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On the Geometrical Conditions to Determine the Flat Behaviour of the Rotational Curves in Galaxies

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Observational data establish that in large samples of disks galaxies the tangent velocity of tests particles in a coplanar orbit is radii independent. With this knowledge, we construct a theorem that furnishes the geometrical conditions on the metric coefficients of an axisymmetric stationary space-time in order to tests particles to obey these data. After this, we apply this theorem to a dilatonic current-carrying cosmic string and arrive to a constraint on the microscopic gauge model.

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1. Introduction

Observational data measuring the rotational curves in some galaxies show that coplanar orbital motion of gas in the outer part of galaxies maintains a constant velocity up to several luminous radii [6, 7, 8, 9, 10]. The most accepted explanation for this effect is that there exists a spherical halo of dark matter which surrounds the galaxy and accounts for the missing mass needed to produce the flat behavior of the rotational curves.

It is reasonable to suppose that the halo of the dark matter is symmetric with respect to the rotation axis of the galaxy, so we consider here an axisymmetric spacetime. In previous works a cosmic string in scalar-tensor gravities were considered [1, 2, 3]. This kind of source is an example of axisymmetric spacetime.

This work is organized as follows. In section 2 we impose the trajectory of the test particle in this static axisymmetric space-time to be coplanar and radii independent and then obtain its angular velocity in terms of the coefficients of the metric. In section 3 we re-write the line element of this region using the Chandrasekhar form and we calculate the tangential velocity of these test particles. Such calculation leads to a theorem that gives a necessary and sufficient condition on the metric coefficients in order to have tangential velocities of equatorial objects circling the galaxy and whose magnitude is radii independent. In section 4 we apply this theorem to the case of a space-time generated by a dilatonic current carrying cosmic string [1, 2]. Finally, in section 5 we present some conclusions.

2. The Line Element

The line element of an axially symmetric space-time is given in the form [11]:

$$ds^2 = -e^{2\psi}(dt + \omega d\varphi) + e^{-2\psi}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \mu^2 d\varphi^2\right] \tag{2.1}$$

where ψ, ω, γ and μ are functions of (ρ, z).

The Lagrangean for a test particle travelling on the static space-time $(\omega = 0)$ described by (2.1) is given by:

$$2\mathscr{L} = -e^{2\psi}\dot{t}^2 + e^{-2\psi}\left[e^{2\gamma}\left(\dot{\rho}^2 + \dot{z}^2\right) + \mu^2\dot{\varphi}^2\right], \tag{2.2}$$

thus, the associated canonical momenta, $p_{x^a} = \frac{\partial \mathscr{L}}{\partial \dot{x}^a}$, are

$$\begin{aligned} p_t = -E &= -e^{2\psi}\dot{t}, \\ p_\varphi = L &= \mu^2 e^{-2\psi}\dot{\varphi}, \\ p_\rho &= e^{-2(\psi-\gamma)}\dot{\rho}, \\ p_z &= e^{-2(\psi-\gamma)}\dot{z}, \end{aligned} \tag{2.3}$$

where E and L are constants of motion for each geodesic, a fact that comes from the symmetries of the space-time analyzed. As there is no explicit dependence on time t, the Hamiltonian, $\mathscr{H} = p_a\dot{x}^a - \mathscr{L}$ is another conserved quantity, which we normalize to be equal minus one half for timelike geodesics. Also, we restrict the motion to be at the equatorial plane, thus $\dot{z} = 0$. In this way, we obtain the following equation for the radial geodesic motion:

$$\dot{\rho}^2 - e^{2(\psi-\gamma)}\left[E\dot{t} - L\dot{\varphi} - 1\right] = 0. \tag{2.4}$$

In order to have stable circular motion, which is the motion we are interested in, we have to satisfy three conditions:

$i)\ \dot{\rho} = 0$

$ii)\ \frac{\partial V(\rho)}{\partial \rho} = 0$, where $V(\rho) = -e^{2(\psi-\gamma)}\left[E\dot{t} - L\dot{\varphi} - 1\right]$,

$iii)\ \frac{\partial^2 V(\rho)}{\partial \rho^2}|_{extr} > 0$, in order to have a minimum.

With these conditions, from (2.4), we obtain a set of two equations constraining the motion to be circular extrema in the equatorial plane:

$$E\dot{t} - L\dot{\varphi} - 1 \,=\, 0, \tag{2.5}$$

$$\frac{\partial}{\partial \rho}\left(e^{2(\psi-\gamma)}\left[E\dot{t} - L\dot{\varphi} - 1\right]\right) \,=\, 0. \tag{2.6}$$

From (2.3), we can express $\dot{t}$ and $\dot{\varphi}$ in terms of E and L, and the metric coefficients as:

$$\dot{t} \,=\, e^{-2\psi}E, \tag{2.7}$$

$$\dot{\varphi} \,=\, \frac{e^{2\psi}}{\mu^2}L. \tag{2.8}$$

Using these equations in the constraints ones and recalling that E and L are constants for each circular orbit, after some rearranging, we arrive at the following equations:

$$\mu^2 e^{-2\psi}\left(1 - e^{-2\psi}E^2\right) + L^2 \,=\, 0, \tag{2.9}$$

$$-\left(e^{2\psi}\right)_\rho E^2 + \left(\frac{e^{2\psi}}{\mu^2}\right)_\rho \,=\, 0, \tag{2.10}$$

where the subindex stands for derivative with respect to ρ. Solving for E and L, we obtain:

$$E \,=\, e^{\psi}\sqrt{\frac{\frac{\mu_\rho}{\mu} - \psi_\rho}{\frac{\mu_\rho}{\mu} - 2\psi_\rho}},$$

$$L \,=\, \mu e^{-\psi}\sqrt{\frac{\psi_\rho}{\frac{\mu_\rho}{\mu} - 2\psi_\rho}}. \tag{2.11}$$

The second derivative of the potential $V(\rho)$ evaluated at the values of E and L which constraint the motion to be circular and extrema, is given by:

$$V_{\rho\rho}|_{extr} = \frac{2e^{2(\psi-\gamma)}}{\frac{\mu_\rho}{\mu} - 2\psi_\rho}\left(\frac{\mu_\rho}{\mu}\psi_{\rho\rho} - \frac{\mu_{\rho\rho}}{\mu}\psi_\rho + 4\psi_\rho^3 - 6\frac{\mu_\rho}{\mu}\psi_\rho^2 + 3\left(\frac{\mu_\rho}{\mu}\right)^2\psi_\rho\right). \tag{2.12}$$

We can now obtain an expression for the angular velocity of a test particle, Ω, moving in a circular motion in the orbital plane, in terms of the metric coefficients, recalling that:

$$\Omega = \frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}}, \tag{2.13}$$

thus, using Eqs. (2.8) and (2.11) in this last equation for the angular velocity, we obtain that:

$$\Omega = \frac{e^{2\psi}}{\mu}\sqrt{\frac{\psi_\rho}{\frac{\mu_\rho}{\mu} - \psi_\rho}}. \tag{2.14}$$

3. The Tangential Velocity

We now want to express the tangential velocity of the test particles in circular motion in the equatorial plane, in terms of the metric coefficients, following [5], we rewrite the line element (2.1) as:

$$ds^2 = -e^{2\psi}dt^2 + e^{-2\psi}\mu^2 d\varphi^2 + e^{-2(\psi-\gamma)}d\rho^2 \tag{3.1}$$

thus, in terms of the proper time, $d\tau^2 = -ds^2$, we have that

$$d\tau^2 = e^{2\psi}dt^2\left[1 - e^{-4\psi}\mu^2\left(\frac{d\varphi}{dt}\right)^2 - e^{2\gamma}e^{-\psi}\left(\frac{d\rho}{dt}\right)^2\right]. \tag{3.2}$$

from which we can write

$$1 = e^{2\psi}u^{0^2}\left[1 - v^2\right], \tag{3.3}$$

where $u^0 = \frac{dt}{d\tau}$ is the usual time component of the four velocity, and a definition of the spatial velocity, v^2, comes out naturally in this way.

This spatial velocity is the 3-velocity of a particle measured with respect to an orthonormal reference system, thus has components:

$$v^2 = e^{-4\psi}\mu^2\left(\frac{d\varphi}{dt}\right)^2 + e^{2\gamma}e^{-4\psi}\left(\frac{d\rho}{dt}\right)^2. \tag{3.4}$$

The orthogonal velocity is the 3-velocity of a particle measured with respect to an orthonormal reference system, thus has components:

$$v^2 = v^{(\varphi)^2} + v^{(\rho)^2}. \tag{3.5}$$

From these last two expressions we obtain for the φ-component the spatial velocity:

$$v^{(\varphi)} = e^{-2\psi}\mu\Omega, \tag{3.6}$$

and replacing Ω from Eq. (2.14), we finally obtain an expression for the tangential velocity of a test particle in stable circular motion, in terms of the metric coefficients of the general line element given by Eq. (2.1), such tangential velocity has the form:

$$v^{(\varphi)} = \sqrt{\frac{\psi_\rho}{\frac{\mu_\rho}{\mu} - \psi_\rho}}. \tag{3.7}$$

It was our goal to obtain this expression for the tangential velocity for a general axisymmetric static space-time, and to be able to describe it in terms of the metric coefficients alone, because now we can impose conditions on this tangential velocity, and deduce a constraint equation among the metric coefficients, which has to be satisfied in order to fulfill the condition imposed on the velocity. In particular, the tangential velocity for a trajectories in each orbit is constant, that is $v_\rho^{(\varphi)} = 0$, thus $v^{(\rho)} = v_c^{(\varphi)}$, with $v_c^{(\varphi)}$ a constant, representing the value of the velocity, from Eq. (3.7), we have that:

$$\frac{\mu_\rho}{\mu} = \frac{1 + v_c^{(\varphi)^2}}{v_c^{(\varphi)^2}}\psi_\rho \tag{3.8}$$

Theorem: *The tangential velocity of circular stable equatorial orbits is constant iff the coefficient metric are related as*

$$e^{\psi} = \left(\frac{\mu}{\mu_0}\right)^l \tag{3.9}$$

with $l = cte$.

We can see that this is a **necessary and sufficient condition** for the velocity $v_c^{(\varphi)}$ to be the same for two orbits at different radii at the equatorial plane, provided that $l = \left(v_c^{(\varphi)}\right)^2 / \left(1 + \left(v_c^{(\varphi)}\right)^2\right)$.

In order to have tangential velocities of equatorial objects circling the galaxy, and whose magnitude is radii independent, the form of the line element in the equatorial plane has to be

$$ds^2 = -\left(\frac{\mu}{\mu_\rho}\right)^{2l} dt^2 + \left(\frac{\mu}{\mu_\rho}\right)^{-2l} \left[e^{2\gamma} d\rho^2 + \mu^2 d\varphi^2\right]. \tag{3.10}$$

4. Stable Circular Geodesics Around a Dilatonic Electrically Charged Cosmic String

The metric of a electrically charged cosmic string is [1, 2]:

$$\begin{aligned} ds_E^2 &= \left(\frac{r}{r_0}\right)^{2l^2-2n} W^2(r)(dr^2 + dz^2) + \left(\frac{r}{r_0}\right)^{-2n} W^2(r) B^2 r^2 d\theta^2 \\ &- \left(\frac{r}{r_0}\right)^{2n} \frac{1}{W^2(r)} dt^2 \end{aligned} \tag{4.1}$$

where

$$W(r) \equiv \frac{\left(\frac{r}{r_0}\right)^{2n} + k}{1+k}.$$

The constants m, n, k and B will be determined after the inclusion of matter fields. Our objective in this section will be to derive the geodesic equations in the equatorial plane $(\dot{z} = 0)$, where dot stands for “derivative with respect to the proper time τ". First of all, let us re-write the metric (4.1) in a more compact way:

$$ds^2 = A(r)\left[dr^2 + dz^2\right] + B(r) d\theta^2 - C(r) dt^2, \tag{4.2}$$

with

$$\begin{aligned} A(r) &= \left(\frac{r}{r_0}\right)^{2m^2-2n} W^2(r), \\ B(r) &= \left(\frac{r}{r_0}\right)^{-2n} W^2(r)\beta^2(r), \\ C(r) &= \left(\frac{r}{r_0}\right)^{2n} W^{-2}(r) \end{aligned}$$

The Lagrangian for a test particle moving in this space-time is given by:

$$2\mathscr{L} = A(r)\left[\dot{r}^2 + \dot{z}^2\right] + B(r)\dot{\theta}^2 - C(r)\dot{t}^2 \tag{4.3}$$

The associated canonical momenta, $p_\alpha = \frac{\partial \mathscr{L}}{\partial \dot{x}^\alpha}$, are:

$$\begin{aligned} p_t &= -E = -C(t)\dot{t}, \\ p_\theta &= L = B(r)\dot{\theta}, \\ p_r &= A(r)\dot{r}, \\ p_z &= A(r)\dot{z}. \end{aligned} \tag{4.4}$$

Because of the symmetries of this particular space-time, the quantities E and L are constants for each geodesic and, because this space-time is static, the Hamiltonian, $\mathscr{H} = p_\alpha \dot{x}^a - \mathscr{L}$, is a constant. Combining this information with the restriction of a motion in an equatorial plane, we arrive to the following equation for the radial geodesic:

$$\dot{r}^2 - A^{-1}\left[E\dot{t} - L\dot{\theta} - 1\right] = 0. \tag{4.5}$$

In this work, we will concentrate on stable circular motion. Therefore, we have to satisfy three conditions simultaneously. Namely:

- $\dot{r} = 0$;
- $\frac{\partial V(r)}{\partial r} = 0$, where $V(r) = -A^{-1}\left[E\dot{t} - L\dot{\theta} - 1\right]$;
- $\frac{\partial^2 V(r)}{\partial r^2}|_{ext} > 0$, in order to have a minimum.

Consequently, we have:

$$\begin{aligned} E\dot{t} - L\dot{\theta} - 1 &= 0 \qquad (4.6) \\ \frac{\partial}{\partial r}\left\{A^{-1}\left[E\dot{t} - L\dot{\theta} - 1\right]\right\} &= 0 \end{aligned}$$

Expressing $\dot{t}$ and $\dot{\theta}$ in terms of the constant quantities E and L respectively, we can re-write the above equations as:

$$\begin{aligned} \left(\frac{1}{C}\right)E^2 - \left(\frac{1}{B}\right)L^2 - 1 = 0 \\ \left(\frac{1}{C}\right)' E^2 - \left(\frac{1}{B}\right)' L^2 = 0 \end{aligned} \tag{4.7}$$

where prime means “derivative with respect to the coordinate r", which finally gives us expressions for E and L:

$$\begin{aligned} E &= C\sqrt{\frac{B'}{B'C - BC'}}, \\ L &= B\sqrt{\frac{C'}{B'C - BC'}} \end{aligned} \tag{4.8}$$

Recalling that the angular velocity of a test particle moving in a circular motion in an orbital plane is $\Omega = \frac{d\theta}{dt} = \frac{\dot{\theta}}{\dot{t}}$, we have:

$$\Omega = \sqrt{\frac{C'}{B'}} \tag{4.9}$$

We are now in position to compute the tangential velocity of the motion in an orbit plane. From now on, we will follow the prescription established by Chandrasekhar. Let us re-express the metric (4.2) in terms of the proper time τ, as $d\tau^2 = -ds^2$:

$$d\tau^2 = C(r)dt^2 \left[1 - \frac{A}{C}\left(\frac{dr}{dt}\right)^2 - \frac{B}{C}\left(\frac{d\theta}{dt}\right)^2\right] \tag{4.10}$$

and comparing with the expression

$$1 = C(r)\left(u^0\right)^2\left[1 - v^2\right], \tag{4.11}$$

where $u^0 = \frac{dt}{d\tau}$, we can easily obtain the spatial velocity v^2:

$$v^2 = \left(v^{(r)}\right)^2 + \left(v^{(\theta)}\right)^2, \tag{4.12}$$

whose components are, respectively:

$$\begin{aligned} v^{(r)} &= \sqrt{\frac{A}{C}}\left(\frac{dr}{dt}\right), \\ v^{(\theta)} &= \sqrt{\frac{B}{C}}\left(\frac{d\theta}{dt}\right) = \sqrt{\frac{B}{C}}\Omega. \end{aligned} \tag{4.13}$$

In order to have stable circular orbits, the tangential velocity $v^{(\theta)}$ must be constant at different radii at the equatorial plane. Therefore, we can impose:

$$v^{(\theta)} = \sqrt{\frac{BC'}{B'C'}} = v_c^{(\theta)} = \text{const.} \tag{4.14}$$

Applying the theorem to this case, we get

$$C^{1/2} = \left(\frac{r}{r_0}\right)^l, \tag{4.15}$$

provided $l = \frac{v_c^{(\theta)}}{1+v_c^{(\theta)}}$. This theorem implies that the line element in the equatorial plane must be:

$$ds^2 = -\left(\frac{r}{r_0}\right)^{2l} dt^2 + \left(\frac{r}{r_0}\right)^{-2l}\left[\left(\frac{r}{r_0}\right)^{2m^2} dr^2 + B^2 r^2 d\theta^2\right] \tag{4.16}$$

This form is clearly not asymptotically flat and also does not describe a space-time corresponding to a central black hole. Therefore, we can infer that it describes solely the region where the tangential velocity of the test particles is constant, being probably joined in the interior and exterior regions with other metrics, suitably chosen in order to ensure regularity in the asymptotic limits.

Let us notice, however, that this metric has the form which has been found previously [1, 2], after identifying l with the appropriate constant parameters which depend on the microscopic details of the model. The calculations are straightforward but length. For this particular configuration, consisting of an electrically charged dilatonic string, we have:

$$l = 2G_0\alpha(\phi_0)\left[U+T+I^2\right], \tag{4.17}$$

where U, T and I^2 are the energy per unit length, the tension per unit length and the current of the string, respectively. $\alpha(\phi_0)$ measures the coupling of the dilaton to the matter fields.

5. Conclusion

We found the conditions on the metric coefficients of a static axisymmetric space-time to admit a test particle with a coplanar circular orbit radii independent up to several luminous radii. A remarkably fact is that the results presented in sections 2 and 3 are independent of the type of the energy-matter tensor present in the space-time and curving it. It is a purely geometric analysis. A possible example of this kind of space-time is the one generated by a dilatonic electrically charged cosmic string.

Considering cosmic strings formed at GUT scales, $G_0\left[U+T+I^2\right] \sim 3\times10^{-6}$, and for a coupling $\alpha(\phi_0)$ which is compatible with present experimental data, $\alpha(\phi_0) < 10^{-3}$, the parameter l (and thus the tangential velocity v_c^θ) seems to be too small. The observed magnitude of the tangential velocity being $v_c^{(\theta)} > 3\times10^{-4}$ cannot be explained by a single dilatonic current-carrying cosmic string in this case. As argued by Lee [4], if a bundle of N cosmic strings formed at GUT scales seeded one galaxy, then the total magnitude of the tangential velocity would be $Nv_c^{(\theta)}$. In our case, to be compatible with astronomical observations, one must have a bundle of $N\sim10^5$ strings seeding a galaxy. With such a density, a cosmic string network would be dominating the universe, and its dynamics would be completely different. The only situation where such a huge number of strings could be possible is at much lower energy scales (electroweak scale, say) but then of course the energy scale is far too low to have any relevance for structure formation.

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