensional free theory. The case keeps the splitting not only in the analysis of the mass matrix itself but also in the pole analysis including the kinetic term. According to the improvement with respect to the rotational invariance, the derived 2-component modes can be regarded as the ordinary spinor under the rotation by $\pi/2$. Furthermore, one can find a massless mode in the case unexpectedly. Our future tasks are analyses of interacting theories and the extension of our approach to four dimensions. In particular, it is crucial whether the massless condition is stable under quantum corrections by gauge interactions.

Acknowledgments

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