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Paley-Wiener-Schwartz Theorem and Microlocal Analysis in Theory of Tempered Ultrahyperfunctions

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We give some precisions on the Fourier-Laplace transform theorem for tempered ultrahyperfunctions introduced by Sebastião e Silva and Hasumi, by considering the theorem in its simplest form: the equivalence between support properties of a distribution in a closed convex cone and the holomorphy of its Fourier-Laplace transform in a suitable tube with conical basis. We establish a generalization of Paley-Wiener-Schwartz theorem for this setting. This theorem is interesting in connection with the microlocal analysis, where a description of the singularity structure of tempered ultrahyperfunctions in terms of the concept of analytic wave front set is given. We also suggest a physical application of the results obtained in the construction and study of field theories with fundamental length.

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1. Introduction

Tempered ultrahyperfunctions were introduced in papers of SEBASTIÃO E SILVA [1] and HASUMI [2], under the name of tempered ultradistributions, as the strong dual of the space of test functions of rapidly decreasing entire functions in any horizontal strip. Recently, aside from the mathematical interest of the results presented in Refs. [1]-[7], BRÜNING–NAGAMACHI [6] have conjectured that the properties of tempered ultrahyperfunctions are well adapted for their use in quantum field theory with a *fundamental length*, while BOLLINI–ROCCA [8] have given a general definition of convolution between two arbitrary tempered ultrahyperfunctions in order to treat the problem of singular products of functions Green also in quantum field theory. The present contribution contains a brief statement of the results obtained in Ref. [7], where the principal problem studied was the generalization of Paley-Wiener-Schwartz theorem for the setting of tempered ultradistributions corresponding to a convex cone and its connection with the microlocal analysis.

2. Tempered Ultrahyperfunctions Corresponding to a Convex Cone

An open set $C \subset \mathbb{R}^n$ is called a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. Moreover, C is an open connected cone if C is a cone and if C is an open connected set. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin. A cone C' is called compact in C – we write $C' \Subset C$ – if the projection $\mathrm{pr}\overline{C}' \overset{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathrm{pr}C \overset{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in x-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq 0, \forall x \in C\big\}$. The cone C^* is called the *dual cone* of C.

By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. An important example of tubular radial domain in quantum field theory is the *forward light-cone*

$$V_+ = \left\{ z \in \mathbb{C}^n \mid \operatorname{Im} z_1 > \left(\sum_{i=2}^{n} \operatorname{Im}^2 z_i\right)^{\frac{1}{2}}, \operatorname{Im} z_1 > 0 \right\} .$$

We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \left\{ x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta \right\} ,$$

where $\delta > 0$ is an arbitrary number.

Let C be an open convex cone and let C' be an arbitrary compact cone of C. Let $B[0;r]$ denote a closed ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B[0;r]\big)\big)$. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$\big|f(z)\big| \leq K(C')(1+|z|)^N e^{h_{C^*}(y)} , \quad z = x + iy \in T(C';r) , \tag{1}$$

where $h_{C^*}(y) = \sup_{\xi \in C^*} |\langle \xi, y\rangle|$ is the indicator of C^*, $K(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which are holomorphic in $T(C';r)$ and satisfy the estimate (**1**) will be denoted by $\mathscr{H}_c^o$. In what follows, we shall prove two lemmas which will be important for our extension of Paley-Wiener-Schwartz theorem for the setting of tempered ultrahyperfunctions.

Lemma 2.1. *Let C be an open convex cone, and let C' be an arbitrary compact cone contained in C. Let $h(\xi)=e^{k|\xi|}g(\xi)$, $\xi\in\mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $C'\setminus\big(C'\cap B[0;r]\big)$. Then $e^{-\langle\xi,y\rangle}h(\xi)\in L^2$, as a function of $\xi\in\mathbb{R}^n$.*

Proof. For details see Ref. [7]. □

Definition 2.2. *We denote by $H'_{C^*}(\mathbb{R}^n;O)$ the subspace of $H'(\mathbb{R}^n;O)$ of distributions of exponential growth with support in the cone C^*:*

$$H'_{C^*}(\mathbb{R}^n;O)=\Big\{V\in H'(\mathbb{R}^n;O)\mid \operatorname{supp}(V)\subseteq C^*\Big\}\,.$$

Lemma 2.3. *Let C be an open convex cone, and let C' be an arbitrary compact cone contained in C. Let $V=D^{\gamma}_{\xi}[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi)=k|\xi|$ for a convex compact set $K=\big[-k,k\big]^n$. Consider $V\in H'_{C^*}(\mathbb{R}^n;O)$. Then $f(z)=(2\pi)^{-n}(V,e^{-i\langle\xi,z\rangle})$ is an element of $\mathscr{H}^o_c$.*

Proof. For details see Ref. [7]. □

We define $\mathscr{U}_c=\mathscr{H}^o_c/\Pi$ as being the quotient space of $\mathscr{H}^o_c$ by set of pseudo-polynomials. Here the set $\mathscr{U}_c$ is the space of tempered ultrahyperfunctions corresponding to the open convex cone $C\subset\mathbb{R}^n$. The space $\mathscr{U}_c$ is algebraically isomorphic to the space of generalized functions $\mathfrak{H}'$. This result, which represents a generalization of Hasumi [2, Proposition 5], was obtained by Carmichael [5, Theorem 5] in the case where C is an open cone, but not necessarily connected.

3. A Generalization of the Paley-Wiener-Schwartz Theorem

More can be said concerning the functions $f(z)\in\mathscr{H}^o_c$. It is shown that $f(z)\in\mathscr{H}^o_c$ can be recovered as the (inverse) Fourier-Laplace transform[1] of the constructed distribution $V\in H'_{C^*}(\mathbb{R}^n;O)$. This result is a generalization of the Paley-Wiener-Schwartz theorem.

Theorem 3.1 (PWS-Type Theorem). *Let $f(z)\in\mathscr{H}^o_c$, where C is an open convex cone. Then the Fourier ultrahyperfunction $V\in H'_{C^*}(\mathbb{R}^n;O)$ has a uniquely determined inverse Fourier-Laplace transform $f(z)=(2\pi)^{-n}(V,e^{-i\langle\xi,z\rangle})$ which is holomorphic in $T(C';r)$ and satisfies the estimate*

$$\big|f(z)\big|\leq K(C')(1+|z|)^N e^{h_{C^*}(y)}\,,\quad z=x+iy\in T(C';r)\,.$$

Proof. For details see Ref. [7]. □

[1] The convention of signs in the Fourier transform which is used in Ref. [7] one leads us to consider the inverse Fourier-Laplace transform.

4. Analytic Wave Front Set of Tempered Ultrahyperfunctions

We shall characterize the spectrum of singularities of tempered ultrahyperfunctions via the notion of analytic wave front set [9]. Let us now consider the consequences of Theorem 3.1.

Theorem 4.1. *If $u \in \mathscr{U}_c(\mathbb{R}^n)$ and $V \in H'_{C^*}(\mathbb{R}^n;O)$ (with $O \subseteq \mathbb{R}^n$), then $WF_A(u) \subset \mathbb{R}^n \times C^*$.*

Proof. Let $\{C_j^*\}_{j\in L}$ be a finite covering of closed properly convex cones of C^*. Decompose $V \in H'_{C^*}(\mathbb{R}^n;O)$ as follows:

$$V = \sum V_j\,, \quad \text{such that } V_j \in H'_{C_j^*}(\mathbb{R}^n;O) = \Big\{V_j \in H'(\mathbb{R}^n;O) \mid \operatorname{supp}(V_j) \subseteq C_j^*\Big\}\,. \tag{4.1}$$

Next apply the Theorem 3.1 for each V_j. Then the decomposition (4.1) will induce a representation of u in the form of a sum of boundary values of functions $f_j(z) \in \mathscr{H}^o_{c_j}$, such that $f_j(z) \to \mathscr{F}^{-1}[V_j] \in \mathfrak{H}'$ in the strong topology of $\mathfrak{H}'$ as $y = \operatorname{Im} z \to 0$, $y \in C'_j \subset C_j$. According to Theorem 3.1, the family of functions $f_j(z)$ satisfy the estimate

$$\big|f_j(z)\big| \leq K(C')(1+|z|)^N e^{h_{C^*}(y)}\,, \quad z = x+iy \in T(C'_j;r)\,.$$

unless $\langle \xi, Y\rangle \geq 0$ for $\xi \in C_j^*$ and $Y \in C'_j$, with $|Y| < \delta$. Then the cones of "bad" directions responsible for the singularities of these boundary values are contained in the dual cones of the base cones. So, we have the inclusion

$$WF_A(u) \subset \mathbb{R}^n \times \bigcup_j C_j^*\,. \tag{4.2}$$

Then, by making a refinement of the covering and shrinking it to C^*, we obtain the desired result. □

5. Physical Applications of the Results Obtained

The Paley-Wiener-Schwartz theorem has a conceptual significance for QFT since it determines the analytic structure of n-point correlations functions of the fields, relating this structure to the support properties implied by the basic physical notions of causality and spectral condition. This theorem underlines the derivation of the main results of the axiomatic QFT, such as PCT theorem.

In its standard form, the Paley-Wiener-Schwartz theorem deals with the Fourier-Laplace transform of *tempered* distributions and so with analytic functions which have *polynomial growth* at infinity. It enters in some way or other in the perturbative framework to QFT due to the *temperedness* of the free propagators. However, the behavior of the fields in a QFT with a fundamental length (as the non-commutative quantum field theories (NCQFT)) can be appreciably *more singular*. This implies that the Wightman framework of local QFT turned out to be too narrow for theoretical physicists, who are interested in handling situations involving a QFT with a fundamental length. In particular, for NCQFT some very important evidences to expect that the traditional Wightman axioms must be somewhat modified are:

- *NCQFT are* **nonlocal**.

- *The existence of hard infrared singularities in the non-planar sector of the theory can destroy the* **tempered** *nature of the Wightman functions.*

- *The commutation relations* $[x_\mu, x_\nu] = i\theta_{\mu\nu}$ *also imply uncertainty relations for space-time coordinates* $\Delta x_\mu \Delta x_\nu \sim |\theta_{\mu\nu}|$*, indicating that the notion of space-time point loses its meaning. Space-time points are replaced by cells of area of size* $|\theta_{\mu\nu}|$*. This suggests the existence of a finite lower limit to the possible resolution of distance. The* **nonlocal** *structure of NCQFT manifests itself in a indeterminacy of the interaction regions, which spread over a space-time domain whose size is determined by the existence of a* **fundamental length** ℓ *related to the scale of nonlocality* $\ell \sim \sqrt{|\theta_{\mu\nu}|}$.

Our aim is to apply the results obtained here in the extension of the Wightman axiomatic approach to NCQFT in terms of tempered ultrahyperfunctions (for details see [10]). We note that the class of NCQFT in terms of ultrahyperfunctions allows for the possibility that the off-mass-shell amplitudes can grow at large energies faster than any polynomial (such behavior **is not possible** if fields are assumed to be tempered only). This fact is relevant since NCQFT stands as an intermediate framework between string theory and the usual quantum field theory.

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