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Change of the Bulk's Signature with Change of the Brane-World's Topology

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In the present paper we prove that if we have $Y:(M^n,g)\longrightarrow(\bar{M}^D,\bar{g})$, a local isometric embedding, a topology τ'_η of $Y(W^n)$ different of the induced topology τ_η of the $Y(W^n)$, (W^n a neighborhood of $p\in M^n$) and the determinants of the metric tensor g_{ij} and $g'_{\mu\nu}$ are not equal in sign at a point, then there is a change of signature of the bulk, $(\bar{M}^D,\bar{g})$.We use the Schwarzschild space-time as a brane-world embedded in the six-dimensional bulk and a change of topology via Kruskal metric obtaining in this form a signature change that bulk.

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1. Introduction

The embedding problem has been required in problems linked to minimal class of the embedding, extrinsic gravity, theory of strings and membranes. Nowadays the embedding problem emerged with a new theory - brane-world - a theory that lead us the idea of unification of fundamental interactions using extra dimensions. Such a model has been showing positive in the sense that we find perspectives and probably deep modifications in the physics, such as: unification in a TeV scale, quantum gravity in this scale and deviation of Newton's law of gravity for small distances. A brane-world may be regarded as a space-time locally embedded in a higher dimensional space, the bulk, solution of higher dimensional Einstein's equations. Furthermore, the embedded geometry is assumed to exhibit quantum fluctuations with respect to the extra dimensions at the TeV scale of energies. Finally, all gauge interactions belonging to the standard model must remain confined to the four-dimensional space-time. Contrasting with other higher dimensional theories, the extra dimensions may be large and even infinite, with the possibility of being observed by TeV accelerators. The embedding conditions relate the bulk geometry to the brane-world geometry, as it is clear from the Gauss-Codazzi-Ricci equations, [6].

On this context it is interesting to study the geometric and topological properties of the bulk (environment space) when we modify the geometry or topology of the brane-world, [1]. Specifically, for example: a brane-world embedding into bulk, if we modify the topology of the brane-world, what will happen with the signature of the bulk? In the present paper we show that if we have $Y : (M^n, g) \longrightarrow (\bar{M}^D, \bar{g})$ a local isometric embedding and a topology τ'_η of $Y(W^n)$, (W^n a neighborhood of $p \in M^n$), different of the induced topology τ_η of the $Y(W^n)$ and the determinants of the tensor metric g_{ij} and $g'_{\mu\nu}$ are not equal in sign at a point, then there is a change of signature of the bulk. Furthermore it is made an application that results from an example of general relativity in brane-worlds context. We use the Schwarzschild space-time as a brane-world and we apply a change of topology via extension of Kruskal metric obtaining in this form signature change of the six-dimensional bulk flat, where is embedded this brane-world.

2. Change of Signature

Theorem:

Let (M^n, g) and $(\bar{M}^D, \bar{g})$ pseudo-Riemannian manifolds and consider $Y : (M^n, g) \longrightarrow (\bar{M}^D, \bar{g})$ a local isometric embedding. Let τ_η the topology of pseudo-Riemannian submanifold $Y(W^n)$, W^n a neighborhood of $p \in M^n$. If we change topology of $Y(W^n)$ to τ'_η, if $det(g_{ij})$ and $det(g'_{\mu\nu})$ differ in sign at a point, then there exist a change of signature of form assigned to $\bar{g}$.

Proof:

Suppose that η_1, η_2 are charts of $Y(W^n)$ with intersecting domains U and V. Then $\eta_2 \circ \eta_1^{-1}$ is a diffeomorphism and so its domain $\eta_1(U \cap V)$ must be open in R^n. Since $(U \cap V)$ is a subset of U such that $\eta_1(U \cap V)$ is open in R^n, then $\eta_1 |_{(U \cap V)}$ is a chart of $Y(W^n)$ with domain $(U \cap V)$. These arguments are sufficient to conclude that the collection of coordinate domains of manifold $Y(W^n)$

forms a basis for a topology on the set $Y(W^n)$. The topology thus induced on the set $Y(W^n)$ by its C^∞ structure is called τ_η topology of the manifold $Y(W^n)$. With this topology, a non-empty subset U of $Y(W^n)$ is open iff each point of U has a coordinate neighborhood which lies in U, [2].

Since $Y : (M^n, g) \longrightarrow (\bar{M}^D, \bar{g})$ is a local isometric embedding we can see a neighborhood of one point of M^n, that is W^n, as a copy in $\bar{M}^D$. We note that since $Y : (M^n, g) \longrightarrow (\bar{M}^D, \bar{g})$ is a embedding, $dY_p : T_p(M^n, g) \longrightarrow T_{Y(p)}(\bar{M}^D, \bar{g})$ is injective, $\forall p \in (M^n, g)$ and Y is a homeomorphism on $Y(M^n, g) \subset (\bar{M}^D, \bar{g})$, where (M^n, g) has induced topology by $(\bar{M}^D, \bar{g})$. We note that if $Y : (M^n, g) \longrightarrow (\bar{M}^D, \bar{g})$ is a local isometric embedding, then each point $p \in M^n$ has a neighborhood W^n such that $Y\,|_{W^n}$ is an embedding and if $Y(W^n)$ is assigned the metric tensor such that the induced map $W^n \longrightarrow Y(W^n)$ is an isometry, then $Y(W^n, g)$ is a pseudo-Riemannian of $(\bar{M}^D, \bar{g})$. In other words we say that locally an isometric immersion is essentially a pseudo-Riemannian submanifold.

Denote τ_η the topology of $Y(W^n, g)$ assigned to atlas A and another topology τ'_η of $Y(W^n, g)$ assigned to new atlas A', both induced topologies on the set $Y(W^n, g)$ by its C^∞ structure.

Let g_{ij} the components of metric tensor on the differential quadratic form, $g_{ij}dx^i dx^j$. By a real transformation this quadratic form at a point $p \in W^n$ is represented by

$$(dx^1)^2 + ... + (dx^r)^2 - (dx^{r+1})^2 - ... - (dx^n)^2,$$

where $S = 2r - n$ is the signature of the form.

Suppose it exists a transformation such that

$$g'_{\mu\nu} = g_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu},$$

from the rule for multiplication of determinants we have

$$det(g'_{\mu\nu}) = det(g_{ij})J^2,$$

where J is the Jacobian of the transformation from g_{ij} to $g'_{\mu\nu}$. Now suppose that $det(g_{ij})$ and $det(g'_{\mu\nu})$ differ in sign at a point, for instance $p \in (W^n, g)$. Thus we have an imaginary transformation from g_{ij} to $g'_{\mu\nu}$ at $p \in (W^n, g)$, [3]. For this transformation at p the form $g'_{\mu\nu}dx'^\mu dx'_\nu$ can be represented by

$$(dx'^1)^2 + ... + (dx'^{r'})^2 - (dx'^{r'+1})^2 - ... - (dx'^n)^2,$$

where $S' = 2r' - n$ is the signature of this form, clearly $S \neq S'$.

We note that g is the induced metric of $\bar{g}$, thus we have $Y^*(\bar{g}) = g$, where $Y^*(\bar{g})$ is called the pullback of $\bar{g}$ by Y, that is, $g(u, v) = \bar{g}(dY(u), dY(v)), \forall u, v \in T_p(W^n, g)$. Using the facts before it is easy to see that exists a change of signature of form assigned to $\bar{g}$ at point p. We remember that for real transformations the signature of $\bar{g}$ is invariant.

3. The Schwarzschild Brane-World

Randall and Sundrum have proposed an interesting scenario of extra non-compact dimensions in which four-dimensional gravity emerges as a low energy effective theory, to solve the hierarchy problems of the fundamental interactions. This proposal is based on the assumption that ordinary matter and its gauge interactions are confined within a four dimensional hypersurface, the physical brane, embedded in a five-dimensional space. In order to describe the real world, the Randall-Sundrum scenario has to satisfy all the existing tests of General Relativity, with base in the motion of material particles within the Schwarzschild brane-world that is given by four-dimensional geodesic equation. That is an excellent and famous model, but from the point of view of mathematics and experimental some constraints exist, [4],[5]. An interesting alternative to the model of Randall-Sundrum is the interesting possibility with an infinite flat extra dimension was proposed by Dvali et al, [6]. For simplicity, we will consider the case of six-dimensional flat bulk where the Schwarzschild space-time is embedded.

The physical space outside an approximately spherical body with mass M is modeled through a 4-dimensional space-time, solution of Einstein equations, whose geometry is described with good approximation by Schwarzschild's solution, [7].

We define the following regions:[8]

a) The exterior Schwarzschild space-time (V_4, g):

$$V_4 = P_I^2 \times S^2\ ;\ P_I^2 = \{(t,r) \in R^2 |\ r > 2m\}.$$

b) The Schwarzschild black hole (B_4, g):

$$B_4 = P_{II}^2 \times S^2\ ;\ P_{II}^2 = \{(t,r) \in R^2 |\ 0 < r < 2m\}.$$

In both cases, S^2 is the sphere of radius r and the metric g is the usually Schwarzschild metric. So we define the space-time $(E,g) = ([P_I^2 \cup P_{II}^2] \times S^2, g)$ and the Kruskal extension (E', g') is established via immersion, [11]. The Q^2 (plane Kruskal, [9] and all characteristics of this metric are given by $(E' = Q^2 \times S^2, g')$, without a singularity at $r = 2m$.

Note that (E,g) is disconnected because, it is composed by two connected components. We have that the topology of (E,g), $(R^2 - \{(t,r) \in R^2 |\ r = 2m\}) \times S^2$, is different from the topology of (E', g') which is equal to $R^2 \times S^2$. This latter is due to the **"theorem"**: *The topology of a gravitational field outside of a body with spherical symmetry is given by* $R^2 \times S^2$, [11].

Suppose that space-time (E,g), with topology $(R^2 - \{(t,r) \in R^2 |\ r = 2m\}) \times S^2$, is immersed into a $(\bar{M}^6, \bar{g})$ pseudo-Euclidean manifold of six dimensions, and signature $S = -2$, the immersion of Kasner, [10].

Now from the new immersion coordinates on Kruskal metric, [11], suppose that space-time (E', g'), with topology $R^2 \times S^2$, is immersed into a $(\bar{M}^6, \bar{g})$ pseudo-Euclidean manifold of six dimensions, where we now must have signature $S' = -4$, the immersion of Fronsdal, [12].

4. Comments

We prove that beginning from a space-time brane-world embedded in an environment space of higher dimensional it is possible to change its signature from change of topology of brane-world. This result shows us that we can have nature geometric and topological constraints which can to enjoin some link with physical proprieties of the space-time. For instance the fact that we assume the space-time to be connected. This is because disconnected components of the universe cannot

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interact by means of any signals and the observations are confined to the connected component wherein the observer is situated.

Observing carefully a special concept of topology change for space-times, our result motives us to investigate the cosmic censorship conjecture problem in brane-world context, i.e., the space-time is viewed as a brane-world, and the conjecture tell us that the complete gravitational collapse of a body always produces a black hole, all space-time singularities resulting from collapse are contained within the black hole and thus hidden from the view of outside observers. No naked singularities occur on the brane. If topological change is allowed for some physical reason, it can affect the structure of space-time brane-world in the sense of arising naked singularities, [13].

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