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MAGNETIC SOLITON FROM POINT-LIKE ELECTRIC CHARGE AT REST ON AN INERTIAL FRAME

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The present work investigates the electrostatic and magnetostatic fields generated by a point-like electrical charge at rest in an inertial frame, under the classical Abelian Born-Infeld non-linear electrodynamic. Satisfying the standard Maxwell equations, without *Anzätze* and with a few constraints, general analytical solutions were found for fields with radial (r) and polar angle (θ) dependence, breaking the radial symmetry of the problem. Apparently, non-linearity is responsible for the emergence of that anomalous magnetostatic field and, at a first glance, its interpretation suggests a connection with the field generated by an intrinsic magnetic dipole. In situations where b (an important parameter of the theory that represents the upper limit of the field strength) is free to become infinite, Maxwell regime takes over, indicating that the magnetic sector vanishes and the electric field assumes its usual Coulomb-type behavior. Our results could be of interest in connection with string theory.

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1. INTRODUCTION

Many people tried to develop theories that could describe point-like charged elementary particles. The idea was to modify and generalize Maxwell's theory to describe strong fields and to avoid singularity at the charge. Since Coulomb's law it was clear that the electric forces become infinite for point-like particle representation. The Mie's [1] theory avoids this difficulty and introduces the concept of a maximal field strength, but breaks the Lorentz invariance and the magnetic field contribution was absent. The model could be considered as the most successful one at the time its publication. He considered that Maxwell's Electrodynamics should be a linear regime of a certain non-linear theory, for weak fields and far away from the source that could be true and the particle could be considered point-like. But for small distances, the non-linear effects become dominant and the extended nature of the objects must be taken into account. Born and Infeld [2] gave the first step in towards the construction of this nonlinear Electrodynamics. Introduced in 1934, its Lagrangian is one of the general non-derivative among others. It depends only on the two algebric Maxwell invariants and preserves the gauge symmetry. Maybe, it is the most significant nonlinear theory of Classical Electrodynamics. In B-I theory, the self-energy of a point-like charge is also finite. It reduces to Maxwell Lagrangian for small field strengths and, like in Mie's theory, allows an upper bound value for the electric field. The advent of Dirac's description for the electron and the birth of QED in the forties caused the forgetfulness of B-I theory. Nowadays there is a new interest due to connections with string theory [3, 4, 5, 6]. It turns out that some objects in this theory, called D-branes, are described by a kind of nonlinear B-I action, but interest has also been enhanced by its compact and elegant form. The B-I parameter, b, has a connection with the critical electric field of the string theory ($E_{crit} = 1/2\pi\alpha \equiv b$). In the present work, Born-Infeld (B-I) non-polynomial Lagrangian has been employed to investigate the classical fields electrostatic and magnetostatic generated by a point-like electric charge at rest in an inertial frame. As it is known the classical Born-Infeld's theory is built up by means of two vectors: the electric field and the magnetic induction, both fundamental fields of the Maxwell tensor. Maxwell linear Electrodynamics excludes the existence of a magnetic sector from a pure electric charge at rest. There no room to accommodate it. But, things work different for a nonlinear theory, in particular for one like B-I electrodynamics. What can it say about magnetic sector? Are there such stable, well-behaved and physically acceptable solution? In other words: is it possible that an electric charge at rest produces something more than electric field? In order to search for non-trivial solutions, the magnetic sector has not been assumed to be zero. The main motivation is to evaluate the non-linearity effects on possible field configurations. The problem here studied presents radial symmetry but the solution founded breaks it. The solutions is analytical and no *Anzätze* was evocated.

2. CLASSICAL BORN-INFELD EQUATIONS IN MINKOVSKI SPACE-TIME

The B-I non-linear Electrodynamics action proposed in 1934 is given as follows:

$$S = \int d^4x b^2 \left[\sqrt{-g} - \sqrt{-\det\left(g_{\mu\nu} + \frac{f_{\mu\nu}}{b^2} \right)} \right] \tag{2.1}$$

The parameter b, like the speed of light in relativity theory, is the maximum field strength allowed by the theory and has a large estimated value (about 10^{15} esu). Setting its value to infinite, leads to Maxwell's linear Electrodynamics. This means that there is no limit to the field strength in linear electrodynamics. g is the determinant of metric tensor $g_{\mu\nu}$ and $f_{\mu\nu}$ is the Maxwell electromagnetic tensor given by $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Enclosed in the action integral is the Born-Infeld Lagrangian density. Open string theory loop calculations lead to this Lagrangian with $b^{-1} = 2\pi\alpha$, where α is the inverse of the string tension. From the string theory point of view, the source has a natural interpretation as being associated with a string ending on a three-brane. In flat space the metric tensor $g_{\mu\nu}$ reduces to $\eta_{\mu\nu}$ with (+1,-1,-1,-1) signature and the Greek indices run over 0, 1, 2 and 3. Evaluating the determinants of eq. 2.1 the Lagrangian density, in Minkovski spacetime, can be written in terms of the two invariants, $f_{\mu\nu}f^{\mu\nu}$ and $f_{\mu\nu}\widetilde{f}^{\mu\nu}$, with the $\widetilde{f}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}f_{\lambda\rho}$ being the dual of $f^{\mu\nu}$.

$$L = b^2 \left[1 - \sqrt{1 + \frac{f_{\mu\nu}f^{\mu\nu}}{2b^2} - \frac{\left(f_{\mu\nu}\widetilde{f}^{\mu\nu}\right)^2}{16b^4}} \right] \tag{2.2}$$

The energy-momentum tensor may also be written as

$$T_{\mu\nu} = \frac{1}{R}\left[f_\mu{}^\alpha f_{\alpha\nu} + b^2 \left(R - 1 - \frac{f_{\sigma\lambda}f^{\sigma\lambda}}{2b^2} \right) \eta_{\mu\nu} \right] \tag{2.3}$$

As well the canonical second rank tensor

$$p^{\mu\nu} = -\frac{1}{2}\frac{\partial L}{\partial f_{\mu\nu}} = \frac{1}{R}\left[f_{\mu\nu} - \frac{\left(f_{\mu\nu}\widetilde{f}^{\mu\nu}\right)\widetilde{f}^{\mu\nu}}{4b^2} \right] \tag{2.4}$$

with

$$R = \sqrt{1 + \frac{f_{\mu\nu}f^{\mu\nu}}{2b^2} - \frac{\left(f_{\mu\nu}\widetilde{f}^{\mu\nu}\right)^2}{16b^4}} \tag{2.5}$$

Finally the Lagrangian, equation 2.2, take the definitive form below.

$$L = b^2 \left[1 - \sqrt{1 - \frac{E^2 - B^2}{b^2} - \frac{\left(\vec{E}\cdot\vec{B}\right)^2}{b^4}} \right] \tag{2.6}$$

In addition, the canonical relations from equation 2.4, give:

$$\overrightarrow{D} = \frac{\partial L}{\partial \overrightarrow{E}} = \frac{\overrightarrow{E} + \left(\frac{\overrightarrow{E}.\overrightarrow{B}}{b^2}\right)\overrightarrow{B}}{\sqrt{1 - \frac{E^2-B^2}{b^2} - \frac{\left(\overrightarrow{E}\cdot\overrightarrow{B}\right)^2}{b^4}}} \qquad \overrightarrow{H} = -\frac{\partial L}{\partial \overrightarrow{B}} = \frac{\overrightarrow{B} - \left(\frac{\overrightarrow{E}.\overrightarrow{B}}{b^2}\right)\overrightarrow{E}}{\sqrt{1 - \frac{E^2-B^2}{b^2} - \frac{\left(\overrightarrow{E}\cdot\overrightarrow{B}\right)^2}{b^4}}} \tag{2.7}$$

The interaction with other charged particles is introduced by adding a term $j^{\mu}A_{\mu}$ to the Lagrangian, where j^{μ} is the current and A_{μ} is the potencial vetor. The equations of motion are the standard Maxwell equations and the non-linearity is hidden in the canonical relations above. Let's recall that for a static point-like charge these electrodynamics are defined by the static Maxwell field equations.

$$\overrightarrow{\nabla}\cdot\overrightarrow{D} = e\delta(\overrightarrow{x}) \qquad \overrightarrow{\nabla}\times\overrightarrow{E} = \overrightarrow{0} \tag{2.8}$$

$$\overrightarrow{\nabla}\cdot\overrightarrow{B} = 0 \qquad \overrightarrow{\nabla}\times\overrightarrow{H} = \overrightarrow{0}. \tag{2.9}$$

The solution for a stationary electric monopole, e, taking $\overrightarrow{B} = \overrightarrow{H} = \overrightarrow{0}$, is well-known and identical to the Maxwell solution, singular for electric induction $\overrightarrow{D}$ while the fundamental field $\overrightarrow{E}$ remains well-defined at all points, even at $r = 0$, so that the electric field singularity of the Maxwell theory disapears. In an elegant paper, Lombardo [7] uses an Schwarzschild line element to describe an electric monopole and eliminates the discontinuity at $r = 0$.

$$\overrightarrow{D} = \frac{e}{r^2}\widehat{r}, \qquad \overrightarrow{E} = \frac{e}{\sqrt{r^4 + r_0^4}}\widehat{r}, \tag{2.10}$$

$$r_0 = \sqrt{\frac{e}{b}}, \qquad \widehat{r} = \frac{\overrightarrow{r}}{r}. \tag{2.11}$$

The field $\overrightarrow{E}$ takes the point-like charge as an effective smooth distribuition given by its divergence,

$$\rho_{eff} = \overrightarrow{\nabla}\cdot\overrightarrow{E}. \tag{2.12}$$

What kind of fields could the theory provide for a non-zero magnetic sector? If everywhere regular solutions do exist, then what originates those fields? In order to start to answer such questions, one must consider each component of the canonical equations at a time. One assumes that the magnetic sector has only radial and polar components and that each component is a function of the radial distance from the point-like charge and the polar angle:

$$\overrightarrow{H} = H_r(r,\theta)\widehat{r} + H_\theta(r,\theta)\widehat{\theta} \quad \overrightarrow{B} = B_r(r,\theta)\widehat{r} + B_\theta(r,\theta)\widehat{\theta} \tag{2.13}$$

The canonical equation 2.7 is written for each component leading to an algebric non-linear system of equations mixing all electric and magnetic components. The subscripts r and θ mean radial and polar components of each vector**:**

$$E_r = D_r \frac{R}{\left(1+\frac{B_r^2}{b^2}\right)} \qquad B_r = H_r \frac{R}{\left(1-\frac{E_r^2}{b^2}\right)} \qquad B_\theta = H_\theta R \tag{2.14}$$

$$R = \sqrt{\left(1-\frac{E_r^2}{b^2}\right)\left(1+\frac{B_r^2}{b^2}\right)+\frac{B_\theta^2}{b^2}}. \tag{2.15}$$

3. SOLUTION OF THE BORN-INFELD EQUATION

The first option is to evaluate the previous algebraic system of equations 2.14 for the fundamental fields and functions of $\overrightarrow{D}$ and $\overrightarrow{H}$ only. The field is to satisfy $\overrightarrow{\nabla} \times \overrightarrow{H} = \overrightarrow{0}$ and the $\overrightarrow{B}$ field must be such that $\nabla \cdot \overrightarrow{B}$, where B_r and B_θ are not simple functions of the components $H_r(r,\theta)$ and $H_\theta(r,\theta)$. We must impose some approximation otherwise the problem will be impossible to be solved. In order to seek for a solution, one admits variable separation as the first assumption on the magnetic sector. The components of $\overrightarrow{H}$ and $\overrightarrow{B}$ field look like below:

$$\overrightarrow{H}(r,\theta) = h_r(r)G(\theta)\widehat{r} + h_\theta(r)J(\theta)\widehat{\theta} \quad \overrightarrow{B}(r,\theta) = b_r(r)G(\theta)\widehat{r} + b_\theta(r)J(\theta)\widehat{\theta} \tag{3.1}$$

If one replaces in $\overrightarrow{\nabla} \times \overrightarrow{H} = \overrightarrow{0} \rightarrow \frac{\partial(rH_\theta)}{\partial r} - \frac{\partial(H_r)}{\partial\theta} = 0$, then we get two differential equations:

$$\frac{1}{h_r(r)}\frac{d\,(rH_\theta)}{dr} = \frac{1}{J(\theta)}\frac{dG(\theta)}{d\theta} = \lambda, \tag{3.2}$$

where λ is a constant. The second assumption establishes that the magnetostatic field components ($B_r$and B_θ) must satisfy, at all points, the following constraints $B_r \ll b$ and $B_\theta \ll b$. This is reasonable, because , like a non-linear effect, cannot be of the same order of b everywhere. Those two assumptions are needed to assure that variable separation remains also valid everywhere and the system could be integrable:

$$B_r(r,\theta) = \frac{h_r(r,\theta)}{\sqrt{1-\frac{E^2}{b^2}}} = h_r(r,\theta)\sqrt{1+\frac{D^2}{b^2}}, \tag{3.3}$$

$$B_\theta(r,\theta) = h_\theta(r,\theta)\sqrt{1-\frac{E^2}{b^2}} = \frac{h_\theta(r,\theta)}{\sqrt{1+\frac{D^2}{b^2}}}. \tag{3.4}$$

The electric field, equation 2.10, was preserved and remains like in Born-Infeld original theory as a consequence of the constraints imposed above. Proceeding analogously for $\overrightarrow{B}$,

$$\overrightarrow{\nabla}\cdot\overrightarrow{B}=0\rightarrow\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B_r\right)+\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)B_\theta\right)=0, \tag{3.5}$$

one gets another pair of equations

$$\sin(\theta)G(\theta)\frac{d}{dr}\left[r^2h_r(r)\sqrt{1+\frac{D^2}{b^2}}\right]+\frac{rh_\theta(\theta)}{\sqrt{1+\frac{D^2}{b^2}}}\frac{d}{d\theta}\left(\sin(\theta)J(\theta)\right)=0 \tag{3.6}$$

To assure, once more, a complete separation of variables the terms involving the polar angle must satisfy the following differential equation**:**

$$\frac{1}{\sin(\theta)G(\theta)}\frac{d}{d\theta}\left(\sin(\theta)J(\theta)\right)=\zeta. \tag{3.7}$$

With the help of equation 3.2 one eliminates $J(\theta)$ on equation 3.7 above and one arrives at a second order differential equation for $G(\theta)$.

$$\frac{d}{d\theta}\left[\sin(\theta)\frac{d}{d\theta}G(\theta)\right]=\lambda\zeta\sin(\theta)G(\theta) \tag{3.8}$$

One physical accepatable solution is $G(\theta)=\cos(\theta)$. The other has a complex term. As a consequence, $\lambda\zeta=-2$ and $J(\theta)=-\frac{\sin(\theta)}{\lambda}$ can be checked by direct substitution. The choices of λ or ζ have no effect on the radial equation that reduces to

$$\frac{d}{dr}\left[r^2h_r(r)\sqrt{1+\frac{D^2}{b^2}}\right]+\frac{2rh_\theta(r)}{\sqrt{1+\frac{D^2}{b^2}}}=0. \tag{3.9}$$

Moreover, the components of H may be written as a simple product of functions of one variable.

$$H_\theta(r,\theta)=h_\theta(r)\sin(\theta)\ \textit{and}\ H_r(r,\theta)=h_r(r)\cos(\theta) \tag{3.10}$$

The simple angular dependence in the solution allows variable separation and the first equation gives the relation between $h_r(r)$ and $h_\theta(r)$**:**

$$h_r(r) = -\frac{d}{dr}\left[r h_\theta(r)\right]. \tag{3.11}$$

Finally, an expression can be obtained for $h_\theta(r)$ when the function $h_r(r)$ is substituted in the previous equation. This is our main diferential equation and the solution will yield the fields we are trying to get:

$$\frac{d}{dr}\left[\sqrt{r^4+r_0^4}\frac{d\Psi(r)}{dr}\right] - \frac{2r^2\Psi(r)}{\sqrt{r^4+r_0^4}} = 0. \tag{3.12}$$

$$\Psi(r) = r h_\theta(r) \tag{3.13}$$

The general analytical solution for $\Psi(r)$ is given in terms of associated Legendre functions of first kind, $P(\mu,\nu,z)$, Legendre functions of second kind, $Q(\mu,\nu,z)$. When the first two parameter is integer, one has a polynomial.

$$\Psi(r) = C_1 P\left(\frac{1}{4},\frac{1}{4},\frac{\sqrt{r^4+r_0^4}}{r_0^2}\right)\sqrt{r} + C_2 Q\left(\frac{1}{4},\frac{1}{4},\frac{\sqrt{r^4+r_0^4}}{r_0^2}\right)\sqrt{r} \tag{3.14}$$

The second function in the general solution leads to an imaginary term and the constant C_2 is set to zero. Choosing appropriate boundary conditions, so that C_1 is equal to one we arrive to the following solution.

$$\Psi(r) = P\left(\frac{1}{4},\frac{1}{4},\frac{\sqrt{r^4+r_0^4}}{r_0^2}\right)\sqrt{r} \tag{3.15}$$

We have now the tools to write down the fundamental magnetic field components and the complete magnetic induction:

$$b_\theta(r) = \frac{h_\theta(r)\, r^2}{\sqrt{r^4+r_0^4}} = \frac{r\Psi(r)}{\sqrt{r^4+r_0^4}}, \tag{3.16}$$

$$b_r(r) = \frac{h_r(r)\sqrt{r^4+r_0^4}}{r^2} = -\frac{d\ln\Psi(r)}{dr}\frac{\sqrt{r^4+r_0^4}}{r^2}, \tag{3.17}$$

$$\left|\vec{B}(r,\theta)\right| = \sqrt{\left(b_r(r)\cos(\theta)\right)^2 + \left(b_r(r)\sin(\theta)\right)^2}. \tag{3.18}$$

Fig 1 depicts the behavior of magnetic field strength as a function of the distance from the charge in units of r_0. Both are regular at the origin and fall down quickly with the distance. Far away from the charge it is possible to show that the behavior is r^{-3} times the angular dependence for thye magnetic field and is similar to the field produced by a classical magnetic dipole.

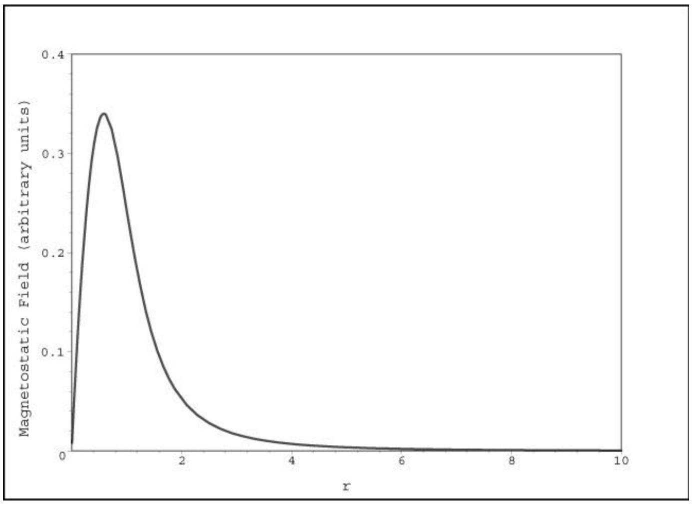

Figure 1: Magnetic field as a function of distance of the charge (r_o units).

No energy flow out but the Poyting vector has a circulating dipole current in $\widehat{\varphi}$ direction around the polar axis. Fig 2 ilustrates the bahavior of this current as a function of the distance from the charge.

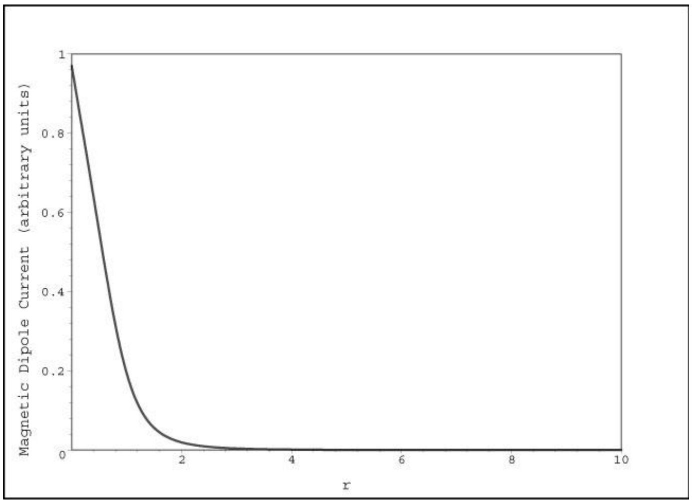

Figure 2: Magnetic dipole stationary current as function of distance of the charge (r_o units).

4. DISCUSSIONS AND CONCLUDING REMARKS

In this work, a new solution to the classical Abelian Born-Infeld Electrodynamics is presented and it has been shown to predict the existence of a real and well-behaved magnetostatic field associated with an electric point-like charge at rest in an inertial frame. It is certainly a non-linear effect that is simply ruled out by Maxwell‘s Electrodynamics. Up to now, B-I non-linear Electrodynamics has not been experimentally confirmed, but, if it can describe the nature, it may be possible to find a place that accommodate such anomalous magnetic field. The B-I theory predicts and assures this field for a standstill electric charge. There is no mathematical or physical contradiction. The present result does not imply that those fields indeed are real objects. The present work does not require additional assumptions as well the introduction of a new term in the B-I Lagrangian. The attempt to describe the electron properties by using only the structure of the field was the main motivation of Born and Infeld to develop this theory. The nature of the fields that come out rather than resembling Dirac‘s or t’Hooft‘s magnetic monopoles, reminds us a magnet seen through a macroscopic point of view that exhibits however a more complex structure at a microscopic level. The breaking of the radial symmetry contributed to the existence of the magnetostatic mathematical solutions. We must also keep in mind that for distances for which quantum effects prevail over the classical description. The Compton wavelength $\lambda = \frac{h}{mc}$ of the electron is about $2.42631058 \times 10^{-12}$ m and we are beyond the limit of the classical validity.

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