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Free Parameters in Quantum Theories: an Analysis with the Variational Approximation

Fábio L. Braghin*

Instituto de Física, Universidade de São Paulo, CP 66318, CEP 05315-970, São Paulo, Brazil

E-mail: braghin@if.usp.br

The so-called asymmetric phase of the massive $\lambda\phi^4$ model in 3+ 1 dimensions (where the condensate $< vac|\phi|vac > \neq 0$) is considered for the search of privileged values of its parameters (mass and coupling constant) and possible conditions they can be expected to satisfy. For this, a successful renormalization procedure for the variational approximation with a trial Gaussian ansatz is re-analysed as a departing framework. The extremization of the renormalized energy density with relation to the renormalized mass, coupling and $\bar{\phi} \equiv < \phi >$ is done. These minimizations do not yield the same expressions of the regularized equations as it is done in the variational approach. Some expressions have an "energy scale" invariance unless for a term usually proportional to an energy scale. A different view on the restoration of symmetry issue is presented. The transcendental character of the GAP equation can be reduced or even eliminated by placing some variables in the complex plane. With this ansatz it is possible to go from a phase in which $\bar{\phi} = 0$ to the other phase by a transformation (which can be a rotation) of the mass parameters in the complex plane. The eventual relevance of the method and of the results for some specific systems as well as some applications are pointed out.

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*Speaker.

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1. Introduction

It is usually highly desirable to predict the values of the free parameters of a physical model, such as masses and couplings, from the theory itself before comparisons or fits to experimental observations. In particular, quantum many body theories (for example in nuclear physics and condensed matter) and effective models for quantum field theories have free parameters which are expected to be fitted to reproduce observables which are measured and known from experiments. Usually for the determination of these parameters there is a large range of values which can be acceptable and used for investigating the properties of the particular system. Therefore in another level of reasoning, although the degrees of freedom do not necessarily correspond to fully physical degrees of freedom (or rather effective ones) it is highly desirable to obtain them from a more fundamental theory when the understanding of the relationship among the (effective) parameters acquire a deeper meaning. Besides that the relationships among the parameters might be of great interest for the understanding of the structure of the theory itself. For the programs discussed above it may be possible to predict values, either exact values or a range of them for which the theory exhibits a particular or special behavior. These "privileged" values can be associated to the validity of the approximation method used to treat the theory, to the applicability of the model and also to points in which particular physical effects can be expected. The main aim of the present work is to suggest and investigate some ways according to which values (or range of values) for these parameters could be found. Eventually this may suggest sort of “constraints” between the parameters inside the model. The basic ideas are: to search for renormalized couplings and masses which extremize (minimize/maximize) the renormalized energy density. Besides that the parameters are placed in the complex plane to investigate whether relationships among them, which can be highly transcendental in non perturbative methods, can be either simplified or more precisely defined. In some sense this procedure can be considered as complementary to the renormalization group method [1].

The $\lambda\phi^4$ model has been extensively studied for different reasons among which to shed light on non perturbative effects in quantum field and many body theories (QFT, QMBT). It corresponds to one of the simplest self interacting model whose structure is expected to be (partially) present in several more elaborated theories and it presents interesting features [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. It has also been considered for the investigation of the Higgs sector of particle theories [6, 11], cosmological models [12], some aspects of Bose-Einstein-Condensation [1, 13, 14] among other systems. Besides that this models shares several properties with the linear sigma model (LSM) which is an effective model for low energy QCD. Although it strongly seems to exhibit the property of asymptotic freedom in the asymmetric phase [3, 4, 7], the model is “trivial” in the symmetric phase [8, 1, 15].

The framework adopted to carry out the investigations mentioned above is the variational Gaussian approximation in the Schroedinger representation. It is equivalent to the Hartree Bogoliubov approach [14], in which a Bogoliubov transformation can yield a non equivalent basis of Fock states [16, 17, 18], and also to the leading order large N approximation [20, 21]. The choice of a particular basis of the Fock space is due to the infinite number of degrees of freedom or conversely, to the non unitarity of the underlying Bogoliubov transformation. In this approach the ground state of the system is determined by (GAP) equations for the variational parameters, which are choosen to be a mass and the classical expected value of a field characteristic from a SSB state (for a scalar field $<\phi>\equiv\bar{\phi}$, which will be referred to as condensate even because it appears due to long range interaction of the model [14]). The Bogoliubov transformation is also appropriated the investigation of non equibrium time dependent situations [14]. In some sense we also hope that the present work can provide some insight on a possible way of investigating these issues because of the (eventual) time dependence (and eventual in medium dependence for many body systems) of the parameters of an interacting field or particle.

In the present work the usual renormalization scheme of the Gaussian approach as carried out, for example, in [22] for the $\lambda\phi^4$ model is used as starting point for further investigation. It is proposed the extremization of the renormalized energy density with relation to the renormalized parameters (coupling constant and mass). Besides that some variables are placed in the complex plane to search suitable (physical) values and eventual conditions for these parameters. The work is organized as follows. In the next section the Gaussian approximation is summarized: the GAP equation (transcendental) is derived, obtained from the regularized theory (with a cutoff) and the renormalization procedure is sketched as done in [22]. In sections 3 and 4 values of the renormalized mass, condensate and coupling constant which extremize the energy density are searched and analysed. For some ranges of the values of the parameters instabilities of the theory can appear. In section 5 transcendental solutions for the equations which define the ground state are investigated by allowing some parameters to be complex such that the imaginary part disappears in the end of the calculation to keep real values of mass parameters and coupling constant. The last section presents a summary.

2. Gaussian approximation for the $\lambda\phi^4$ model

The Lagrangian density for a self interacting scalar field $\phi(\mathbf{x})$ is given by:

$$\mathscr{L}(\mathbf{x})=\frac{1}{2}\left\{\partial_\mu\phi(\mathbf{x})\partial^\mu\phi(\mathbf{x})-m_0^2\phi^2(\mathbf{x})-\frac{\lambda}{12}\phi^4(\mathbf{x})\right\}, \tag{2.1}$$

where bare mass is m_0^2 and coupling constant λ. The theory is quantized in the (functional) Schrodinger representation [18, 19]. The action of the field and momentum operators over a functional state $|\Psi[\phi]>$ are given respectively by: $\hat{\phi}|\Psi>=\phi|\Psi>$ and $\hat{\pi}=-i\hbar\delta/\delta\phi|\Psi>$.

In the static Gaussian approximation at zero temperature the trial ground state wave functional Ψ can be parametrized by a Gaussian like:

$$\Psi[\phi(\mathbf{x})]=Nexp\left\{-\frac{1}{4}\int d\mathbf{x}d\mathbf{y}\delta\phi(\mathbf{x})G^{-1}(\mathbf{x},\mathbf{y})\delta\phi(\mathbf{y})\right\}, \tag{2.2}$$

Where $\delta\phi(\mathbf{x})=\phi(\mathbf{x})-\bar{\phi}(\mathbf{x})$ is the field shifted by the condensate, the point where the wave function is centered; the normalization factor is N, the variational parameters are the (classical) expected value of the field, $\bar{\phi}(\mathbf{x})=<\Psi|\phi|\Psi>$, and the quantum fluctuations represented by the two point function, i.e., the width of the Gaussian: $G(\mathbf{x},\mathbf{y})=<\Psi|\phi(\mathbf{x})\phi(\mathbf{y})|\Psi>$. In variational calculations the averaged energy calculated with $\Psi[\phi(\mathbf{x})]$ is to be minimized to obtain the GAP equations. In principle it yields a maximum bound for the ground state (averaged) energy, although ultraviolet divergences make this not necessarily (completely) reliable. The minimization of the renormalized theory is useful for this theoretical bound of the variational principle. Each of these variational parameters represents one component of the scalar field: the expected value in the ground state ("classical" part) and the two-point Green's function with the mass of the quantum which is decomposed into creation and annihilation operators [14]. Considering this wavefunctional yields a self energy in which there is a resummation of "cactus" type loop diagrams [4, 15, 23, 24, 26].

The averaged value of the Hamiltonian is calculated and expressed in terms of the variational parameters by means of expressions written above. The variational procedure requires the minimization of the averaged energy density with respect to the variational parameters to produce the following GAP and condensate equations:

$$\begin{array}{ll} \dfrac{\delta \mathscr{H}}{\delta G(\mathbf{x},\mathbf{y})} \rightarrow \;\; 0 = -\dfrac{1}{8} G^{-2}(\mathbf{x},\mathbf{y}) + \dfrac{\Gamma(\mathbf{x},\mathbf{y})}{2} + \dfrac{\lambda}{2} \bar{\phi}(\mathbf{x})^2 \quad (i) \\ \dfrac{\delta \mathscr{H}}{\delta \phi(\mathbf{x})} \rightarrow \;\; 0 = \Gamma(\mathbf{x},\mathbf{y}) \bar{\phi}(\mathbf{y}) + \dfrac{\lambda}{6} \bar{\phi}^2(\mathbf{x}), \quad (ii) \end{array} \tag{2.3}$$

Where $\Gamma(\mathbf{x},\mathbf{y}) = -\Delta + \left(m_0^2 + \frac{\lambda}{2} G(\mathbf{x},\mathbf{x}) \right) \delta(\mathbf{x}-\mathbf{y})$. The Green's function G can be written from expressions above as:

$$G_0(\mathbf{x},\mathbf{y}) = <\mathbf{x}| \frac{1}{\sqrt{-\Delta + m^2}} |\mathbf{y}> \tag{2.4}$$

where m^2 is given by the self consistent (transcendental) GAP equation (expression (2.3)):

$$m^2 = m_0^2 + \frac{\lambda}{2} TraceG(x,x,m^2) + \frac{\lambda}{2} \bar{\phi}^2. \tag{2.5}$$

An analogous expression holds for the case in which $\bar{\phi} = 0$, i.e.,

$$\mu^2 = m^2(\bar{\phi} = 0) = m_0^2 + \frac{\lambda}{2} TraceG(x,x,\mu^2).$$

Expression (2.4) is equivalent to the Feynman Green's function with time integrated and with sign changed in the imaginary part by replacing the self consistent mass by the bare mass m_0^2. The physical masses in the different phases can assume different values from each other. The condition of minimum for this procedure and its stability was partially investigated in [4] and it corresponds to analysing the second order variation of the energy density with respect to the variational parameters.

The above expression for the Gaussian width (2.4) (and its inverse G_0^{-1}) can be calculated in the momentum space with a regulator Λ (cutoff). which will be eliminated latter The renormalization procedure has been performed in three dimensions for example in [22, 15, 3, 4, 9]. According to the procedure shown below the non equivalence of the Fock basis for each of the phases (explicited through the Gaussian covariances or equivalently m^2 and μ^2) is exhibitted as discussed in [16].

2.1 Renormalized parameters

The renormalization procedure of the parameters of the model is done as follows [22, 4]. The energy density of the symmetric phase, as well as its GAP equation (2.5), is subtracted from the corresponding expression of the asymmetric phase. The GAP equation as defined in expression (2.5) can be rewritten as:

$$\mu^2 = m^2 + g_R \left(\bar{\phi}^2 + \frac{m^2}{8\pi^2} Ln \left(\frac{m}{\mu} \right) \right), \tag{2.6}$$

where the renormalized parameters were defined as:

$$\mu^2 = m_R^2 \equiv \frac{m_0^2 + \frac{\lambda\Lambda^2}{16\pi^2}}{1 + \frac{\lambda}{16\pi^2} log\left(\frac{d\Lambda}{\mu}\right)}, \qquad g_R = \frac{-\frac{\lambda}{2}}{1 + \frac{\lambda}{16\pi^2} log\left(\frac{d\Lambda}{\mu}\right)}. \tag{2.7}$$

In the first of these expressions $m_R^2 \equiv \mu^2$ was chosen to produce the usual effective potential [15, 22]. It is seen from the second of these expressions that in the limit of $\Lambda \to \infty$ the bare coupling constant would go to zero in order to keep g_R finite if μ is kept constant. This is the "triviality" problem. It does not make any consideration of how the ratio Λ/μ behaves as $\Lambda \to \infty$.

The resulting subtracted energy density, $\mathscr{H}_{sub} = \mathscr{H}(\bar{\phi}) - \mathscr{H}(\bar{\phi} = 0)$, is re-written in terms of the renormalized mass, coupling constant and the mass scale eliminating the cutoff. It is given by:

$$\mathscr{H}_{sub} = \frac{m^2}{2}\bar{\phi}^2 + \frac{1}{4g_R}\left(m^2 - \mu^2\right)^2 + \frac{1}{128\pi^2}\left(m^4 Ln\left(\frac{m^4}{\mu^4}\right) - m^4 + \mu^4\right). \tag{2.8}$$

The mass scale μ^2 is not a free parameter for the ground state in fact, it can be considered to be a function of the mass m^2 and the coupling g_R by the GAP expression (2.6). Other approaches can be of interest for investigating the variational method in the Schrodinger picture [25]. In the ground state the parameters $\bar{\phi}, m^2, \mu^2$ (for a given g_R) are related by the GAP and condensate expressions shown above. Any deviation of these values fixed in the GAP equation induce temporal evolution or can correspond to excited states (stable or not).

However the integration of the GAP equation with relation to m^2 should also yield an expression equal to (2.8) in the case the order of performing renormalization and extracting the ground state does not change results. This does not happen [10].

3. Energy density and renormalized mass

In the following the renormalized energy density $\mathscr{H}_{sub}$ is minimized with relation to the renormalized (physical) mass is searched:

$$\frac{\partial \mathscr{H}_{sub}}{\partial m} = 0. \tag{3.1}$$

Considering that $\bar{\phi}^2$ is in fact dependent on m_R^2 by expression (2.6) the resulting expression is given by:

$$0 = m^3\left[Ln^2\left(\frac{m}{\mu}\right)a_1 + Ln\left(\frac{m}{\mu}\right)a_2 + a_3\right], \tag{3.2}$$

where a_i can be given in terms of

$$J = 1 - \frac{g_R}{(8\pi)^2} = 1 - G_R,$$

and:

$$\begin{aligned} a_1 &= \frac{1}{g_R}J^2 + \frac{1}{32\pi^2}, \qquad a_3 = \frac{1}{32\pi^2}\left(1 + \frac{g_R}{32\pi^2}\right). \\ a_2 &= \frac{2}{g_R}\left(-1 + J + \frac{J^2}{(32\pi^2)}\right) + \frac{1}{128\pi^2}\left(1 + \frac{2J^2}{(8\pi)^2}\right), \end{aligned} \tag{3.3}$$

The expression (3.2) is not equal to the GAP (3.2) obtained from the minimization of the regularized energy density with relation to $G(m^2)$, i.e. the variational parameter. There are therefore five solutions for the renormalized mass m^2 which can be written in the following form:

$$m^3 = 0, \qquad m^{\pm} = \mu \; exp(H^{\pm}), \tag{3.4}$$

where:

$$H^{\pm} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}. \tag{3.5}$$

These solutions for $m^{\pm}$ can be viewed as having corrections for the value of μ due to the self interaction through the parameters $H^{\pm}$ due to the appearance of $\bar{\phi} \neq 0$. In these expressions there is an energy scale invariance for $m^{\pm}$ with simultaneous changes in the mass renormalization parameter μ.

The particular case of $m^2 = \mu^2$, for which the GAP equation is trivially satisfied with $\bar{\phi} \to 0$, is found for $a_3 = 0$ yielding $g_R = -32\pi^2$. This point can correspond to a restoration of the symmetry breaking when $\bar{\phi} \neq 0$. According to expression (3.5) there are two solutions for the above value of $a_3 = 0$. The first one, in which $m^+ = 0$, is not consistent as it will be discussed below, and the other corresponds to following solution:

$$\frac{m^-}{\mu} = exp\left(-\frac{a_2}{a_1}\right). \tag{3.6}$$

These resulting values can be compared to the limit for the stability of a two-body bound state, when $m = \mu$, found by Kerman and Lin: $g < -8\pi^2$ [9].

A similar case is obtained when $a_1 = 0$. In this point $g_R = 64\pi^2$ yielding undefined ratio m^+/μ (i.e., this ratio can be defined by the GAP equation as usually for the coupling above).

The zero mass solutions correspond to a saddle point, they are not minima neither maxima of the energy density in agreement with [4, 9]. If the others solutions are minima is checked via the positiveness of the second derivative:

$$\frac{\partial^2 \mathscr{H}_{sub}}{\partial m^2} = \frac{m^2}{(8\pi)^2} \left(2Ln\left(\frac{m}{\mu}\right) a_1 + a_2\right) > 0. \tag{3.7}$$

For the derivation of these expressions the complete self consistency of the Gaussian equations was not completely considered. There has been used a truncation on the dependence on μ, i.e., the dependence of $Ln(\mu/m)$ on μ (self consistency) was considered only for μ not very different from m, i.e. $\mu^2 = m^2 + \delta$ where $\delta << m^2$. Out of this range the above solutions are not expected to be valid. Considering that in each of the phases the corresponding particle can be expected to have different masses, eventually in a Higgs-like picture, this means that these masses in the different phases are not very different.

In the limits of $g_R \to \pm\infty$ it is obtained analytically that either $m = \mu$ or $m = 0$. For $\mu \to \infty$ the renormalized coupling constant $G_R \to 0$. While the solution m_R^- in the weak coupling regime can be identified to the renormalization point usually considered (for $\mu >> m$ and/or the cutoff going to infinite), although in the present analysis $\delta << \mu^2$, there is a consistent stable solution m^+ for which $\mu \simeq m^+$. Therefore the ground state can impose several restrictions on the values that mass and coupling can assume. It is worth to remind that the masses in the different phases (m^2 and μ^2) can be expected to be different, even though very close, giving rise to $\bar{\phi} \neq 0$ according to the GAP equation. This is actually intensively investigated in strong interacting matter at finite density and temperature where the QCD scalar condensate, that varies considerably with energy density and temperature, which varies accordingly for example in [27] and references therein. These equations are numerically investigated in [10].

Analogously to what was done for the renormalized mass in the preceeding section the extremization of the renormalized energy density with respect to the renormalized coupling constant is done in [10]. Moreover, one relevant subject for any approximation method is the understanding of the range of values of the parameters of the model (as mass and mainly coupling constants) for which the approximation is more appropriated. The extremization is found from:

$$\frac{\partial \mathscr{H}_{sub}}{\partial g_R} = 0.$$

Some numerical solutions are presented in [10]. Most of the solutions however correspond to maxima of the energy density because $\frac{\partial^2 \mathscr{H}_{sub}}{\partial g_R^2} < 0$. This analysis provides an assessment of the amount of energy involved in scattering process of two scalars at the threshold (this energy is maximized by particular values of the couplling constant shown in [10]), whose amplitude reduces to the scattering length. These couplings which maximize the energy density are positive and do not give rise to bound states. Conversely the two-scalar scattering is favored in energy ranges which are associated to a different range of values of the coupling constant.

4. The condensate: $\bar{\phi}$

In the framework of the variational approximation the variational equation for the condensate (expression (2.3 (ii))) (a variational parameter from the trial wavefunctional) is obtained from the regularized energy density $\mathscr{H}_{reg}$. Alternatively it will be argued that the minimization of the renormalized energy density can also provide reliable information about the model in the framework of the approach. The minimization equation is done as:

$$\frac{\partial \mathscr{H}_{sub}}{\partial \phi} = 0. \tag{4.1}$$

For this derivation the GAP equation provides the dependence of the mass on the condensate, i.e., $m^2(\bar{\phi})$ and $\mu^2 \equiv m^2(\bar{\phi} = 0)$ is kept constant. It yields the following expressions:

$$\bar{\phi} = 0, \qquad \bar{\phi}^2 = -\frac{m^2}{g_R}\left(1 + \frac{1}{8\pi^2} Ln\left(\frac{m}{\mu}\right)\right). \tag{4.2}$$

This last expression can be imposed to be equal to the expression of $\bar{\phi}_0$ obtained from the minimization of the regularized energy density depending on the relation between λ and mass scale μ as it will be shown below. However it is not completely consistent with the GAP equation (2.6) which is obtained from the minimization of the reguralized energy density with respect to the mass m in the asymmetric phase and then renormalized. To make these expressions compatible it would be necessary to consider the following alternatives for these expressions:

$$\mu^2 \neq m_R^2, \qquad \text{or} \qquad \mu^2 = (g_R - 1)\frac{m^2}{8\pi^2} Ln\left(\frac{m}{\mu}\right), \tag{4.3}$$

where m_R^2 is the one of expression (2.7). It is not clear whether these identifications are reasonable or if they imply a meaningful loss of generality. The limit of $g_R = 1$ does not seem to be reasonable, being a very particular point. Furthermore to be meaningful it requires $\mu^2 = 0$ (and thus $m^2 = 0$) or $m^2 \to \infty$ according to these expressions (4.3). Therefore the two minimization procedures (of the regularized and the renormalized energy densities with respect to the regularized and renormalized parameters respectively) do not seem to yield necessarily the same expressions for the parameters in the ground state. Nevertheless it is worth to remember that renormalization is performed essencially from the regularized GAP equation. These points are discussed further latter.

>From the expression (4.2) the following conditions to obtain non zero real values of $\bar{\phi}$ can be considered:

$$\begin{aligned} if: g_R > 0 \to Ln\left(\frac{m}{\mu}\right) < -8\pi^2, \\ if: g_R < 0 \to Ln\left(\frac{m}{\mu}\right) > -8\pi^2. \end{aligned} \tag{4.4}$$

The energy density is expected be stable for the condensate values found in expression (4.2). This minimum is verified by calculating the second derivative of the energy density with relation to $\bar{\phi}$, i.e.: $\partial^2 \mathscr{H}_{sub} / \partial \bar{\phi}^2 > 0$. Its positiveness corresponds to the condition:

$$g_R \left(1 + \frac{g_R}{32\pi^2}\right) > 0. \tag{4.5}$$

>From this it is seen that for positive coupling constant g_R, it can assume any value (from this stability criterium) whereas if $g_R < 0$ one would have to consider $g_R < -32\pi^2$. Again, this value can be compared to the value obtained in [9] for the threshold of the two particle bound state given by: $g < -8\pi^2$. Expressions (4.4) and (4.5) can correspond to constraints for the values that the renormalized coupling assumes in order to yield stable real ground states.

Expression (4.2) can be written as:

$$g_R \bar{\phi}^2 = -m^2 \left(1 + \frac{1}{8\pi^2} Ln\left(\frac{m}{\mu}\right)\right). \tag{4.6}$$

When $\mu = m\, exp(8\pi^2)$ it follows that either $\bar{\phi} = 0$ or $g_R = 0$ in the asymmetric phase of the potential. This can correspond to the so called symmetry restoration when the condensate disappears at a particularly high excitation energy, i.e., the symmetry is restored. A different solution for the particular limit of $\bar{\phi} = 0$ was found in expression for a_3 shown above where the energy density is minimum with relation to the mass for $m^2 = \mu^2$.

4.1 Further comments

The above expression for the condensate (4.2) can be equated to the previous (regularized) one. Taking into account the expression of the renormalized coupling constant in terms of the bare one (expression (2.7)) this can be written as:

$$\lambda = \frac{16\pi^2}{Ln\left(\frac{\Lambda d}{\mu}\right)} \left(-1 + \frac{3}{2\left(1 + \frac{1}{8\pi^2} Ln\left(\frac{m}{\mu}\right)\right)} \right). \tag{4.7}$$

If the cutoff is sent to infinite the bare coupling constant assumes different values depending on the ratio of μ/m. For example, there is a case in which $\lambda = 0$ if either $\Lambda \to \infty$ for finite μ or:

$$\frac{m}{\mu} = exp(4\pi^2), \tag{4.8}$$

being therefore $m^2 >> \mu^2$. Varying μ together with Λ it can yield solutions with non zero λ. For Λ/μ finite, the coupling λ can even diverge when:

$$\frac{m}{\mu} = exp(-8\pi^2). \tag{4.9}$$

This is the same point found above (for expression (4.6)) for the possible restoration of the symmetry.

It is worth emphasizing that it has been assumed, according to the variational principle, that the minimum of the effective potential with relation to the condensate necessarily defines the ground state together with its minimum in respect to the (physical) mass m^2 in the regularized theory. The different results found in this work from the minimization of the regularized and renormalized theory may indicate that these assumptions are not (complete) correct. Furthermore in the renormalized theory the bare mass and coupling which determine the effective potential at the tree level are eliminated in favor of the renormalized (physical) ones.

5. The GAP equation in the complex plane, explicit solutions

The logarthmic term of the GAP equation introduces non linearities which hinders the extraction of (numerical) explicit solutions. It is proposed in the following part of this work an heuristic trick to extract analytical non transcendental solutions for the GAP equation. It can also relate further the parameters reducing the number of free parameters, making them "constrained" in the ground state of the model. Firstly it is considered that the the mass scale and renormalized mass develop imaginary parts:

$$\mu^2 \to \nu^2 = re^{i\theta}, \qquad m^2 \to \tau^2 = te^{i\omega}, \tag{5.1}$$

where r, s, θ, ω are respectively modulus and phases. Depending on the relative values of these parameters this parametrization corresponds simply to a rotation. With these parametrizations the GAP equation (2.6) can be written as:

$$\begin{aligned} &\left(rcos\theta - tcos\omega - g_R\bar{\phi}^2 - D(t\,cos\omega\, Ln(t/r) - (\omega - \theta)tsin\omega)\right) + \\ &+ i\left[t\ sin\omega - r\ sin\theta + D(t\ cos\omega\ (\omega - \theta) + t\ sin\omega\ Ln(t/r))\right] = 0, \end{aligned} \tag{5.2}$$

where $D = \frac{g_R}{8\pi^2}$. Both the real and the imaginary parts in this expression have basically the same structure of the usual GAP equation. It is worth to emphasize that requiring the GAP equation to be real is equivalent to keep masses as real number parameters.

The parametrization in the complex plane can be just a trick to reduce the transcendental character of the GAP equation (2.6). However an imaginary part for the covariance of the Gaussian (G or correspondingly m^2, μ^2) can introduce time dependence, instability of the system, if considered in the wavefunctional. However in this case other considerations are needed to keep unitarity [28]. *Therefore the phases θ, ω can have different roles, eventually corresponding to (dynamical) corrections to the calculation of the ground state from virtual unstable states.* This will not be discussed further here.

Several cases can be analysed separatedly in the following.

1a) Firstly for $\theta = \omega$ the GAP equation is given by:

$$\begin{aligned} &\left((r-t)cos\omega - g_R\bar{\phi}^2 - D(t\,cos\omega\,Ln(t/r))\right) + \\ &+i\left[(t-r)\;sin\omega + Dt\;sin\omega\;Ln(t/r)\right] = 0. \end{aligned} \tag{5.3}$$

The real part of the GAP equation is the usual one (apart from the $cos\theta$ term which corresponds to a normalization of the mass parameters) and the imaginary part, like a rotation in this complex plane, is the GAP equation in the symmetric phase.

2a) For $r = t$ the GAP equation is rewritten as:

$$\begin{aligned} &r(cos\theta - cos\omega) - g_R\bar{\phi}^2 - D(\theta-\omega)rsin\omega + \\ &+i\left[r\;(sin\omega - sin\theta) + Dr\;cos\omega\;(\omega-\theta)\right] = 0. \end{aligned} \tag{5.4}$$

In this case, these two expressions yield values for ω and θ and they can be solved as function of r and $D \propto g_R$. The resulting (real) values for m^2 and μ^2 still can be different. However the GAP expression has not the logarithmic term anymore. It is decomposed in two trigonometric equations.

Other limits yield interesting features as well.

3a) Requiring the GAP equation (5.2) to have only real component (this is considered to be a stable system) the imaginary part is set to zero. The expression still is quite complicated but the analysis of some particular cases will be very useful. For $\omega = 0$ it follows that:

$$rsin\theta = -Dt\,\theta, \quad or \quad cos\theta = \sqrt{1 - \frac{B^2\theta^2}{r^2}}. \tag{5.5}$$

In this case the self consistent character of the GAP equation remains strong. The real part of expression (5.2) keeps the same form of expression (2.6) basically with the mass parameters m^2, μ^2 replaced by r, t.

4a) For $\theta = 0$ (and $\omega \neq 0$) the resulting expressions for the real part of the GAP equation and its imaginary part (to be equated to zero) can be obtained from expression (5.2). They can be written as:

$$\begin{aligned} &r - t\,cos\omega - g_R\bar{\phi}^2 - D(t\,cos\omega\,Ln(t/r) - \omega t\,sin\omega) = 0, \\ &+i\left[t\,sin\omega + D(\omega t\,cos\omega + tLn(t/r)\;sin\omega)\right] = 0. \end{aligned} \tag{5.6}$$

It does not provide simpler solutions and therefore they are not shown. The resulting number of free parameters is not reduced because although there is one more expression ($\Im m(GAP)$) there also is one extra variable (ω).

Since the phases are auxiliar parameters it is reasonable to assume they are very small without (great) loss of generality for the results. The expression for the imaginary part of the GAP equation in the limit when $sin(\theta) \sim \theta$ and $sin(\omega) \sim \omega$ is given by:

$$\omega t \left(1 + D + DLn\left(\frac{t}{r}\right)\right) = \theta\,(r + Dt)\,. \tag{5.7}$$

This expression can be regarded as fixing the ratio θ/ω.

Several particular cases are analyzed below although the more interesting case is obtained for ω, θ non zero and very small.

(1b) Assuming the phases are equal $\theta = \omega$ expression (5.7) reduces to:

$$r - t = Dt\,Ln\left(\frac{t}{r}\right), \tag{5.8}$$

which fixes the ratio r/t or correspondently m^2/μ^2. This expression is only consistent with the renormalized GAP equation 2.6 for $\bar{\phi} = 0$ (which is obtained from the minimization of the regularized energy density). Besides that it was mentioned above that, since $\omega \neq 0$ and $\theta \neq 0$, it is not clear whether μ^2 and m^2 remain real although the GAP equation is necessarily real. This happens because, in this case, the imaginary part of both parameters can cancel with each other to result a real GAP equation instead of allowing for independent cancelation. On the other hand each angle (ω or θ) can be set to zero separatedly as done below.

(2b) For $\omega = 0$ it follows from expression (5.7):

$$r \simeq -Dt, \tag{5.9}$$

which also fixes the ratio m^2/μ^2 being a real number only for $g_R < 0$.

(3b) For $\theta = 0$, expression (5.7) is re-computed up to the order of $O(\omega^2)$ and it reduces to:

$$\omega^2 = \frac{6 + 6D + 6DLn\left(\frac{t}{r}\right)}{1 + 3D + Ln\left(\frac{t}{r}\right),}, \tag{5.10}$$

where it has been assumed that $sin\omega \sim \omega$. In this case it is reasonable to consider $\omega^2 \sim 0$ leading to the expression:

$$\frac{t}{r} = exp\left(\frac{1+D}{D}\right). \tag{5.11}$$

For $g_R = -8\pi^2$ it follows that $t = r$, and therefore $m^2 \simeq \mu^2$.

If $\omega \neq 0$ it will appear in the real part of the GAP equation and therefore the number of free parameters in the renormalized equation does not diminish with the new parametrization. Therefore $\omega = 0$ would be the only possibly interesting case. This does not happens because of expression (5.9) which imposes negative coupling $g_R < 0$.

The real part of the GAP equation for small angles keeps nearly the form of the original GAP, it can be written as:

$$t - r + g_R\bar{\phi}^2 + D\left[tLn\left(\frac{t}{r}\right) - t\,\omega\,(\omega - \theta)\right] = 0, \tag{5.12}$$

where either r or t can be written as a function of the other by means of the constraints of the imaginary parts from the expressions (5.8), (5.9) or (5.10). In this third case the auxiliar parameter ω was not eliminated (although $\theta = 0$). However for very small phases the expression (5.12) reduces to the usual real GAP equation (2.6). In this case the real part of the GAP equation is the same as expression (2.6) written as:

$$t - r + g_R\bar{\phi}^2 + Dt\,Ln\left(\frac{t}{r}\right) = 0. \tag{5.13}$$

5.1 The energy density

Simultaneously the renormalized energy (density) can be required to be a real number. However it is easy to notice from expression (5.7) that the resulting expression for the imaginary part of $\mathscr{H}_{sub}$ will be quite complicated. Below it will be assumed that the phases have small values. Although this might impose limitations in the results depending on what kind of system one deals with, it will be just considered that they are "auxiliar" parameters eliminated in the end. With this assumption several simplifications occurs because: $sin(\theta) \sim \theta$ and $sin(\omega) \sim \omega$. The result for the imaginary part of the energy density, up to first order in the phases, will be given by:

$$\Im m(\mathscr{H}_{sub}) = \omega \left(\frac{t\bar{\phi}^2}{2} + 2t^2 A_- + \frac{tr}{2g_R} + \frac{r^2}{64\pi^2} \right) + \theta \left(2A_+ r^2 - \frac{rt}{2g_R} + \frac{r^2}{32\pi^2} Ln\left(\frac{t}{r}\right) - r^2 \right) \to 0, \tag{5.14}$$

Where

$$A_{\pm} = \frac{1}{4g_R} \pm \frac{1}{128\pi^2}.$$

One of these variables (A_+) can be identified with a solution for g_R if ones consider a fixed value of $\mathscr{H}_{sub}(\omega = \theta = 0)$ for $\mu = m$ given by expression:

$$A_+ = - \left. \frac{\mathscr{H}_{sub}}{m^4} \right|_{\mu=m}. \tag{5.15}$$

Expression (5.14) still is very complicated and it can also be used to fix the ratio θ/ω which can be equated to the same ratio obtained from expression (5.7). However this has been written for $\mathscr{H}_{sub}$ in the form given by expression (2.8), which can be written differently by means of the GAP equation for $m^2 = m^2(\mu^2)$. This allows to re-arrange an equation of r as a function of t and to eliminate one of these variables. The resulting identity reads:

$$\frac{\omega}{\theta} = -\frac{2A_+ r^2 - \frac{rt}{2g_R} + \frac{r^2}{32\pi^2} Ln\left(\frac{t}{r}\right) - r^2}{\frac{t\bar{\phi}^2}{2} + 2t^2 A_- + \frac{tr}{2g_R} + \frac{r^2}{64\pi^2}} = -\frac{\frac{r}{t} + D}{\left(1 + D + DLn\left(\frac{t}{r}\right)\right)}. \tag{5.16}$$

In this expression the same parameter is used: $D = g_R/(8\pi^2)$. This (highly transcendental) expression appears in addition to the usual real part of the GAP equation, expression (5.13), making a system of two algebraic expressions with two variables (r,t). g_R is a remaining input/free parameter although the previous sections might provide elements to the range in which it acquires acceptable values.

6. Discussion, Summary

A further analysis of the usual variational Gaussian approximation was done to propose some ways to explore the behavior of the model (the $\lambda\phi^4$ was chosen) with respect to its parameters. The renormalized energy density was extremized with respect to the renormalized mass and coupling and to the condensate. Concerning the extremization with respect to the mass, five solutions were found, two of which which can correspond to stable vacua in specific ranges of the renormalized coupling constant. For this it was considered that the mass scale μ is close to the physical mass. A sort of "energy scale" invariant algebraic expression was found in this calculation. In other words, a variation in the renormalized (physical) mass m^2 with a corresponding variation in the renormalization mass scale parameter μ^2 yield the same solutions. The minimization (maximization) of the energy density was also discussed.

Furthermore the minimization of the renormalized energy density with relation to $\bar{\phi}$, in the vacuum, was also discussed. The resulting expression is not completely consistent with the renormalized GAP equation unless the expression (2.7) is modified such that $\mu^2 \neq m_R^2 \to 0$. >From this expression it was pointed out that either the "condensate" or g_R disappears when the mass scale (introduced in the renormalization procedure) assumes the value

$$\bar{\phi}\left(\mu = m\, exp(8\pi^2)\right) = 0.$$

This can be seen as a restoration of the spontaneous symmetry breaking. With this value for μ, the bare coupling λ may also diverge for Λ/μ finite, as shown in expression (4.7).

Finally the mass parameters were placed in the complex plane. With this non-transcendental solutions for the GAP equation were found besides other relations among the parameters reducing the number of free parameters. The imaginary part of the masses can be required to be zero at the end of the calculation producing another expression which relates the mass, coupling and the renormalization scale parameter. This parametrization for the imaginary part may lead to new relation between the parameters reducing the number of free variables. The imaginary parameters may be required to be very small ($sin(\omega) \sim \omega$ or $sin(\theta) \sim \theta$). The same parametrization is applied to the energy density which also must be a real number. The number of free parameters (m^2 or μ^2, and g_R) is reduced and non transcendental solutions may result such as that of expression (5.9). However the imaginary parts can also acquire physical meaning in the case of unstable or time dependent situations although the results would need more input to respect unitarity and other fundamental properties.

It is worth to remind that the ground state in the framework of the variational approximation is found by the minimization with respect to the two point function $G(\mathbf{x},\mathbf{y},m^2)$ (which is a function of the physical mass m^2 or $\mu^2 = m^2(\bar{\phi} = 0)$) and to the condensate $\bar{\phi}$ - they are the variational parameters (given in expressions (2.3)). Although they are regarded initially as independent variables, the GAP equation (for ground state) relates them and eventually the mass scale (μ^2) might be eliminated. While the GAP equation is used for the renormalization of the bare parameters in the vacuum, expression (4.2) was calculated from renormalized expressions for the ground state. However there is nothing really defined about the behavior of the renormalized parameters in excited states. It was shown with sections 2.1, 3 and 4 that the minimizations of the renormalized energy with relation to the mass and $\bar{\phi}$ yield different ground state (GAP) expressions from the ones obtained by the usual variational procedure for the regularized theory. This may have several meanings. It might not be evident whether these variational parameters are really or completely suitable as independent parameters for the Gaussian approximation and extensions (or leading order large N, Hartree Bogoliubov) or even in the *exact ground state*, i.e., the energy must be minimum with respect to particular combination(s) of these (or other) (physical?) variables. Notwithstanding the minimization of the regularized energy may not be equivalent to the minimization of the renormalized one because in the regularized theory there still are other (bare) parameters which are eliminated in the renormalization procedure corresponding to a sort of "hidden dependences" among them. It may also be that the renormalization procedure has to be improved such as to make both ways of obtaining the ground state expressions equivalent. In this case the renormalization procedure would be allowed to be done at any moment independently of the order of the the variation, renormalization and extraction of observables within a certain (non)perturbative approach.

Some physical situations in which these ranges of the space of parameters of the model can be of relevance were proposed. Other aspects were raised including the eventual equivalence to the Bogoliubov transformation for describing superfluid systems and the possibility of modification in the physical mass of the particle due to the presence of the condensate $m^2(\bar{\phi}) \neq \mu(\bar{\phi} = 0)$. In particular issues of relevance for systems investigated in the Many body problem, finite density formalisms and effective field theories were raised and discussed such as the role of "constraining" the parameters to define the ground state of the corresponding system.

At last, it is interesting to look at the procedures adopted here as a complementary investigation to the renormalization group method in which the behavior of the renormalized/bare parameters with the scale parameter can be established. Although they provide different insights into the issue of the behavior of the theory and its structure and observables at different energy scales, most part of the methods presented here is not directly (nor necessarily) concerned about the variation of the renormalization energy scale. It is rather (at first order) concerned about the total amount energy (or energy density) involved in a physical process with the corresponding coupling constants, condensate and masses. A full analysis "coupling" the renormalization group equations to those found above will be done elsewhere.

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