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Boson-Boson Bound States in Higher-Derivative Electromagnetism Augmented by a Chern-Simons Term

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The Chern-Simons term alone is unable to form “scalar Cooper pairs". Nonetheless, there exist charged-scalar-boson — charged-scalar-boson bound states in the framework of Maxwell-Chern-Simons theory. Numerical calculations indicate that there are also Cooper pairs within the context of three-dimensional electromagnetism with a cutoff (three-dimensional electromagnetism with higher derivatives) supplemented by a Chern-Simons term. We show that it is always possible to find an interval where the models with higher derivatives have a total number of bound states greater than those with lower derivatives.

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1. Introduction

It is well-known that “scalar Cooper pairs" do not occur in the framework of pure Chern-Simons theory [1]. Nonetheless, charged-scalar-boson — charged-scalar-boson bound states do exist in the framework of Mawell-Chern-Simons theory [2]. Interesting enough, numerical calculations show that there are also scalar Cooper pairs within the context of three-dimensional electromagnetism with a cutoff [3, 4] (three-dimensional electromagnetism with higher derivatives) enlarged by a Chern-Simons term. The electromagnetic part of this model is defined by the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda},$$

where $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic tensor field, and a is a cutoff. This Lagrangian is, of course, gauge and Lorentz invariant; in addition it leads to local field equations which are linear in the field quantities. Moreover, at distances much larger than the cutoff, the fields described by it become essentially equivalent to the Maxwell fields. The classical and quantum formalism for the constrained Hamiltonian related to the singular higher-order Lagrangian in (2 +1) dimensions mentioned above were constructed by Greco *et al.* [5], and afterward the canonical and the path-integral quantization were performed [5, 6]. The latter was accomplished by extending the Faddeev-Senjanovic method [7]. The massive spin-1 part of the electromagnetic field unluckily has negative energy, which implies that three-dimensional electromagnetism with a cutoff is nonunitary due to the presence of a ghost state with negative norm. On the other hand, the breakdown of causality may perhaps only occur on a microscopic scale if the parameter a is small enough to make the massive field only important on distance scales near the Planck length $\sim 10^{-33}$ cm. It is therefore not unlikely that a higher-derivative model can represent an *effective* theory of electromagnetism at more familiar lengths.

It is worth noticing that recently it was shown that in the framework of four-dimensional electromagnetism with a cutoff, the electromagnetic mass of a point charge occurs in the equation of motion in a form consistent with special relativity; furthermore, the exact equation of motion does not exhibit runaway solutions or non-causal behavior, when the cutoff is larger than half of the classical radius of the electron [8, 9].

Our main objective here is to analyze whether or not higher-derivative terms may be used as a mechanism for increasing the number of charged-scalar-boson — charged-scalar-boson bound states (scalar Cooper pairs) related to the usual three-dimensional Maxwell-Chern-Simons theory.

The paper is organized as follows. In section 2, we discuss the occurrence of bound states is both Maxwell-Chern-Simons and higher-derivative models. A rough estimate of the number of bound states in the context of the preceding models is made in section 3. Finally, the concluding remarks are presented in section 4.

2. Scalar Cooper Pairs

To begin with, we derive an expression that allows us to calculate the three-dimensional nonrelativistic potential for the interaction of two identical charged scalar bosons via a photon exchange.

This expression is used afterward to find the bound states for both Maxwell-Chern-Simons and higher-derivative models.

2.1 Charged-Scalar-Boson — Charged-Scalar-Boson Low energy Potential or the Marriage of Nonrelativistic Quantum Mechanics and Quantum Field Theory in the Nonrelativistic Limit

Nonrelativistic quantum mechanics tells us that in the first Born approximation the cross section for the scattering of two indistinguishable massive particles, in the center-of-mass frame (CoM), is given by

$$\frac{d\sigma}{d\Omega} = \left| \frac{m}{4\pi} \int e^{-i\mathbf{p}'\cdot\,\mathbf{r}} V(r) e^{i\mathbf{p}\cdot\,\mathbf{r}} d^2\,\mathbf{r} \right|^2,$$

where $\mathbf{p}$ $(\mathbf{p}')$ is the initial (final) momentum of one of the particles in the CoM.

In terms of the transfer momentum, $\mathbf{k} \equiv \mathbf{p}' - \mathbf{p}$, it reads

$$\frac{d\sigma}{d\Omega} = \left| \frac{m}{4\pi} \int V(r) e^{i\mathbf{k}\cdot\,\mathbf{r}} d^2\,\mathbf{r} \right|^2. \tag{2.1}$$

On the other hand, from quantum field theory we know that the cross section, in the CoM, for the scattering of two identical charged scalars bosons by an electromagnetic field, can be written as

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{16\pi E}\,\mathscr{M} \right|^2,$$

where E is the initial energy of one of the bosons and $\mathscr{M}$ is the Feynman amplitude for the process at hand, which in the nonrelativistic limit $(\mathrm{N}.R.)$ reduces to

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{16\pi m}\,\mathscr{M}_{\mathrm{N}.R.} \right|^2. \tag{2.2}$$

From Eqs. (2.1) and (2.2) we come to the conclusion that the expression that enables us to compute the three-dimensional effective nonrelativistic potential has the form

$$V(r) = \frac{1}{4m^2}\frac{1}{(2\pi)^2}\int d^2\,\mathbf{k}\,\mathscr{M}_{\mathrm{N}.R.}\,e^{-i\mathbf{k}\cdot\,\mathbf{r}}, \tag{2.3}$$

which clearly shows how the potential from quantum mechanics and the Feynman amplitude obtained via quantum field theory are related to each other.

2.2 Bound States for Maxwell-Chern-Simons Model

In the Lorentz gauge the Maxwell-Chern-Simons electromagnetism coupled to a charged-scalar field is described by the Lagrangian

$$\begin{aligned}\mathscr{L} \;=\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^{\mu}\partial^{\nu}A^{\rho} - \frac{1}{2\lambda}(\partial_{\nu}A^{\nu})^2 \\ & + (D_{\mu}\phi)^{*}D^{\mu}\phi - m^2\phi^{*}\phi,\end{aligned} \tag{2.4}$$

where $D_\mu \equiv \partial_\mu + iqA_\mu$.

It follows that the nonrelativistic potential is given by

$$V(r) = -\frac{Q^2}{m\pi s}\left[\frac{1}{r^2} - \frac{sK_1(sr)}{r}\right]\mathbf{L} + \frac{Q^2}{2\pi s}K_0(sr), \tag{2.5}$$

where $\mathbf{L} \equiv \mathbf{r}\wedge\mathbf{P}$ is the orbital angular momentum, and K is the modified Bessel function.

Let us then investigate whether or not this potential can bind a pair of identical charged-scalar bosons. In this case, the corresponding time-independent Schrödinger equation can be written as

$$\begin{aligned} \mathscr{H}_l\mathscr{R}_{nl} &= -\frac{1}{m}\left(\frac{d^2}{dr^2}\mathscr{R}_{nl} + \frac{1}{r}\frac{d}{dr}\mathscr{R}_{nl}\right) + V_l^{\text{eff}}\mathscr{R}_{nl} \\ &= E_{nl}\mathscr{R}_{nl}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} V_l^{\text{eff}} &\equiv \frac{l^2}{mr^2} + V(r) \\ &= \frac{l^2}{mr^2} - \frac{Q^2}{m\pi s}\left[\frac{1}{r^2} - \frac{sK_1(sr)}{r}\right]\mathbf{L} + \frac{Q^2}{2\pi s}K_0(sr), \end{aligned}$$

where $\mathscr{R}_{nl}$ is the nth normalizable eigenfunction of the radial Hamiltonian $\mathscr{H}_l$ whose corresponding eigenvalue is E_{nl} and V_l^{eff} is the lth partial wave effective potential. Note that V_l^{eff} behaves as $\frac{l^2}{mr^2}$ at the origin and as $\frac{l}{m}\left[l - \frac{Q^2 s}{\pi s}\right]\frac{1}{r}$ asymptotically.

On the other hand,

$$\begin{aligned} \frac{d}{dr}V_l^{\text{eff}} &= -\frac{2l}{m}\left[l - \frac{Q^2}{\pi s}\right]\frac{1}{r^3} - \frac{Q^2 sl}{m\pi}\frac{1}{r}K_0(sr) \\ &\quad -\left[\frac{Q^2 2l}{m\pi r^2} + \frac{Q^2 s}{2\pi}\right]K_1(sr). \end{aligned}$$

Assuming, without any loss of generality, that $l > 0$, it is trivial to see that, if $l > \frac{Q^2}{\pi s}$, the potential is strictly decreasing, which precludes the existence of bound states. The remaining possibility is $l < \frac{Q^2}{\pi s}$. In this interval V_l^{eff} approaches $+\infty$ at the origin and 0^- for $r \to +\infty$, which is indicative of a local minimum. Consequently, the existence of charged-scalar-boson — charged-scalar-boson bound states is subordinated to the condition $0 < l < \frac{Q^2}{\pi s}$.

In terms of the dimensionless parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $\beta \equiv \frac{m}{s}$, and $\tilde{E}_{nl} \equiv \frac{mE_{nl}}{s^2}$, Eq. (2.6) reads

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right]\mathscr{R}_{nl} + \left[\tilde{E}_{nl} - \tilde{V}_l^{\text{eff}}\right]\mathscr{R}_{nl} = 0, \tag{2.7}$$

with

$$\tilde{V}_l^{\mathrm{eff}} \equiv -\frac{l(\alpha-l)}{y^2} + \frac{\alpha\beta}{2}K_0(y) - \frac{\alpha l}{y}K_1(y).$$

Of course, Eq. (2.7) cannot be solved analytically; nevertheless, it can be solved numerically. To accomplish this, we rewrite the radial function as $\mathscr{R}_{nl} \equiv \frac{u_{nl}}{\sqrt{y}}$. As a consequence, Eq. (2.7) takes the form

$$\left[\frac{d^2}{dy^2} + \frac{1}{4y^2}\right]u_{nl} + \left[\tilde{E}_{nl} - \tilde{V}_l^{\mathrm{eff}}\right]u_{nl}. \tag{2.8}$$

Using the Numerov algorithm [10], we have solved Eq. (2.8) numerically for several values of the parameters α, β, and l. In Fig. 1 we present our numerical results for the potential in the specific case of $l = 6$. The corresponding ground-state energy is -1.68×10^{-8} MeV. The graphic shown in Fig. 1 exhibits the generic features of the potential, although it has been composed using particular values of the parameters α, β, and l.

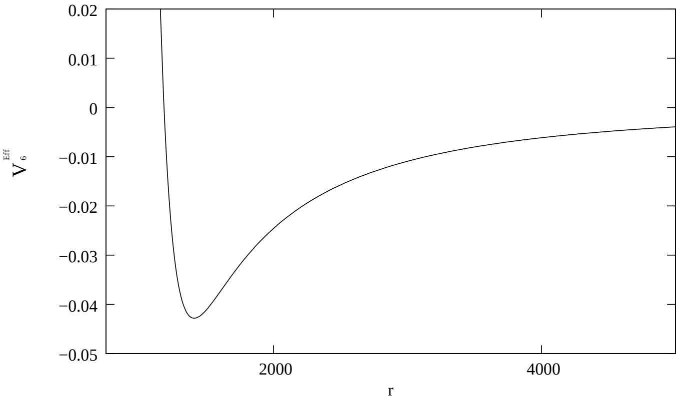

Figure 1: Attractive effective nonrelativistic potential corresponding to the eigenvalue $l = 6$. Here $[V_6^{eff}] = eV$, $[r] = MeV^{-1}$, $\alpha = 7.6$, and $\beta = 7000$.

2.3 Bound States for the Model with Higher Derivatives

2.3.1 The Lagrangian

$$\begin{aligned}\mathscr{L} \;=\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\alpha F_{\mu\alpha} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 \\ & + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho,\end{aligned} \tag{2.9}$$

where $s > 0$ is the topological mass.

2.3.2 The Nonrelativistic Potential

$$V(r) = -\frac{sQ}{\pi m a^4}\left[\frac{a^4}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j\sqrt{|x_j|}\,K_1(\sqrt{|x_j|}\,r)\right] l + \frac{Q}{2\pi a^4}\left[\sum_j A_j K_0(\sqrt{|x_j|}\,r)\right], \tag{2.10}$$

where

$$A_1 \equiv \frac{1+a^2x_1}{(x_1-x_2)(x_1-x_3)}, A_2 \equiv \frac{1+a^2x_2}{(x_2-x_1)(x_2-x_3)}, A_3 \equiv \frac{1+a^2x_3}{(x_3-x_1)(x_3-x_2)},$$

$$B_1 \equiv \frac{-(1+a^2x_1)^2}{s^2(x_1-x_2)(x_1-x_3)}, B_2 \equiv \frac{-(1+a^2x_2)^2}{s^2(x_2-x_1)(x_2-x_3)}, B_3 \equiv \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)},$$

and x_1, x_2, and x_3 are the roots of the equation

$$x^3 + \frac{2x^2}{a^2} + \frac{x}{a^4} + \frac{s^2}{a^4} = 0. \tag{2.11}$$

We are supposing $a < \frac{2\sqrt{3}}{9s}$, which implies that Eq. (2.11) has three distinct negative real roots.

2.3.3 Bound States

Employing the dimension parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $\beta \equiv \frac{m}{s}$, $X_j \equiv \frac{|x_j|}{s}$, $b_j \equiv \frac{s^2}{a^4}B_j$, and $a_j \equiv \frac{A_j}{a^4}$, the effective nonrelativistic potential assumes the form

$$\tilde{V}_l^{\text{eff}} \equiv -\frac{l(\alpha-l)}{y^2} + \frac{\alpha\beta}{2}\sum_j a_j K_0(X_j y) - \frac{\alpha l}{y}\sum_j b_j X_j^2 K_1(X_j y). \tag{2.12}$$

However, only if $a \ll 1$ will the well-recognized properties of QED_3 be preserved. In this case it can be shown that the existence of scalar Cooper pairs is subordinated to the condition $0 < l < \alpha$, where we have assumed $l > 0$, without any loss of generality.

In Fig. 2 we present our numerical results for the potential and the corresponding radial eigenfunctions concerning the first three bound states in the specific case of $l = 4$. The associated energies are $E_{14} = -6.4 \times 10^{-7}$ MeV, $E_{24} = -1.3 \times 10^{-7}$ MeV, $E_{34} = -5.2 \times 10^{-9}$ MeV. These graphics exhibit, in a sense, the generic features of the potential, although they have been composed using particular values of the parameters α, β, l, and a.

3. A Rough Estimate of the Number of Bound States

We derive here approximate expressions for the maximal number of bound states related to both higher-derivative and Maxwell-Chern-Simons models.

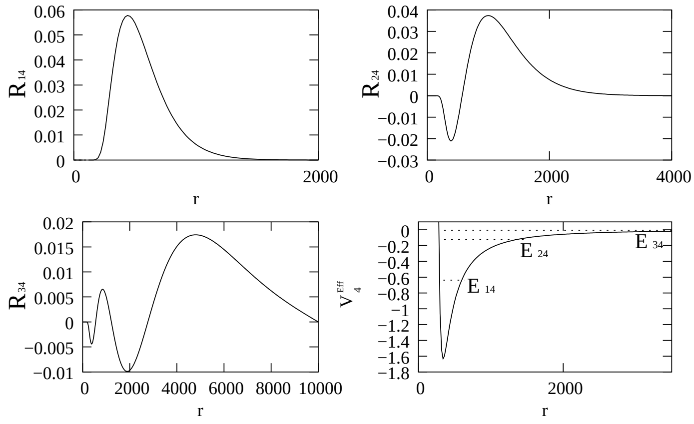

Figure 2: V_4^{eff} with the lowest three allowed energies and the corresponding energy eigenfunctions. Here $\left[V_4^{\text{eff}}\right] = eV$, $[r] = MeV^{-1}$, $\alpha = 8$, $\beta = 2000$, and $a = 0.00952\, MeV^{-1}$.

3.1 Bargmann's Bound in Two Dimensions

It was shown by Bargmann [11, 12] that for a central potential the number of bound states in two dimensions is given by

$$N_0(V) = \sum_0^{\infty} 2n_l(V), \tag{3.1}$$

where

$$n_l(V) \leq \frac{1}{2l} \int_0^{\infty} V_-(r) r dr.$$

Here $V_-(r) = sup(V(r), 0)$. However, as far as our potentials are concerned, the analytical computation of $V_-(r)$ is rather involved. But, since $V_-(r) = sup(V(r), 0) \geq -|V(r)|$, we get that $N_0(-|V|) \geq N_0(V)$. Since we only want to make a rough estimate of the total number of bound states we replace the sum in Eq. (3.1) by an integral. Noting that for $l = 0$ the bound is divergent reflecting the fact that a negative potential always has a bound state in two dimensions, we arrive at the expression

$$N(l_{\max}) \leq \left| \int_{l=1}^{l_{max}} \int_0^{\infty} \frac{V(r) r}{l}\, dr dl \right|, \tag{3.2}$$

where $l_{\max}$ is the maximal angular momentum. This inequality, unlike Bargmann's one, is especially suitable for our purposes.

3.2 Finding $l_{\max}$

3.2.1 The Model with Higher Derivatives

In order to find $l_{\max}$ for this model, we have to solve the inequality

$$
\begin{aligned}
& l_{\max}\left(\frac{l_{\max}}{m}-\frac{Q}{\pi m s}\right)\ln(r)|_{r_0}^{\infty} \\
& +\frac{sQl_{\max}}{\pi m a^4}\sum_j B_j K_0(\sqrt{|x_j|}\,r)|_{r_0}^{\infty} \\
& -\frac{Q}{2\pi a^4}\sum_j \frac{A_j}{\sqrt{|x_j|}} r K_1(\sqrt{|x_j|}\,r)|_{r_0}^{\infty} < 0.
\end{aligned} \tag{3.3}
$$

The radial variable was limited to the interval $r_0 < r < \infty$ to avoid the usual infrared divergences.

From (3.3) we obtain the constraints

$$
l_{\max} = \frac{Q}{\pi s}, \tag{3.4}
$$

$$
\frac{Q}{\pi m}\sum_j B_j ln(\sqrt{|x_j|}) < -\frac{1}{2}\sum_j \frac{A_j}{|x_j|}. \tag{3.5}
$$

3.2.2 The Maxwell-Chern-Simons Model

In this case the constraint on $l_{\max}$ is the same as in **3.2.1**. We have also a constraint on m; however, since we want to compare **3.2.1** with **3.2.2**, we shall use here the constraint on the mass found in **3.2.1**.

3.3 An Estimate of the Number of Bound States

3.3.1 The Model with Higher Derivatives

$$
N(l_{\max}) \leq F\ ,
$$

where

$$
\begin{aligned}
F\ \approx & \left|\left(1-\frac{Q}{\pi s}\right)\left[\frac{Q}{\pi s m} ln\left(\frac{r_{max}}{r_0}\right) - \frac{sQ}{\pi m a^4}\sum_j B_j ln(\sqrt{|x_j|})\right]\right. \\
& \left. +\ \frac{Q}{2\pi a^4} ln\left(\frac{Q}{2\pi s}\right)\sum_j \frac{A_j}{|x_j|}\right|.
\end{aligned}
$$

3.3.2 The Maxwell-Chern-Simons Model

$$
\bar{N}(l_{\max}) \leq G\ ,
$$

where,

$$
G \approx \left|\tfrac{Q}{2\pi s^2} ln\left(\tfrac{Q}{\pi s}\right) - \tfrac{Q}{\pi m s}\left(\tfrac{Q}{\pi s}-1\right) ln(s r_{max})\right|.
$$

Note that in both **3.3.1** and **3.3.2** we have assumed that $r \leq r_{\max}$ in order to avoid that $\ln r$ blows up at infinity.

4. Concluding Remarks

If we choose, for instance, $\frac{Q}{s} = 38MeV^{-1}$, we promptly obtain $l_{\max} \approx 12$. On the other hand, taking $a = 0.02\ MeV^{-1}$, we see that $m \approx 0.18\ MeV$, satisfies the constraint (3.5). For r (in MeV) varying in the range $1\times 10^{-23} < r < 8\times 10^{-6}$, we get $N_{\max} \approx 4\bar{N}_{\max}$.

At first sight, it seems that the models with higher-derivatives will have a total number of bound states greater than that of the Chern-Simons model. However, this is a misleading conclusion. Indeed, if we fix $r_{\max}$, say, equal to $5\times 10^{-3} MeV$ (10 *Ansgtrom*), and vary r_0 keeping the values of $\frac{Q}{s}, l_{\max}$, and m, equal to those of the example above, as it is shown in Fig. 3, we see that if $0 < r_0 < 0.1$, higher derivatives win the game; now, if $r_0 = 0.1$, the game ends in a tie, and , finally, if $r_0 > 0.1$, higher derivatives lose the Cup.

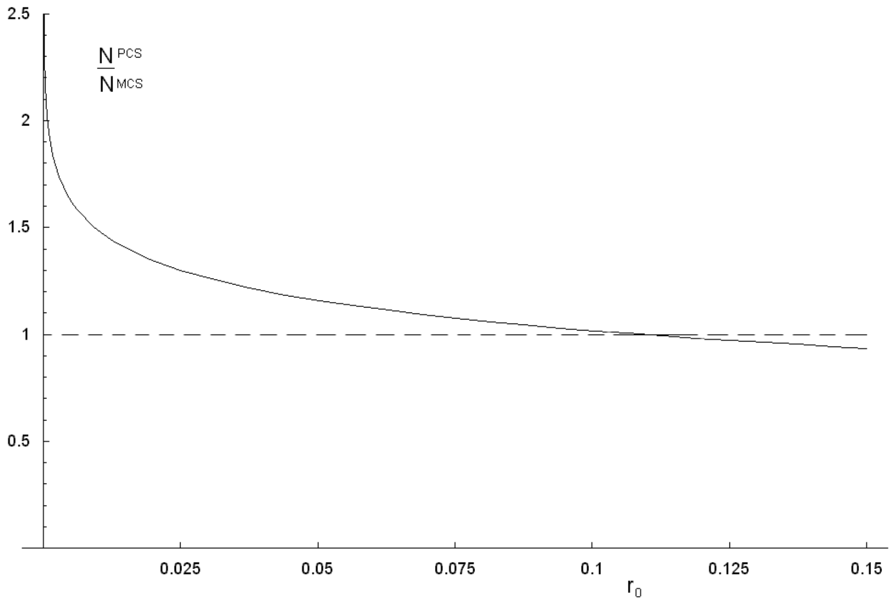

Figure 3: $N_{\max}/\bar{N}_{\max}$ versus r_0. Here $[r_0] = Ansgtrom$, and $r_{\max} = 10\ Ansgtrom$.

In conclusion, we may say that our rough calculations seem to indicate that it is possible to find an interval $I \subseteq [r_0, r_{\max}]$ where higher derivatives win the Cup.

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