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NOTE ON QUANTUM CORRECTION TO BTZ INSTANTON ENTROPY

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We find a zeta function expression of the effective action and the Mann-Solodukhin quantum correction to the entropy of the BTZ instanton with a conical singularity. We also compute the instanton topology.

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1. Introduction

In the paper [4], R. Mann and S. Solodukhin calculate the entropy, with a one-loop ultraviolet correction term, of the BTZ instanton with a conical singularity by a suitable differentiation of the effective action. In [7], we raised the question of whether one could capture this quantum correction by way of a suitable zeta function deformation, and we announced an affirmative result there. This note presents an elaboration on that result, with a simpler deformation of zeta, and an expression of the effective action in terms of zeta. The topology of the instanton is also computed-which one can view as a deformation of the topology of the regular BTZ black hole.

2. Topology of the instanton

Let H^3 denote hyperbolic 3-space consisting of points (x,y,z) in $\mathbb{R}^3$ with $z>0$. Using spherical-type coordinates (ψ,χ,θ) we write $x=e^{\psi}\sin\chi\cos\theta$, $y=e^{\psi}\sin\chi\sin\theta$, $z=e^{\psi}\cos\chi$, $0<\chi<\pi/2$. We fix $\alpha\in\mathbb{R}$ satisfying $0<\alpha\leq 1$, and for a whole number $n\in\mathbb{Z}$ we define $(x_{\alpha,n},y_{\alpha,n},z_{\alpha,n})\in H^3$ by

$$\begin{bmatrix} x_{\alpha,n} \\ y_{\alpha,n} \\ z_{\alpha,n} \end{bmatrix} = \begin{bmatrix} \cos 2\pi n\alpha & -\sin 2\pi n\alpha & 0 \\ \sin 2\pi n\alpha & \cos 2\pi n\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2.1}$$

$M>0$, $J\geq 0$, and $\Lambda<0$ will denote the *Euclidean* BTZ black hole mass, angular momentum, and cosmological constant, respectively, so that the outer and inner horizons $r_+>0$, $r_-\in i\mathbb{R}$ are given by

$$r_+^2=\frac{M\sigma^2}{2}\left[1+\left(1+\frac{J^2}{M^2\sigma^2}\right)^{\frac{1}{2}}\right], r_-=-\frac{\sigma J i}{2r_+} \tag{2.2}$$

for $\sigma:=+1/\sqrt{-\Lambda}$. We define

$$a=\frac{\pi r_+}{\sigma}>0, b=\frac{\pi|r_-|}{\sigma}=\frac{\pi J}{2r_+}\geq 0. \tag{2.3}$$

The Euclidean BTZ instanton $B(\alpha)$ with conical singularity and defect angle $2\pi(1-\alpha)$ is obtained from H^3 via the identification $(x,y,z)\sim(x_{\alpha,n},y_{\alpha,n},z_{\alpha,n})$, and the "Schwarzschild identification" $(x,y,z)\sim(e^{2an}(x\cos(2bn)-y\sin(2bn)), e^{2an}(x\sin(2bn)+y\cos(2bn)), e^{2an}z)$, for $x_{\alpha,n},y_{\alpha,n},z_{\alpha,n}$ in (2.1), a,b in (2.3), for any $n\in\mathbb{Z}$.

Consider the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by (2.1); that is

$$n\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} x_{\alpha,n} \\ y_{\alpha,n} \end{bmatrix}. \tag{2.4}$$

The corresponding quotient space will be denoted by $\left(\mathbb{Z}\backslash\mathbb{R}^2\right)(\alpha)$. Then for the circle group S^1, the following theorem can be proved, which computes the topology of the Euclidean instanton $B(\alpha)$ with conical singularity; $0<\alpha\leq 1$.

Theorem 1. *$B(\alpha)$ is homeomorphic to the product topological space $\left(\mathbb{Z}\backslash\mathbb{R}^2\right)(\alpha)\times S^1$.*

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For $\alpha = 1$, $B(\alpha)$ reduces to the regular BTZ black hole $B(1)$ [1] with no singularity, and Theorem 1 reduces to the familiar statement that the topology of $B(1)$ is $\mathbb{R}^2 \times S^1$. For $\alpha \neq 1$, the topology of $\left(\mathbb{Z}\backslash\mathbb{R}^2\right)(\alpha)$, and hence of $B(\alpha)$, can be quite notorious. For example, if α is irrational then one has the existence of dense orbits.

3. The Mann-Solodukhin quantum correction, and effective action

Quantum corrections to black hole entropy have been considered by many authors. In [2–4], for example, various sums appear which we can provide a zeta function meaning for. We shall focus on the sum $\sum_{n=1}^{\infty} s_n$, for s_n given in equation (5.3) of [4], which gives a quantum correction to BTZ entropy.

In the paper [7] we announced the construction of a family of zeta functions $\left\{Z^{(\alpha)}(s;a,b)\right\}_{0<\alpha\leq 1}$ such that for $\alpha = 1$

$$Z^{(1)}(s;a,b) = \prod_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1,k_2\geq 0}}^{\infty} \left[1-\left(e^{2bi}\right)^{k_1}\left(e^{-2bi}\right)^{k_2} e^{-(k_1+k_2+s)2a}\right] \tag{3.1}$$

is the zeta function presented in [6]; also see [5]. a and b are the numbers given in (2.3). Namely, $Z^{(\alpha)}(s;a,b)$ is given by

$$Z^{(\alpha)}(s;a,b) = \prod_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1,k_2\geq 0}}^{\infty} \left[1-\left(e^{\frac{2bi}{\alpha}}\right)^{k_1}\left(e^{-\frac{2bi}{\alpha}}\right)^{k_2} e^{-(k_1+k_2+\alpha s)\frac{2a}{\alpha}}\right] \cdot \prod_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k_1,k_2,k_3\geq 0}}^{\infty} \frac{\left[1-e^{-2(k_3+1)2a}\left(e^{\frac{2bi}{\alpha}}\right)^{k_1}\left(e^{-\frac{2bi}{\alpha}}\right)^{k_2} e^{-(k_1+k_2+\alpha s)\frac{2a}{\alpha}}\right]}{\left[1-e^{-2(k_3+\frac{1}{\alpha})2a}\left(e^{\frac{2bi}{\alpha}}\right)^{k_1}\left(e^{-\frac{2bi}{\alpha}}\right)^{k_2} e^{-(k_1+k_2+\alpha s)\frac{2a}{\alpha}}\right]}. \tag{3.2}$$

A simpler expression of this zeta function has been obtained recently. Namely, we can show that

$$\begin{aligned} Z^{(\alpha)}(s;a,b) &= \prod_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1,k_2\geq 0}}^{\infty} \left[1-e^{(ib-a)\frac{2k_1}{\alpha}} e^{-4ak_2-2sa}\right] \cdot \\ &\quad \prod_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1,k_2\geq 0}}^{\infty} \left[1-e^{-(ib+a)\frac{2(k_1+1)}{\alpha}} e^{-4ak_2-2sa}\right]. \end{aligned} \tag{3.3}$$

Formula (3.3) shows that $Z^{(\alpha)}(s;a,b)$ is in fact an *entire* function of s, but it is *not* clear from (3.3) that $Z^{(1)}(s;a,b)$ is given by the right hand side of equation (3.1). That is, one needs the more complicated expression (3.2) to see that the family $\left\{Z^{(\alpha)}(s;a,b)\right\}_{0<\alpha\leq 1}$ is a deformation of the zeta function in [6]. One computes that for $Re\, s > 0$,

$$\begin{aligned} \log Z^{(\alpha)}(s;a,b) &= -\sum_{n=1}^{\infty} \frac{\sinh\left(\frac{2an}{\alpha}\right) e^{-(s-1)2an}}{4n\sinh(2an)\left[\sinh^2\left(\frac{an}{\alpha}\right)+\sin^2\left(\frac{bn}{\alpha}\right)\right]} \\ &= -\sum_{n=1}^{\infty} \frac{\sinh\left(\frac{2an}{\alpha}\right) e^{-(s-1)2an}}{2n\sinh(2an)\left[\cosh\left(\frac{2an}{\alpha}\right)-\cos\left(\frac{2bn}{\alpha}\right)\right]}. \end{aligned} \tag{3.4}$$

That is, $Z^{(\alpha)}(s;a,b)$, for $Re\, s > 0$, is obtained by exponentiating either sum in (3.4) - sums that appear in [2–4], for certain values of s, but with no zeta function meaning - which (3.4) therefore provides. Using the second sum in (3.4), for example, one can compute the *quantum correction function* $S^{(c)}(s;a,b)$ defined by

$$S^{(c)}(s;a,b) := \left[\alpha \frac{\partial}{\partial \alpha} - 1\right] \log Z^{(\alpha)}(s;a,b).\Big|_{\alpha=1}. \tag{3.5}$$

Namely, for $Res > 0$,

$$S^{(c)}(s;a,b) = \sum_{n=1}^{\infty} \frac{e^{-(s-1)2an}}{2n\left[\cosh(2an) - \cos(2bn)\right]} \cdot \left[1 + 2an\coth(2an) - \left(\frac{2an\sinh(2an) + 2bn\sin(2bn)}{\cosh(2an) - \cos(2bn)}\right)\right]. \tag{3.6}$$

At this point, we specialize the choice of s: $s = 1 + \sqrt{\mu}$ for $\mu > 0$.

Theorem 2.

$$\left[\alpha \frac{\partial}{\partial \alpha} - 1\right] \log Z^{(\alpha)}(1 + \sqrt{\mu};a,b)\Big|_{\alpha=1},$$

for a,b in (2.3), is the quantum correction to the classical BTZ black hole entropy given in equation (5.3) of [4].

Theorem 2 follows from formula (3.6) (since $S^{(c)}(1+\sqrt{\mu};a,b)$ is the sum $\sum_{n=1}^{\infty} s_n$ in [4]) and it provides for a zeta function expression of quantum corrected black hole entropy.

Using the parameters $\sigma, \mu > 0$, one obtains a solution

$$K_\mu(r,t) := \frac{\frac{r}{\sigma} e^{-\frac{r^2}{4t} - \frac{\mu t}{\sigma^2}}}{(4\pi t)^{3/2}\sinh\left(\frac{r}{\sigma}\right)}, \tag{3.7}$$

where $t > 0$, of the heat equation

$$\left[\frac{\partial}{\partial t} - \Box - \frac{(1-\mu)}{\sigma}\right] K_\mu(r,t) = 0, \tag{3.8}$$

where $\Box$ is the Laplacian of the de-Sitter metric $ds^2 = -dT_1^2 + dT_2^2 + dX_1^2 + dX_2^2 = dr^2 + \sigma^2\sinh^2(\frac{r}{\sigma}) \cdot \left[d\lambda^2 + \sin^2(\lambda)d\delta^2\right]$ in the coordinates (r,λ,δ) with $T_1 = \sigma\cosh\left(\frac{r}{\sigma}\right)$, $T_2 = \sigma\sinh\left(\frac{r}{\sigma}\right)\sin(\lambda)\cos(\delta)$, $X_1 = \sigma\sinh\left(\frac{r}{\sigma}\right)\cos(\lambda)$, $X_2 = \sigma\sinh\left(\frac{r}{\sigma}\right)\sin(\lambda)\sin(\delta)$: $-T_1^2 + T_2^2 + X_1^2 + X_2^2 = -\sigma^2$, and

$$\Box = \frac{\partial^2}{\partial r^2} + \frac{2\coth\left(\frac{r}{\sigma}\right)}{\sigma}\frac{\partial}{\partial r} + \frac{\operatorname{csch}^2\left(\frac{r}{\sigma}\right)}{\sigma^2}\frac{\partial^2}{\partial \lambda^2} + \frac{\cot(\lambda)\operatorname{csch}^2\left(\frac{r}{\sigma}\right)}{\sigma^2}\frac{\partial}{\partial \lambda} + \frac{\operatorname{csch}^2\left(\frac{r}{\sigma}\right)\csc(\lambda)}{\sigma^2}\frac{\partial^2}{\partial \delta^2}. \tag{3.9}$$

The heat kernel $K_{B(1)}$ of the regular BTZ black hole can be obtained by "averaging" the heat kernel $K_\mu(r,t)$. In turn, one obtains the heat kernel $K_{B(\alpha)}$ of $B(\alpha)$ in terms of $K_{B(1)}$ via a Sommerfeld formula, and one expresses the effective action of $B(\alpha)$ in terms of the trace of $K_{B(\alpha)}$; see formula (4.5) of [4]. By the latter formula and the first sum in (3.4), one sees that the *non-divergent* part of the $B(\alpha)$ effective action coincides, in fact, exactly with the logarithm of $Z^{(\alpha)}(s;a,b)$ at the point $s = 1 + \sqrt{\mu}$. Thus a close connection has been indicated between the family $\left\{Z^{(\alpha)}(s;a,b)\right\}_{0<\alpha\leq 1}$ of the zeta functions and BTZ black hole thermodynamics.

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